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Strong Converse Results for Baskakov-Durrmeyer Operators

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The purpose of this paper is to prove a strong converse result for Baskakov-Durrmeyer operators in the terminology of [4]. Together with a direct result by Berdysheva [2] this establishes equivalence between the error of approximation and an appropriate K-functional.

Keywords and Phrases: Baskakov-Durrmeyer operators, strong Voronovskaja-type theorem, strong converse result of type A.

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1. Introduction

The type of modification for the Bernstein operators introduced by Durrmeyer [6] and independently developed by Lupaş [11] was carried over to many other classical operators.

Here we consider the so-called Baskakov-Durrmeyer operators which were introduced by Sahai and Prasad in [12] and independently by one of the authors in a more general setting in [7].

Definition 1. Let $f \in L_p[0,\infty)$, $1 \le p \le \infty$, $n \in \mathbb{N}$, $n \ge 2$. Then the n-th Baskakov-Durrmeyer operator is defined by

$$(B_n f)(x) = \sum_{k=0}^{\infty} b_{nk}(x)(n-1)\int_0^{\infty} b_{nk}(t)f(t)\,dt,$$

where we denote

$$b_{nk}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \qquad x \in [0,\infty).$$

In [7] (see also [9]) the approximation behaviour of the operators B_n was studied by using the Ditzian-Totik modulus of smoothness and an equivalent K-functional, respectively, (see [5]), given by

$$K_\varphi^2(f,t^2)_p = \inf\left\{\|f-g\|_p + t^2\|\varphi^2 D^2 g\|_p : g \in AC_{loc},\ \varphi^2 D^2 g \in L_p[0,\infty)\right\},$$

where the step-weight function φ is defined by $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$, and D denotes the ordinary differentiation operator.

It was proved in [7, Satz 5.14] that the error of approximation can be estimated by

$$\|B_n f - f\|_p \le C\big\{K_\varphi^2(f,n^{-1})_p + n^{-1}\|f\|_p\big\}, \qquad f \in L_p[0,\infty),\ 1 \le p < \infty.$$

By replacing $\varphi^2 D^2$ in the above given K-functional by $\widetilde{D}^2 = D\varphi^2 D$, i. e., considering

$$\widetilde{K}_\varphi^2(f,t^2)_p = \inf\big\{\|f-g\|_p + t^2\|\widetilde{D}^2 g\|_p : g, \widetilde{D}^2 g \in L_p[0,\infty)\big\},$$

Berdysheva proved in [2, Theorem 7] the direct result

$$\|B_n f - f\|_p \le 2\widetilde{K}_\varphi^2\big(f,(2(n-1))^{-1}\big)_p, \qquad f \in L_p[0,\infty),\ 1 \le p < \infty. \tag{1}$$

The advantages of Berdysheva's estimate are the nice constant 2 and that there is no additional term on the right-hand side. It could be regarded as a disadvantage that, as far as we know, there is no information about an equivalent modulus of smoothness until now.

The main result of our paper is a strong converse result with a good constant which corresponds to the direct estimate by Berdysheva.

In the following we collect some notations and well-known properties of the Baskakov-Durrmeyer operators which will be used frequently throughout this paper.

It was shown in [9, Theorem 3.2] that the operators B_n yield a positive, linear approximation method in the spaces $L_p[0,\infty)$, $1 \le p < \infty$.

They are contractive, i. e.

$$\|B_n f\|_p \le \|f\|_p, \qquad f \in L_p[0,\infty),\ 1 \le p \le \infty. \tag{2}$$

The operators posses nice commutativity properties. It was proved in [8] that

$$B_n(B_l f) = B_l(B_n f), \qquad f \in L_p[0,\infty),\ 1 \le p \le \infty,\ n,l \in \mathbb{N},\ n,l \ge 2. \tag{3}$$

They also commute with the differential operators $\widetilde{D}^2$, i. e.

$$B_n(\widetilde{D}^2 f) = \widetilde{D}^2(B_n f), \qquad f \in L_p[0,\infty),\ 1 \le p \le \infty,\ \widetilde{D}^2 f \in L_p[0,\infty), \tag{4}$$

which was proved in [10, Lemma 3.1].

