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On Recent Progress in Gasca-Maeztu Conjecture

Hakop Hakopian

In this paper we present an overview of some of recent main results in the area of Gasca-Maeztu conjecture. We mainly consider the multivariate (dimension ≥ 3) case where the generalized conjecture is due to Carl de Boor. Many results we present in more general setting. Namely, instead of standard assumption of geometric characterization property we assume just independence. We provide proofs of some recent basic results. Usually they are modified and shorter versions of the original ones. Some new results also are presented. In particular, the classification of GC_2 sets having 3 maximal planes in $\mathbb{R}^3$ is obtained, which is the last step in classification of all GC_2 sets in $\mathbb{R}^3$.

Keywords and Phrases: multivariate interpolation, poised set, geometric characterization, Gasca-Maeztu conjecture, de Boor conjecture, maximal hyperplane.

1. Introduction

Denote by $\Pi_n^d = \Pi_n(\mathbb{R}^d)$ the space of algebraic polynomials in d variables of total degree not exceeding n:

$$\Pi_n^d = \{p(\mathbf{x}) = \sum_{|\alpha|\leq n} a_\alpha \mathbf{x}^\alpha : a_\alpha \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d\},$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d,$$

and

$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \qquad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

Let

$$N := N_n := N_n^d := \dim \Pi_n^d = \binom{n+d}{d}.$$

Let us fix a set of points called *knots*:

$$\mathcal{X}_s := \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(s)}\} \subset \mathbb{R}^d.$$

Denote

$$\mathcal{X} := \mathcal{X}_N.$$

The problem of finding $p \in \Pi_n^d$ satisfying the conditions

$$p(\mathbf{x}^{(k)}) = c_k, \qquad k = 1, 2, \ldots, s \tag{1}$$

is called *interpolation problem* and denoted briefly $(\Pi_n^d, \mathcal{X}_s)$; p is called an *interpolating polynomial.*

Definition 1. The interpolation problem $(\Pi_n^d, \mathcal{X}_s)$ is called *solvable*, if for any set of values $\{c_1, c_2, \ldots, c_s\}$ there exists a polynomial $p \in \Pi_n^d$ satisfying the conditions (1).

An interpolating polynomial p satisfying the conditions

$$p \in \Pi_n^d \;\text{ and }\; p(\mathbf{x}^{(j)}) = \delta_{jk}, \qquad 1 \le j \le N,$$

where δ_{jk} is the Kronecker symbol, is called *an n-fundamental polynomial* for $A = \mathbf{x}^{(k)} \in \mathcal{X}_s$. In the sequel we will denote this fundamental polynomial by

$$p_A^\star := p_{A,\mathcal{X}_s}^\star := p_{k,\mathcal{X}_s}^\star := p_k^\star.$$

Definition 2. The set $\mathcal{X}_s$ is called Π_n^d*-independent* (or briefly *n-independent*), if for any its knot there exists an n-fundamental polynomial.

The following characterization is easily seen by using elementary linear algebra.

Proposition 1. *The interpolation problem $(\Pi_n^d, \mathcal{X}_s)$ is solvable if and only if $\mathcal{X}_s$ is n-independent.*

Obviously fundamental polynomials are linearly independent. Thus if the set $\mathcal{X}_s$ is n-independent then

$$s \le N. \tag{2}$$

Furthermore, an interpolation polynomial can be expressed by the Lagrange formula

$$p = \sum_{j=1}^{N} p(\mathbf{x}^{(j)}) p_j^\star. \tag{3}$$

Next we consider the poisedness of a knot set.

Definition 3. The interpolation problem $(\Pi_n^d, \mathcal{X}_s)$ is called *poised*, if for any set of values $\{c_1, c_2, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n^d$ satisfying the conditions (1).

The following two propositions are elementary linear algebra facts:

Proposition 2. *The interpolation problem* $(\Pi_n^d, \mathcal{X}_s)$ *is poised if and only if* $\mathcal{X}_s$ *is n-independent and*

$$s = N,$$

i.e., $\mathcal{X}_s = \mathcal{X}$.

Proposition 3. *The interpolation problem* $(\Pi_n^d, \mathcal{X})$ *is poised if and only if*

$$p \in \Pi_n^d \quad \textit{and} \quad p(\boldsymbol{x}^{(j)}) = 0, \quad j = 1, 2, \ldots, N \quad \Rightarrow \quad p = 0.$$

Thus maximal possible number of n-independent points in $\mathbb{R}^d$ is N.

