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Strong Convergence of Cubic Partial Sums of Two-Dimensional Walsh-Fourier Series

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We prove that certain means of the cubic partial sums of the two-dimensional Walsh-Fourier series are uniformly bounded operators from dyadic Hardy space H_1 to the space L_1. As a consequence we obtain strong convergence theorems concerning cubic partial sums.

Keywords and Phrases: Walsh function, Hardy space, strong means.

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1. Introduction

It is known [1] that the Walsh-Paley system is not a Schauder basis in $L_1(G)$ (for the definition of G, see Section 2). Moreover, there exists a function in the dyadic Hardy space $H_1(G)$, the partial sums of which are not bounded in $L_1(G)$. Simon [4] proved strong convergence result for one-dimensional Walsh-Fourier series. In particular, the following is true

Theorem S. *Let $f \in H_1(G)$. Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0.$$

For the two-dimensional Walsh-Fourier series Weisz [7] generalized the result of Simon and proved that the following result is true.

Theorem W1. *Let $f \in H_1(G \times G)$. Then*

$$\sum_{2^{-\alpha} \le k/l \le 2^{\alpha},\, (k,l) \le (n,m)} \frac{1}{\log n \log m} \sum_{k=1}^{n} \frac{\|S_{k,l} f\|_1}{k\, l} \le c\|f\|_{H_1}.$$

We prove that certain means of the cubic partial sums of the two-dimensional Walsh-Fourier series are uniformly bounded operators from dyadic Hardy space H_1 to the space L_1. As a consequence we obtain strong convergence theorems concerning cubic partial sums.

2. Definitions and Notation

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called *the Walsh group.* A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G,$$
$$I_n(x) = I_n(x_0, \ldots, x_{n-1}) := \{y \in G : \ y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$$
$$(x \in G, \ n \in \mathbb{N}).$$

These sets are called *dyadic intervals.* Let $0 = (0 : i \in \mathbb{N}) \in G$ denotes the null element of G and $I_n := I_n(0)$, $n \in \mathbb{N}$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$. Let $\overline{I}_n := G \setminus I_n$.

For $k \in \mathbb{N}$ and $x \in G$ denote by

$$r_k(x) := (-1)^{x_k} \qquad (x \in G, \ k \in \mathbb{N})$$

the k-th Rademacher function. For $n \in \mathbb{N}$, let $n = \sum_{i=0}^{\infty} n_i 2^i$ with $n_i \in \{0, 1\}$, i.e., n is represented in the binary numeral system. Denote $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} \left(r_k(x)\right)^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (x \in G, \ n \in \mathbb{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [5])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } \ x \in I_n \\ 0, & \text{if } \ x \in \overline{I}_n, \end{cases} \tag{1}$$

$$D_n(x) = w_n(x) \sum_{j=0}^{\infty} n_j w_{2^j}(x) D_{2^j}(x). \tag{2}$$

The rectangular partial sums of the 2-dimensional Walsh-Fourier series are defined as follows:

$$S_{M,N} f(x,y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j) w_i(x) w_j(y),$$

where

$$\widehat{f}(i,j) = \iint_{G\times G} f(x,y) w_i(x) w_j(y)\, d\mu(x,y)$$

is said to be the (i,j)-th Walsh-Fourier coefficient of the function f.

Denote

$$S_M^{(1)} f(x,y) := \int_G f(s,y) D_M(x+s)\, d\mu(s)$$

and

$$S_N^{(2)} f(x,y) := \int_G f(x,t) D_N(y+t)\, d\mu(t).$$

The norm in the space $L_1(G\times G)$ is defined by

$$\|f\|_1 := \iint_{G\times G} |f(x,y)|\, d\mu(x,y).$$

Let $f \in L_1(G\times G)$. Then the dyadic maximal function is given by

$$f^*(x,y) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x)\times I_n(y))} \Big| \iint_{I_n(x)\times I_n(y)} f(s,t)\, d\mu(s,t) \Big|, \quad (x,y) \in G\times G.$$

The dyadic Hardy space $H_1(G\times G)$ consists of all functions for which

$$\|f\|_{H_1} := \|f^*\|_1 < \infty.$$

A bounded measurable function a is said to be *atom*, if there exists a dyadic two-dimensional cube $I\times I$, such that

a) $\int_{I\times I} a\, d\mu = 0$;

b) $\|a\|_\infty \le \mu(I\times I)^{-1}$;

c) $\operatorname{supp} a \subset I\times I$.

We will use the following decomposition theorem (see Weisz [6]).