Next we are going to find a similar number for any k-dimensional flat in $\mathbb{R}^d$, where $k \leq d$. By *k-dimensional flat* we mean a shift of k-dimensional linear subspace of $\mathbb{R}^d$. We call $(d-1)$-dimensional flats *hyperplanes.*

Denote the restriction (i.e., trace) of a polynomial $p \in \Pi_n^d$ on F by $p|_F$.

Definition 4. We call knot set $\mathcal{Y} \subset F$ *n-complete in F* if

$$p \in \Pi_n, \quad p|_{\mathcal{Y}} = 0 \quad \Rightarrow \quad p|_F = 0.$$

We call $\mathcal{Y} \subset F$ *n-basic in F* if it is both n-complete in F and n-independent.

The following is a well-known fact in linear algebra: If the set $\mathcal{X}_s$ is n-complete in $\mathbb{R}^d$ then

$$s \geq N. \tag{4}$$

Proposition 4. *Suppose that F is a k-dimensional flat and $\mathcal{Y} \subset F$.*
(i) If $\mathcal{Y}$ is n-independent, then

$$\#\mathcal{Y} \leq N_n^k. \tag{5}$$

(ii) If $\mathcal{Y}$ is n-complete in F, then

$$\#\mathcal{Y} \geq N_n^k. \tag{6}$$

In particular, if $\mathcal{Y}$ is n-basic in F, then

$$\#\mathcal{Y} = N_n^k. \tag{7}$$

(iii) If (7) *holds then $\mathcal{Y}$ is n-complete in F if and only if $\mathcal{Y}$ is n-independent.*

Proof. Suppose that F is given by the linearly independent system

$$H_i(\mathbf{x}) = 0, \qquad i = 1, \ldots, d-k,$$

where $H_i \in \Pi_1$.

Without loss of generality assume that the above system is solved with respect to the last $d-k$ variables:

$$x_i = h_i(x_1, \ldots, x_k), \qquad i = k+1, \ldots, d.$$

For $A = (x_1, \ldots, x_d)$ denote by $\breve{A}$ its projection on $\mathbb{R}^k$:

$$\breve{A} = (x_1, \ldots, x_k).$$

Denote

$$\breve{\mathcal{Y}} := \{\breve{A} : A \in \mathcal{Y}\} \subset \mathbb{R}^k.$$

We are going to verify that $\mathcal{Y}$ is n-independent in $\mathbb{R}^d$ if and only if $\breve{\mathcal{Y}}$ is n-independent in $\mathbb{R}^k$.

For a polynomial $p \in \Pi_n^d$ denote by $\breve{p}$ the polynomial from Π_n^k which represents the trace of p on the k-dimensional flat F:

$$\breve{p}(x_1, \ldots, x_k) = p(x_1, \ldots, x_k, h_{k+1}(x_1, \ldots, x_k), \ldots, h_d(x_1, \ldots, x_k)).$$

Now suppose that $\mathcal{Y}$ is n-independent and $p_A^\star(x_1, \ldots, x_d)$ is a fundamental polynomial of $A \in \mathcal{Y}$. Then it is easily seen that $\breve{p}_A^\star = p_{\breve{A}}^\star(x_1, \ldots, x_k)$, i.e., $\breve{p}_A^\star$ is a fundamental polynomial of $\breve{A} \in \breve{\mathcal{Y}}$. Therefore $\breve{\mathcal{Y}}$ is n-independent. Now in view of (2) we get that (5) holds.

Next suppose that $\breve{\mathcal{Y}}$ is n-independent and fix $A \in \mathcal{Y}$, then $\breve{A} \in \breve{\mathcal{Y}}$. Let $p_{\breve{A}}^\star(x_1, \ldots, x_k)$ be its n-fundamental polynomial. Then considering this as polynomial of d-variables, i.e., by setting

$$q(x_1, \ldots, x_d) := p_{\breve{A}}^\star(x_1, \ldots, x_k),$$

we readily observe that it is n-fundamental for A with respect to the set $\mathcal{Y}$, i.e., the set $\mathcal{Y}$ is n-independent.

Next, let us verify similarly that $\mathcal{Y}$ is n-complete in F if and only if $\breve{\mathcal{Y}}$ is n-complete in $\mathbb{R}^k$.

Suppose that $\mathcal{Y}$ is n-complete in F and $p(x_1, \ldots, x_k)$ vanishes at $\breve{\mathcal{Y}}$. Then considering this as polynomial of d-variables, i.e., by setting

$$q(x_1, \ldots, x_d) := p(x_1, \ldots, x_k),$$

we readily observe that it vanishes at the set $\mathcal{Y}$. Since the latter is n-complete in F, we conclude that $q = 0$, thus $p = 0$ or, in other words, $\breve{\mathcal{Y}}$ is n-complete in $\mathbb{R}^k$. In view of (4) we get that (6) holds.

Thus, we have in particular that $\mathcal{Y}$ is n-basic in F if and only if $\breve{\mathcal{Y}}$ is n-basic in $\mathbb{R}^k$.

Now let us turn to (iii). Suppose that $\mathcal{Y}$ is n-independent and $\#\mathcal{Y} = N_n^k$. Then $\breve{\mathcal{Y}}$ is n-independent and has the same cardinality. Therefore $\breve{\mathcal{Y}}$ is n-basic in $\mathbb{R}^k$, which means that $\mathcal{Y}$ is n-complete in F. Similarly, we verify the reverse implication. $\square$

Remark 1. As it follows from the above proof, a set of points on a k-dimensional flat F in $\mathbb{R}^d$ is n-independent (or n-complete in F) if and only if the set of their projections on some k-dimensional coordinate flat is n-independent (or n-complete in $\mathbb{R}^k$, respectively).

2. Maximal Hyperplanes and Flats

Definition 5. A k-dimensional flat F in $\mathbb{R}^d$ is called *n-maximal* (or just *maximal*) for the set $\mathcal{X}$ if $\mathcal{X} \cap F$ is n-basic in F. In particular,

$$\#\{\mathcal{X} \cap F\} = N_n^k. \tag{8}$$

Notice that, in view of Proposition 4 (iii), F is maximal for n-independent set $\mathcal{X}$ if (8) holds.

Proposition 5 (Apozyan [2]). *Suppose $\mathcal{X}$ is an n-poised set. Then a k-dimensional flat F is maximal for it if and only if for each knot $A \in \mathcal{X} \setminus F$ the fundamental polynomial $p_A^\star$ vanishes on F.*

Proof. Indeed, the direct implication is a corollary of above Definition 5, since the fundamental polynomials mentioned here vanish at $\mathcal{X} \cap F$. The converse implication follows from the Lagrange formula (3), where the fundamental polynomials with nonzero coefficients also vanish at $\mathcal{X} \cap F$. □

Proposition 6. *Suppose $\mathcal{X}$ is an n-independent (poised) set. Suppose also that a k-dimensional flat F is n-maximal for it. Next the set $\mathcal{X}'$ is obtained from $\mathcal{X}$ by changing the positions of knots from $\mathcal{X} \cap F$ inside F such that they remain n-independent. Then the set $\mathcal{X}'$ is an n-independent (poised) set.*

Proof. Without loss of generality suppose that $\mathcal{X}$ is an n-poised set, since otherwise we could enlarge it to such a set. Now in view of Proposition 3 it suffices to verify that

$$p \in \Pi_n, \quad p|_{\mathcal{X}'} = 0 \quad \Rightarrow \quad p|_{\mathcal{X}} = 0.$$

The latter follows readily from Proposition 4 (iii). □

Next we consider the case of maximal hyperplanes. We use same letter, say H to denote both the hyperplane and the linear polynomial $H \in \Pi_1^d$ which takes part in the equation of the hyperplane: $H(\mathbf{x}) = 0$.

Definition 6. Suppose $\mathcal{X}$ is an n-independent set. We say that *a point $A \in \mathcal{X}$ uses a hyperplane H* if the n-fundamental polynomial $p_A^\star$ contains the linear factor H, or equivalently, $p_A^\star|_H = 0$.

Now we get readily from Proposition 5 the following corollaries:

Corollary 1 (C. de Boor [3]). *Suppose $\mathcal{X}$ is an n-poised set. Then the hyperplane H is maximal for $\mathcal{X}$ if and only if each knot $A \in \mathcal{X} \setminus H$ uses the hyperplane H.*

Corollary 2. *Suppose that the hyperplane H is maximal for n-independent set $\mathcal{X}$. Then the set $\mathcal{X}' = \mathcal{X} \setminus H$ is $(n-1)$-independent set. Also, each maximal hyperplane of $\mathcal{X}$ distinct from H is maximal for $\mathcal{X}'$. Moreover, if the set $\mathcal{X}$ is n-poised, then $\mathcal{X}'$ is $(n-1)$-poised, too.*

Proof. Indeed, the fundamental polynomials of $A \in \mathcal{X}'$ with respect to $\mathcal{X}$ have the form Hp, where $p \in \Pi_{n-1}$. It is readily noticed that p is a fundamental polynomial of A with respect to $\mathcal{X}'$. Also according to Proposition 5, p contains factor H', where H' is a maximal, distinct from H and $A \in \mathcal{X} \setminus H'$. The moreover part follows from the equality

$$\#\mathcal{X}' = \#\mathcal{X} - N_n^{d-1} = N - N_n^{d-1} = N_{n-1}. \qquad \square$$

For an n-poised set $\mathcal{X}$ and given hyperplane H denote by $\mathcal{N}_H$ the set of all knots of $\mathcal{X}$ that do not lie on H and do not use it.