Theorem W2. *A function $f \in L_1(G \times G)$ is in the dyadic Hardy space $H_1(G \times G)$ if and only if there exists a sequence $\{a_k : k \in \mathbb{N}\}$ of atoms and a sequence $\{\lambda_k : k \in \mathbb{N}\}$ of real numbers such that*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k \tag{3}$$

and

$$\sum_{k=0}^{\infty} |\lambda_k| < \infty.$$

Moreover, the following equivalence of norms holds

$$\|f\|_{H_1} \sim \inf \sum_{k=0}^{\infty} |\lambda_k|$$

where the infimum is taken over all decomposition of f of the form (3).

Denote by $E_{lr}(f)_1$ the best approximation of a function $f \in L_1(G \times G)$ by Walsh polynomials of degree $\leq l$ in the variable x and of degree $\leq r$ in the variable y, and let $E_l^{(1)}(f)_1$ be the partial best approximation of the function $f \in L_1(G \times G)$ by Walsh polynomials of degree $\leq l$ in the variable x, whose coefficients are integrable functions of the other variable y. Analogously, we define $E_r^{(2)}(f)_1$.

Let $2^L \leq l < 2^{L+1}$ and $E_{2^L,2^L}(f)_1 := \|f - T_{2^L,2^L}\|_1$. Since

$$\|S_{2^L,2^L}(f)\|_1 \leq \|f\|_1$$

we can write

$$\begin{aligned} E_{2^L,2^L}(f)_1 &\leq \|f - S_{2^L,2^L}(f)\|_1 = \left\|f - S_{2^L}^{(1)}\left(S_{2^L}^{(2)}(f)\right)\right\|_1 \\ &\leq \|f - S_{2^L}^{(1)}(f)\|_1 + \left\|S_{2^L}^{(1)}\left(S_{2^L}^{(2)}(f) - f\right)\right\|_1 \\ &\leq \left\|f - S_{2^L}^{(1)}(f)\right\|_1 + \left\|S_{2^L}^{(2)}(f) - f\right\|_1. \end{aligned} \tag{4}$$

Let $T_{2^L}^{(1)}(x,y)$ be a polynomial of the best approximation $E_{2^L}^{(1)}(f)_1$. Then

$$\begin{aligned} \left\|S_{2^L}^{(1)}(f) - f\right\|_1 &\leq \left\|f - T_{2^L}^{(1)}\right\|_1 + \left\|S_{2^L}^{(1)}(f - T_{2^L}^{(1)})\right\|_1 \\ &\leq 2\left\|f - T_{2^L}^{(1)}\right\|_1 = 2E_{2^L}^{(1)}(f)_1. \end{aligned} \tag{5}$$

Analogously, we can prove that

$$\left\|S_{2^L}^{(2)}(f) - f\right\|_1 \leq 2E_{2^L}^{(2)}(f)_1. \tag{6}$$

Combining (4), (5) and (6), we conclude that

$$E_{2^L,2^L}(f)_1 \leq 2E_{2^L}^{(1)}(f)_1 + 2E_{2^L}^{(2)}(f)_1. \tag{7}$$

3. Formulation of Main Results

Theorem 1. *Let $f \in H_1(G \times G)$. Then*

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_1}{n \log^2(n+1)} \le c\|f\|_{H_1}.$$

Corollary 1. *Let $f \in H_1(G \times G)$. Then*

$$\frac{1}{\log(n+1)} \sum_{j=1}^{n} \frac{\|S_{j,j}f - f\|_1}{j \log(j+1)} \to 0 \qquad \text{as } n \to \infty.$$

Corollary 2. *Let $f \in H_1(G \times G)$. Then*

$$\sum_{n=1}^{\infty} \frac{|S_{n,n}f(x,y)|}{n \log^2(n+1)} < \infty \qquad \text{a. e. in } G \times G.$$

Theorem 2. *Let $\{\alpha_k : k \in \mathbb{P}\}$ be a non-negative and non-increasing sequence. Then for any $f \in L_1(G \times G)$ the following estimation holds true:*

$$\sum_{n=0}^{\infty} \Big\| \sum_{k=2^n}^{2^{n+1}-1} \alpha_k (S_{k,k}f - f) \Big\|_1 \le c \sum_{k=1}^{\infty} \alpha_k \big(E_k^{(1)}(f)_1 + E_k^{(2)}(f)_1\big).$$

Corollary 3. *Let $f \in L_1(G \times G)$ and*

$$\sum_{k=1}^{\infty} \alpha_k \big(E_k^{(1)}(f)_1 + E_k^{(2)}(f)_1\big) < \infty. \tag{8}$$

Then

$$\sum_{n=0}^{\infty} \Big| \sum_{k=2^n}^{2^{n+1}-1} \alpha_k (S_{k,k}f(x,y) - f(x,y)) \Big| < \infty \qquad \text{a. e. in } G \times G.$$

Theorem 2 in the one-dimensional case for trigonometric Fourier series was announced in [3].