Corollary 1 means that

$$H \text{ is maximal if and only if } \mathcal{N}_H = \emptyset. \tag{9}$$

Proposition 7 (Carnicer and Gasca [5]). *Suppose $\mathcal{X}$ is n-poised and H is a non-maximal hyperplane. Then $\mathcal{N}_H$ is a nonempty $(n-1)$-dependent set. Moreover, no knot of $\mathcal{N}_H$ has an $(n-1)$-fundamental polynomial.*

Proof (see [9]). In view of (9) we have that $\mathcal{N} := \mathcal{N}_H \neq \emptyset$. Suppose conversely that $A \in \mathcal{N}$ has an $(n-1)$-fundamental polynomial $p^\star_{A,\mathcal{N}}$. Consider the polynomial

$$p = p^\star_{A,\mathcal{X}} - \gamma H p^\star_{A,\mathcal{N}} \in \Pi_n, \tag{10}$$

where γ is chosen so that $p(A) = 0$. Thus p vanishes on the whole $\mathcal{N}$ and due to its form, also on $\mathcal{X} \cap H$. According to the Lagrange formula (3) it is a linear combination of fundamental polynomials of knots, where p does not vanish. Thus we get

$$p = \sum_{B \in Z} c_i p^\star_{B,X}, \tag{11}$$

where $Z = X \setminus \{\mathcal{N} \cup H\}$. It is easily seen that the fundamental polynomials in the right-hand side are exactly those which use H. Thus p and therefore, in view of (10), also $p^\star_{A,X}$ uses H, which contradicts the definition of the set $\mathcal{N}$. $\square$

The Vandermonde determinant $V_n(\mathcal{X})$ for the point set $\mathcal{X} = \{\mathbf{x}^i\}_{i=1}^N$ and Π_n^d is the following determinant:

$$V_n(\mathcal{X}) = \begin{vmatrix} 1 & x_1^1 & \cdots & x_d^1 & \cdots & (x_1^1)^n & (x_1^1)^{n-1}x_2^1 & \cdots & (x_d^1)^n \\ 1 & x_1^2 & \cdots & x_d^2 & \cdots & (x_1^2)^n & (x_1^2)^{n-1}x_2^2 & \cdots & (x_d^2)^n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_1^N & \cdots & x_d^N & \cdots & (x_1^N)^n & (x_1^N)^{n-1}x_2^N & \cdots & (x_d^N)^n \end{vmatrix}. \tag{12}$$

Thus $V_n(\mathcal{X})$ is an $N \times N$-determinant.

Obviously we have that the interpolation problem $(\mathcal{X}, \Pi_n)$ is n-poised if and only if $V_n(\mathcal{X}) \neq 0$.

Below we consider hyperplanes that contain just one knot less than maximals.

Theorem 1. *Suppose that the hyperplane H contains $N-1$ knots of n-poised set $\mathcal{X}$. Then for each knot $A \in \mathcal{N}_H$ the set $\mathcal{X}' = \mathcal{X} \setminus \{H \cup \{A\}\}$ is an $(n-1)$-poised set. In particular, the set $\mathcal{N}_H \setminus A$ is an $(n-1)$-independent set. Also, each maximal hyperplane of $\mathcal{X}$ not containing the knot A is maximal also for $\mathcal{X}'$.*

Proof. Consider the Vandermonde determinant of the interpolation problem $(\mathcal{X}, \Pi_n)$ with the knot A replaced by variable $\mathbf{x}$ and denote it by $V(\mathbf{x})$. We have that $V(A) \neq 0$, since the set $\mathcal{X}$ is poised. Therefore $V(\mathbf{x})$ is a constant times the fundamental polynomial of A which does not vanish identically on F. Therefore there exists $A' \in F$ such that $V(A') \neq 0$. This means that the above interpolation problem, where A is replaced by A', is poised. But H becomes maximal for the changed set of knots and it remains to apply Corollary 2. $\square$

It is worth mentioning that if $A \notin \mathcal{N}_H$, then the set $\mathcal{X}'$ in Theorem 1 is not $(n-1)$-poised since it contains $(n-1)$-dependent set $\mathcal{N}_H$.

Next we are going to investigate the set of maximal hyperplanes of independent point sets.

Proposition 8 (Apozyan [2]). *Suppose $\mathcal{X}$ is an n-independent set. Suppose H is a maximal hyperplane and F is a maximal k-dimensional flat, $k \geq 1$. Then either H contains F or $E := H \cap F$ is a $(k-1)$-dimensional maximal flat.*

Proof. First let us verify that $E \neq \emptyset$. Indeed, assume conversely that $E = \emptyset$. Then $\mathcal{X}' := \mathcal{X} \setminus H$ is an $(n-1)$-independent set while the k-flat F contains $N_n^k > N_n^{k-1}$ its knots which contradicts (5). Now from linear algebra we have that F is a $(k-1)$-dimensional flat. Then let us extend $\mathcal{X}$ to $\bar{\mathcal{X}}$ which is an n-poised set (see [9]). Finally, in view of Proposition 5, it is enough to verify that the fundamental polynomial of each knot

$$A \in \bar{\mathcal{X}} \setminus E \tag{13}$$

vanishes on E. Indeed, suppose that (13) holds. Then either $A \notin H$ or $A \notin F$. Thus, in view of Proposition 5, the fundamental polynomial vanishes at H or F, respectively. Therefore it vanishes at E. $\square$

Next it is established that the maximal hyperplanes of any independent set are in general position.

Proposition 9 (C. de Boor [3]). *Suppose $\mathcal{X}$ is an n-independent set and $\mathcal{H} := \{H_i,\ i = 1, \dots, s\}$, $s \le d+1$, is a set of distinct maximal hyperplanes. Then*

$$F := \bigcap_{i=1}^{s} H_i$$

is a maximal $(d-s)$-dimensional flat. In the case $s = d$ and $s = d+1$, F is a knot of $\mathcal{X}$ or $\emptyset$, respectively. Moreover, the set $\mathcal{H}$ is determined uniquely by F, if $s \le d$, i.e., there is no other hyperplane H containing F.

Conversely, suppose, taking into account Proposition 8, that F is maximal k-dimensional flat, where $k > d - s$. Then without loss of generality we may assume that

$$F = \bigcap_{i=1}^{s-1} H_i \quad \text{and} \quad F \subset H_s.$$

According to Proposition 2, we have that $\mathcal{X}' := \mathcal{X} \setminus H_s$ is $(n-1)$-independent set and H_i, $i = 1, \dots, s-1$, are still maximal hyperplanes with

$$F \cap \mathcal{X}' = \emptyset. \tag{14}$$

But in view of Proposition 8 we have that F is maximal flat for $\mathcal{X}'$ of dimension

$$\dim F \ge d - (s-1) \ge 0,$$

which contradicts to (14). □

3. Knot Sets with *GC* Property

We now present the construction of poised sets introduced by Chung and Yao [7].

Definition 7. A set of knots $\mathcal{X} \subset \mathbb{R}^d$, $\#\mathcal{X} = N$, is said to satisfy *the geometric characterization GC_n*, or is *a GC_n set* for short, if the fundamental polynomial of each $A \in \mathcal{X}$ is product of linear factors:

$$p_A^\star = \gamma \cdot h_1^A \cdot h_2^A \cdots h_n^A,$$

or geometrically, there exist n hyperplanes $h_1^A, h_2^A, \dots, h_n^A$, such that

$$\mathcal{X} \setminus \{A\} \subset h_1^A \cup h_2^A \cup \cdots \cup h_n^A \quad \text{but} \quad A \notin h_1^A \cup h_2^A \cup \cdots \cup h_n^A.$$

In this case we say that the knot A uses the hyperplanes $h_1^A, h_2^A, \dots, h_n^A$.

Remark 2. Note that a hyperplane that is used by a knot is determined uniquely by the knots from $\mathcal{X}$ through which it passes.

Indeed, otherwise one could replace the hyperplane by another one, passing through same knots, in the knot's fundamental polynomial, thus contradicting its uniqueness.

Note that each set $\mathcal{X} \subset \mathbb{R}^d$ satisfying GC_n is Π_n^d-poised by Proposition 2.

The Gasca-Maeztu conjecture, also known as the GM-conjecture, is the following [8]:

GM**-conjecture.** *If a set $\mathcal{X}$ of knots in $\mathbb{R}^2$ satisfies GC_n, then there is a maximal line, i.e., line passing through $n+1$ knots of $\mathcal{X}$.*

So far the GM-conjecture has been verified only for $n \leq 4$ in [4] (see also [5] and [9]). Actually, the GM-conjecture states that every Chung-Yao set (GC set) is a particular case of another well-known construction, called Berzolari-Radon set: there exist lines $l_0, l_1, \ldots, l_n$, such that $l_i \setminus (l_0 \cup \cdots \cup l_{i-1})$ contains exactly $n+1-i$ knots, $i = 0, 1, \ldots, n$ (see e.g. [9]).