It is not known whether condition (8) yields a. e. convergence of the series

$$\sum_{k=0}^{\infty} \alpha_k |S_{k,k}f(x,y) - f(x,y)|.$$

4. Proofs

Proof of Theorem 1. From Theorem W2 we can write

$$\iint_{G\times G} |S_{n,n}f(x,y)|\, d\mu(x,y) \le \sum_{j=0}^{\infty} |\lambda_j| \iint_{G\times G} |S_{n,n}a_j(x,y)|\, d\mu(x,y).$$

Because of this and Theorem W1 we only have to prove that

$$\sum_{n=1}^{\infty}\frac{\|S_{n,n}a\|_1}{n\log^2(n+1)}\leq c \tag{9}$$

for every atom a.

Let a be any atom with support $I_N(z')\times J_N(z'')$ and $\mu(I_N)=\mu(J_N)=2^{-N}$. We may assume that $z'=z''=0$.

Let $(x,y)\in\overline{I}_N\times\overline{I}_N$, then $D_{2^i}(x+s)1_{I_N}(s)=0$ and $D_{2^i}(y+t)1_{I_N}(t)=0$ for $j\geq N$. Recall that $w_{2^j}(x+t)=w_{2^j}(x)$ for $t\in I_N$ and $j<N$. Consequently, from (2) we obtain

$$\begin{aligned}
S_{n,n}a(x,y) &= \iint_{G\times G} a(s,t)D_n(x+s)D_n(y+t)\,d\mu(s,t)\\
&= \iint_{I_N\times I_N} a(s,t)D_n(x+s)D_n(y+t)\,d\mu(s,t)\\
&= \iint_{I_N\times I_N} a(s,t)w_n(x+s+y+t)\sum_{i=0}^{N-1}n_iw_{2^i}(x+s)D_{2^i}(x+s)\\
&\qquad\times\sum_{j=0}^{N-1}n_jw_{2^j}(y+t)D_{2^j}(y+t)\,d\mu(s,t)\\
&= w_n(x)\sum_{i=0}^{N-1}n_iw_{2^i}(x)D_{2^i}(x)w_n(y)\sum_{j=0}^{N-1}n_jw_{2^j}(y)D_{2^j}(y)\\
&\qquad\times\iint_{I_N\times I_N} a(s,t)w_n(s+t)\,d\mu(s,t)\\
&= w_n(x+y)\sum_{i=0}^{N-1}n_iw_{2^i}(x)D_{2^i}(x)\sum_{j=0}^{N-1}n_jw_{2^j}(y)D_{2^j}(y)\\
&\qquad\times\int_{I_N}\Big(\int_{I_N}a(t+\tau,t)\,d\mu(t)\Big)w_n(\tau)\,d\mu(\tau)\\
&= w_n(x+y)\sum_{i=0}^{N-1}n_iw_{2^i}(x)D_{2^i}(x)\sum_{j=0}^{N-1}n_jw_{2^j}(y)D_{2^j}(y)\\
&\qquad\times\int_{I_N}\Phi(\tau)w_n(\tau)\,d\mu(\tau)\\
&= w_n(x+y)\sum_{i=0}^{N-1}n_iw_{2^i}(x)D_{2^i}(x)\sum_{j=0}^{N-1}n_jw_{2^j}(y)D_{2^j}(y)\widehat{\Phi}(n),
\end{aligned}$$

where

$$\Phi(\tau)=\int_{I_N}a(t+\tau,t)\,d\mu(t).$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n\log^2(n+1)} \iint_{\overline{I}_N \times \overline{I}_N} |S_{n,n}a(x,y)|\, d\mu(x,y)$$
$$\leq \sum_{n=1}^{\infty} \frac{|\widehat{\Phi}(n)|}{n\log^2(n+1)} \Big(\int_{\overline{I}_N} \sum_{i=0}^{N-1} D_{2^i}(x)\, d\mu(x) \Big)^2 \leq N^2 \sum_{n=1}^{\infty} \frac{|\widehat{\Phi}(n)|}{n\log^2(n+1)}.$$

Let $n < 2^N$. Since $w_n(\tau) = 1$ for $\tau \in I_N$ and $n < 2^N$, we have

$$\widehat{\Phi}(n) = \int_G \Phi(\tau) w_n(\tau)\, d\mu(\tau) = \int_G \Big(\int_{I_N} a(t+\tau, t)\, d\mu(t) \Big) w_n(\tau)\, d\mu(\tau)$$
$$= \iint_{I_N \times I_N} a(s,t)\, d\mu(s,t) = 0.$$

Hence, we can suppose that $n \geq 2^N$.

By Hölder and Parseval's inequality we obtain

$$N^2 \sum_{n=1}^{\infty} \frac{|\widehat{\Phi}(n)|}{n\log^2(n+1)} \leq N^2 \Big(\sum_{n=2^N}^{\infty} \frac{1}{n^2 \log^4(n+1)} \Big)^{1/2} \Big(\sum_{n=2^N}^{\infty} |\widehat{\Phi}(n)|^2 \Big)^{1/2}$$
$$\leq \frac{c}{2^{N/2}} \Big(\int_G |\Phi(\tau)|^2\, d\mu(\tau) \Big)^{1/2}$$
$$= \frac{c}{2^{N/2}} \Big(\int_{I_N} \Big| \int_{I_N} a(t+\tau, t)\, d\mu(t) \Big|^2 d\mu(\tau) \Big)^{1/2}$$
$$\leq \frac{c}{2^{N/2}} \|a\|_\infty \frac{1}{2^N} \frac{1}{2^{N/2}} \leq c < \infty. \tag{10}$$

Let $x, y \in \overline{I}_N \times I_N$. Then we have

$$S_{n,n}a(x,y) = w_n(x) \sum_{j=0}^{N-1} n_j w_{2^j}(x) D_{2^j}(x) \iint_{G \times G} a(s,t) w_n(s) D_n(y+t)\, d\mu(s,t)$$
$$= w_n(x) \sum_{j=0}^{N-1} n_j w_{2^j}(x) D_{2^j}(x) \int_G S_n^{(2)} a(s,y) w_n(s)\, d\mu(s)$$
$$= w_n(x) \sum_{j=0}^{N-1} n_j w_{2^j}(x) D_{2^j}(x) \widehat{S}_n^{(2)} a(n,y)\,.$$

Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{n\log^2(n+1)} \iint_{\overline{I}_N \times I_N} |S_{n,n}a(x,y)|\, d\mu(x,y)$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n\log^2(n+1)} \iint_{\overline{I}_N \times I_N} \sum_{j=0}^{N-1} D_{2^j}(x) |\widehat{S}_n^{(2)} a(n,y)|\, d\mu(x,y).$$

Let $n < 2^N$. Then by the definition of atom we have

$$\begin{aligned}\widehat{S}_n^{(2)} a(n,y) &= \int_G \left(\int_G a(s,t) D_n(y+t)\, d\mu(t) \right) w_n(s)\, d\mu(s) \\ &= D_n(y) \iint_{I_N \times I_N} a(s,t)\, d\mu(s,t) = 0.\end{aligned}$$

Hence, we can suppose that $n > 2^N$. Then we have

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{1}{n \log^2(n+1)} \iint_{\overline{I}_N \times I_N} |S_{n,n} a(x,y)|\, d\mu(x,y) \\ &\qquad\qquad \le N \sum_{n=2^N}^{\infty} \frac{1}{n \log^2(n+1)} \int_{I_N} \left| \widehat{S}_n^{(2)} a(n,y) \right| d\mu(y).\end{aligned}$$

Since

$$\left\| \widehat{S}_n^{(2)} a(n,y) \right\|_2 \le c \, \|a\|_2$$

from Hölder inequality we can write

$$\begin{aligned}&\int_{I_N} \left| \widehat{S}_n^{(2)} a(n,y) \right| d\mu(y) \le \int_{I_N} \left| \int_G S_n^{(2)} a(s,y) w_n(s) d\mu(s) \right| d\mu(y) \\ &= \int_{I_N} \left| \int_{I_N} \left(\int_{I_N} a(s,t) D_n(y+t)\, d\mu(t) \right) w_n(s)\, d\mu(s) \right| d\mu(y) \\ &\le \int_{I_N} \left(\int_{I_N} \left| \int_{I_N} a(s,t) D_n(y+t)\, d\mu(t) \right| d\mu(y) \right) d\mu(s) \\ &\le \frac{c}{2^{N/2}} \int_{I_N} \left(\int_{I_N} \left| \int_{I_N} a(s,t) D_n(y+t)\, d\mu(t) \right|^2 d\mu(y) \right)^{1/2} d\mu(s) \\ &\le \frac{c}{2^{N/2}} \int_{I_N} \left(\int_{I_N} |a(s,t)|^2\, d\mu(t) \right)^{1/2} d\mu(s) \le \frac{c \|a\|_\infty}{2^{N/2}} \cdot \frac{1}{2^N} \cdot \frac{1}{2^{N/2}} \le c < \infty.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{1}{n \log^2(n+1)} \iint_{\overline{I}_N \times I_N} |S_{n,n} a(x,y)|\, d\mu(x,y) \\ &\qquad\qquad \le c\, N \sum_{n=2^N}^{\infty} \frac{1}{n \log^2(n+1)} \le c < \infty.\end{aligned} \tag{11}$$