C. de Boor [3] generalized the GM-conjecture for $\mathbb{R}^d$:

GM_d**-conjecture.** *If a set $\mathcal{X}$ of knots in $\mathbb{R}^d$ satisfies GC_n, then there is a hyperplane passing through $\dim \Pi_n^{d-1}$ knots of $\mathcal{X}$.*

For $\mathbb{R}^2$ we have the following:

Theorem 2 (Carnicer and Gasca [6]). *If GM-conjecture is true, then there exist at least three maximal lines.*

Note that the mentioned minimal number of maximal lines is attained for the well-known Newton lattice $\mathcal{N}$, which is a GC_n set:

$$\mathcal{N} = \{(i,j) : i + j \leq n,\ i, j \in Z_+\}.$$

On the basis of above result and the fact that the Newton lattice in d-dimension has $d+1$ maximal hyperplanes C. de Boor made the following:

Conjecture (de Boor [3]). *Every GC set in $\mathbb{R}^d$ has at least $d+1$ maximal hyperplanes.*

In the same paper C. de Boor also presented a counterexample, which shows that this at first sight natural conjecture is not true.

In view of this one could doubt whether in higher dimensions, or even in the bivariate case the Gasca-Maeztu conjecture is true.

In [1] the following important result was proved.

Theorem 3 (Apozyan, Avagyan, and Ktryan [1]). *GM_d-conjecture is true for Π_2^3.*

Below we are going to present a modified and shorter proof of Theorem 3 (cf. [1]). First we bring a list of results for Π_2^3 some of which were used in the original proof (see [1]) and some are special cases of results presented in previous sections.

Proposition 10. *If the set of* 10 *knots* $\mathcal{X} = \{A_1, \ldots, A_{10}\} \subset \mathbb{R}^3$*, satisfies the* GC_2 *property, then:*

(a) There is no line passing through 4 *knots.*

(b) If 3 *knots are on a line, then one of the planes for each of remaining* 7 *knots should necessarily pass through that line.*

(c) There can not be three distinct triples of collinear knots.

(d) A plane with exactly 4 *knots cannot be used by two knots.*

(e) A plane with exactly 5 *knots cannot be used by three knots.*

(e) If a plane with exactly 5 *knots is used by* 2 *knots then the remaining* 3 *knots* $(3 = 10 - (5 + 2))$ *are collinear.*

Proof. Indeed, (a) and (b) follow from Proposition 4 and Proposition 5 $(d = n = 2)$, respectively.

For (c), suppose conversely that there are three distinct collinear triples. These nine knots are on two planes used by the tenth knot. Therefore two triples must be coplanar. But then 6 knots of these triples in view of Remark 1 are 2-dependent, which is a contradiction.

For (d), without loss of generality assume that a plane H is used by A_1 and A_2 and passes through just 4 knots $A_3, \ldots, A_6$. Then the remaining 4 knots $A_7, \ldots, A_{10}$ are contained in other two distinct planes – H_1 and H_2 used by A_1 and A_2, respectively. (We have also that H_1 contains A_2 and H_2 contains A_1.) Thus knots $A_7, \ldots, A_{10}$ are on the line of intersection of H_1 and H_2, which contradicts (a).

Now consider the case of twice used plane H with 5 knots. Suppose that H is used by A_1 and A_2 and passes through $A_3, \ldots, A_7$. Then, in view of Proposition 7 we have that

$$\mathcal{N}_H \subset \{A_8, A_9, A_{10}\} \tag{15}$$

must be 1-dependent. Thus necessarily equality takes place in (15) and the three knots there are on one line. In the same way we get that triple usage of H is impossible. Indeed, otherwise $\#\mathcal{N}_H \leq 2$ and therefore it cannot be 1-dependent set. Hence $\mathcal{N}_H = \emptyset$ implying in view of Corollary 1, that H is maximal, which is a contradiction. □

Proof of Theorem 3. Assume conversely that GM_d conjecture is not true for Π_2^3. Let $\mathcal{X} = \{A_1, \ldots, A_{10}\} \subset \mathbb{R}^3$, be a set of 10 knots satisfying the GC_2 property for which there is no plane passing through 6 knots. Then each knot of $\mathcal{X}$ uses 2 planes and at least one of them contains 5 of other 9 knots. Thus we have

$*$ There is a plane, denoted by P', passing through 5 knots.

We also verify readily that

$*$ 3 knots out of 5 on P' are collinear.

Indeed, P' is not maximal. Hence, in view of Corollary 1, there is a knot, not contained in P', say A, which does not use it. One of the two planes used by A, say P'', contains $k \geq 3$ of 5 knots of P'. These k knots are on the line $P' \cap P''$. Now by Proposition 10 (a), we get $k = 3$.

Denote the above line with 3 knots by l. Note that l is a maximal 1-flat.

In the next statement we follow [1], where a bivariate fact from [9] (see Lemma 3.5 therein) is extended to $\mathbb{R}^3$.

$*$ There is a plane, denoted by P, with 5 knots which is used by two knots.

Consider the 7 knots outside the line l which, according to Proposition 5, should use a plane passing through l. First note that, in view of Remark 2, any plane which is used by a knot and is passing through the line l should pass through at least one another knot, too. Next, there are at most 6 such planes – the P' itself and the other 5 passing through l and 5 points outside of P', respectively. Thus for 7 knots there are 6 planes and therefore there should be a plane which passes through l and is used by two knots.

Now in view of Proposition 10 (f) we arrive to the following final configuration:

$*$ 5 knots, say $A_1, \ldots, A_5$, contained in a plane P used by two knots say A_6 and A_7. Moreover, three knots, say A_1, A_2, A_3, are collinear in P and three knots, say A_8, A_9, A_{10}, are collinear outside of P.

Note that in view of Proposition 10 (c) no triple inside $A_4, \ldots, A_7$ is collinear, or in other words, any triple there determines a plane.

Denote the line through A_1, A_2, A_3, by l_{1-3}. Let also

$$\mathcal{X}_{8-10} = \{A_8, A_9, A_{10}\}.$$

Now notice that one of the two planes (let us name it as first) used by a knot from $\mathcal{X}_{8-10}$ passes necessarily through the line l_{1-3} and another knot from A_8, A_9, A_{10}.

We start by considering the case when such a plane passes just through the above mentioned 4 knots (three in l_{1-3}). Without loss of generality assume that it is the first plane of A_8 and passes through l_{1-3} and A_9 only. Then the second plane of A_8, denoted by L, passes through the remaining knots: $A_4, A_5, A_6, A_7, A_{10}$. Let us check that also A_9 and A_{10} are using L. Indeed, the second planes of these knots pass necessarily through A_4, A_5 and at least one from A_6 and A_7. Indeed, if these latter two knots belong to the first plane, then it contains 6 knots. It remains to note that both the planes through A_4, A_5, A_6 or A_4, A_5, A_7 coincide with L.

Therefore each of first planes of knots of $\mathcal{X}_{8-10}$ contains 5 knots – the 4 necessary ones mentioned above plus one additional. It is easily seen that the additional knot must be A_6 or A_7. Thus there are two possible planes through l_{1-3} and A_6 or A_7. Therefore for two knots of $\mathcal{X}_{8-10}$, say for A_8 and A_9, the first planes coincide. Denote it by L'. As was mentioned, it passes through one knot of $\mathcal{X}_{8-10}$, which in this case must be A_{10}, and one of A_6 or A_7, say A_6.

Now in view of Proposition 7

$$\mathcal{N}_H \subset \{A_4, A_5, A_7\}$$

must be 1-dependent, therefore three knots in the right-hand side must be collinear which, as was mentioned, is a contradiction. $\square$

Next we formulate another basic result concerning GC conjecture in $\mathbb{R}^3$.

Theorem 4 (Ktryan [10]). *Any GC_2 set $\mathcal{X}$ in $\mathbb{R}^3$ has at least three maximal planes.*

For the proof we refer the reader to [10].

At the end of this section let us formulate the most important recent result in the considered area which was proved in 2011 by A. Apozyan in his Ph. D. Thesis:

Theorem 5 (Apozyan). *The GM_d-conjecture is not true in general.*

To prove Theorem 5, Apozyan constructed a GC_2 set of 28 points in $\mathbb{R}^6$ which has no maximal plane!

4. The Classification of GC_2 Sets: the Uniqueness of the Counterexample of Carl de Boor

Thus all GC_2 sets in $\mathbb{R}^3$ can be divided into 3 classes: those having $3, 4$ or 5 maximal planes. The case of 5 maximal planes were considered in [3]. To present it let us recall the Chung and Yao natural lattice. Here we start with $n+3$ planes: L_i, $i = 1, \dots, n+3$, which satisfy the following conditions: each three planes have a unique point of intersection and intersection of each 4 planes is empty. The set X consists of all intersection points of these planes. Then it is easily seen that $\#X = N_n^3$ and X satisfies the condition GC_n^3.

They are characterized as follows.