Analogously, we can prove that

$$\sum_{n=1}^{\infty} \frac{1}{n \log^2(n+1)} \int_{I_N \times \overline{I}_N} |S_{n,n} a(x,y)|\, d\mu(x,y) \le c < \infty. \tag{12}$$

Let $(x,y) \in I_N \times I_N$. Then by the definition of atom we can write

$$\begin{aligned}
\int_{I_N\times I_N} |S_{n,n}a(x,y)|\, d\mu(x,y) &\le \frac{1}{2^N}\Big(\int_{I_N\times I_N} |S_{n,n}a(x,y)|^2\, d\mu(x,y)\Big)^{1/2} \\
&\le \frac{1}{2^N}\Big(\int_{I_N\times I_N} |a(x,y)|^2\, d\mu(x,y)\Big)^{1/2} \\
&\le \frac{\|a\|_\infty}{2^{2N}} \le c < \infty.
\end{aligned} \tag{13}$$

Combining (10)–(13) we obtain (9). Theorem 1 is proved. □

Proof of Theorem 2. We have

$$\begin{aligned}
\Big\|\sum_{k=2^n}^{2^{n+1}-1} \alpha_k (S_{k,k}f - f)\Big\|_1 &\le \Big\|\sum_{k=2^n}^{2^{n+1}-1} \alpha_k S_{k,k}(f - T_{2^n,2^n})\Big\|_1 \\
&\quad + \Big\|\sum_{k=2^n}^{2^{n+1}-1} \alpha_k (f - T_{2^n,2^n})\Big\|_1 \\
&\le \Big\|\sum_{k=2^n}^{2^{n+1}-1} \alpha_k S_{k,k}(f - T_{2^n,2^n})\Big\|_1 \\
&\quad + 2^n \alpha_{2^n} E_{2^n,2^n}(f)_1.
\end{aligned} \tag{14}$$

By Abel's transformation we can write

$$\begin{aligned}
\sum_{k=2^n}^{2^{n+1}-1} \alpha_k S_{k,k} g &= \sum_{k=2^n}^{2^{n+1}-1} (\alpha_k - \alpha_{k+1}) \sum_{j=1}^{k} S_{j,j} g \\
&\quad + \alpha_{2^{n+1}-1} \sum_{j=1}^{2^{n+1}-1} S_{j,j} g - \alpha_{2^n} \sum_{j=1}^{2^n-1} S_{j,j} g.
\end{aligned} \tag{15}$$

Since

$$\Big\|\sum_{j=1}^{k} S_{j,j} g\Big\|_1 \le c\, k \|g\|_1,$$

we obtain from (15) that

$$\Big\|\sum_{k=2^n}^{2^{n+1}-1} \alpha_k S_{k,k} g\Big\|_1 \le c\, 2^n \alpha_{2^n} \|g\|_1.$$

Consequently,

$$\Big\|\sum_{k=2^n}^{2^{n+1}-1} \alpha_k S_{k,k}(f - T_{2^n,2^n})\Big\|_1 \le c\, 2^n \alpha_{2^n} E_{2^n,2^n}(f)_1. \tag{16}$$

Combining (7), (14) and (16) we have

$$\sum_{n=1}^{\infty}\Big\|\sum_{k=2^n}^{2^{n+1}-1}\alpha_k(S_{k,k}f-f)\Big\|_1 \le c\sum_{n=1}^{\infty}2^n\alpha_{2^n}E_{2^n,2^n}(f)_1$$
$$\le c\sum_{n=1}^{\infty}2^n\alpha_{2^n}\big(E_{2^n}^{(1)}(f)_1+E_{2^n}^{(2)}(f)_1\big) \le c\sum_{k=1}^{\infty}\alpha_k\big(E_k^{(1)}(f)_1+E_k^{(2)}(f)_1\big).$$

Theorem 2 is proved. □

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