Proposition 11 (C. de Boor [3]). *Any GC_2 set in $\mathbb{R}^3$ with 5 maximal planes corresponds to Chung and Yao natural lattice of degree 2.*

The case of 4 maximal planes was considered in [2]. To present it in the considered special case let us introduce the concept of so called space Δ-structure (see [10], [2]): 4 knots, called black, are the vertices of a pyramid and other 6 knots, called white, lie one by one on 6 edges of the pyramid so that they are not coplanar.

Proposition 12 (Apozyan [2]). *Any GC_2 set in $\mathbb{R}^3$ with exactly 4 maximal planes corresponds to a space Δ-structure.*

Notice that any side of the pyramid with Δ-structure is a maximal plane and contains three white and three black knots.

Now let us discuss the counterexample of de Boor [3]. This configuration is obtained from a space Δ-structure by moving in a fixed side of the pyramid a white knot to the line passing through two other white knots of that side. Let us name the resulting configuration as $\hat{\Delta}$-structure. It is easily seen that there are 3 maximal planes here. The intersection of these latter planes is a black knot which we denote by M.

Next we verify that a $\hat{\Delta}$-structure satisfies the GC_2 property.

Indeed, in view of Proposition 6 it is poised. Therefore for knots which use a maximal plane (with 6 knots) the GC condition holds since the second plane has to pass through just 3 knots. Notice that then, in view of Proposition 3, none of the two planes vanishes at M.

It is easily seen that in $\hat{\Delta}$-structure only M does not use a maximal plane. But one can readily find the two planes it uses. They are the same which M was using in the original Δ-structure, i.e., in the structure before the movement of the white knot. In fact the white knot we move from one plane M is using to another.

Next we show that actually the counterexample of C. de Boor gives the only configuration of GC_2 set with 3 maximal planes. This completes the classification of all GC_2 sets in $\mathbb{R}^3$.

Proposition 13. *Any GC_2 set in $\mathbb{R}^3$ with exactly 3 maximal planes corresponds to a space $\hat{\Delta}$-structure.*

Proof. Assume that $\mathcal{X} = \{A_1, \dots, A_{10}\} \subset \mathbb{R}^3$ satisfies the GC_2 property and has exactly 3 maximal planes. According to Proposition 9 these three planes intersect at a knot which corresponds to the one denoted above by M. This is the first black knot. Next, by the same Proposition 9 each two of these three planes intersect with a maximal line. These lines contain the edges of the pyramid of the $\hat{\Delta}$-structure. Denote these lines by l_i, $i = 1, 2, 3$. Each line passes through three knots, one of which is M. Denote the other knots on l_i, by M_i and M_i', $i = 1, 2, 3$. Denote also by P_{ij} the plane through l_i and l_j, $1 \le i, j \le 3$. These are the 3 maximal planes.

As it was mentioned above, M does not use a maximal plane, since it is the only knot belonging to all three maximal planes. Therefore it uses a plane with 5 knots. Denote that plane by P. This plane passes through exactly one knot from each line l_i, $i = 1, 2, 3$, distinct of M, i.e., exactly one of M_i and M_i', $i = 1, 2, 3$.

Indeed, if it passes through two knots of one of the lines, then it passes through M, while if it does not pass through any knot of one of the lines, then the second plane used by M passes through two knots of that line and therefore it passes through M itself, which is a contradiction.

These three knots on lines l_i, $i = 1, 2, 3$, P passes through will be the 3 other black knots, which together with M form the vertices of the pyramid of the $\hat{\Delta}$- structure. Other three M_i or M_i', $i = 1, 2, 3$, will be white knots. Thus we identified situation of 7 knots: 4 black and 3 white. It remains to identify the 3 remaining white knots.

Note that P is fixed by the condition that it passes through three black knots of the lines l_i, $i = 1, 2, 3$. Next we are to add two of the 3 white knots to P to make it pass through 5 knots.

Notice that from the other hand we are to add the 3 white knots one by one to the maximal planes P_{12}, P_{13}, and P_{23}, which so far contain 5 knots each.

Thus we conclude that the two of three white knots are on the intersection of P with two maximal planes.

Finally, it remains to identify the place of the last white knot on the third maximal plane. It must lie on the second plane used by the knot M, which as it is seen easily, is the plane P' passing through 3 white knots of maximal lines l_i, $i = 1, 2, 3$.

Therefore we conclude finally that the last white knot lies on the intersection of P' and the third maximal plane, which completes the proof. $\square$

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Hakop Hakopian
Department of Informatics and Applied Mathematics
Yerevan State University
A. Manoukian St. 1
Yerevan
ARMENIA
E-mail: hakop@ysu.am