The following identities can be easily verified:

$$\int_0^\infty b_{nk}(t)dt = \frac{1}{n-1}, \tag{5}$$

$$\sum_{k=0}^{\infty} b_{nk}(x) = 1, \tag{6}$$

$$\varphi^2(x)Db_{nk}(x) = (k-nx)b_{nk}(x), \tag{7}$$

$$\varphi^2(x)\widetilde{D}^2 b_{nk}(x) = \{(k-nx)^2 - n\varphi^2(x)\}b_{nk}(x). \tag{8}$$

For $f \in L_p[0,\infty)$, $1 \le p < \infty$, such that $\widetilde{D}^2 f \in L_p[0,\infty)$, we use the representations (see [2, Theorem 6])

$$B_n f - f = \sum_{j=n}^{\infty} \frac{1}{j(j-1)} \widetilde{D}^2 B_j f, \tag{9}$$

and

$$B_n f - B_j f = \sum_{k=n}^{j-1} \frac{1}{k(k-1)} \widetilde{D}^2 B_k f, \qquad j > n. \tag{10}$$

2. Strong Voronovskaja Type Result

In order to derive a strong converse result of type A in the terminology of [4], we have to prove an appropriate strong Voronovskaja type estimate. To do so, we first need some Bernstein type inequalities.

For $f \in L_p[0,\infty)$, $1 \le p \le \infty$, Berdysheva proved in [2, Theorem 2] that

$$\|\widetilde{D}^2(B_n f)\|_p \le 2n\|f\|_p. \tag{11}$$

The next inequality deals with iterates B_n^l of the operators. For $l = 2$ the analogous result for the Bernstein-Durrmeyer operators was proved by Chen, Ditzian and Ivanov (see [3, Theorem 3.2]).

Lemma 1. *Let $f \in L_p[0,\infty)$, $1 \le p \le \infty$, $l \in \mathbb{N}$, $l \ge 2$. Then*

$$\|\widetilde{D}^2(B_n^l f)\|_p \le n\|f\|_p. \tag{12}$$

Proof. First we consider the case $l = 2$ and note that it is sufficient to prove (12) for $p = \infty$ (see [3]). As by (4) the operators B_n and the differential operator $\widetilde{D}^2$ commute, we get with the help of (5)

$$\begin{aligned}
\big|\widetilde{D}^2(B_n^2 f)(x)\big| &= \big|B_n \widetilde{D}^2(B_n f)(x)\big| \\
&= \Big|\sum_{j=0}^{\infty} b_{nj}(x)(n-1) \\
&\quad \times \int_0^{\infty} b_{nj}(u)\Big\{\sum_{k=0}^{\infty} \widetilde{D}^2 b_{nk}(u)(n-1)\int_0^{\infty} b_{nk}(t)f(t)dt\Big\}du\Big| \\
&\le \|f\|_\infty \sum_{j=0}^{\infty} b_{nj}(x)(n-1)\sum_{k=0}^{\infty}\Big|\int_0^{\infty} b_{nj}(u)\widetilde{D}^2 b_{nk}(u)\,du\Big|. \qquad (13)
\end{aligned}$$

Integration by parts, using (7) and the Cauchy-Schwarz inequality leads to the estimate

$$\begin{aligned}
&\Big|\int_0^{\infty} b_{nj}(u)\widetilde{D}^2 b_{nk}(u)\,du\Big| \\
&= \Big|\int_0^{\infty} \varphi^2(u)(Db_{nj}(u))(Db_{nk}(u))\,du\Big| \\
&= \Big|\int_0^{\infty} \frac{(j-nu)(k-nu)}{\varphi^2(u)}\, b_{nj}(u)b_{nk}(u)\,du\Big| \\
&\le \Big\{\int_0^{\infty} \frac{(j-nu)^2}{\varphi^2(u)}\, b_{nj}(u)b_{nk}(u)\,du\Big\}^{1/2}\Big\{\int_0^{\infty} \frac{(k-nu)^2}{\varphi^2(u)}\, b_{nj}(u)b_{nk}(u)\,du\Big\}^{1/2}.
\end{aligned}$$

Together with the Cauchy-Schwarz inequality for infinite sums we now get

$$\begin{aligned}
\sum_{k=0}^{\infty}\Big|\int_0^{\infty} b_{nj}(u)\widetilde{D}^2 b_{nk}(u)\,du\Big| &\le \Big\{\sum_{k=0}^{\infty}\int_0^{\infty} \frac{(j-nu)^2}{\varphi^2(u)}\, b_{nj}(u)b_{nk}(u)\,du\Big\}^{1/2} \\
&\quad \times \Big\{\sum_{k=0}^{\infty}\int_0^{\infty} \frac{(k-nu)^2}{\varphi^2(u)}\, b_{nj}(u)b_{nk}(u)\,du\Big\}^{1/2} \\
&=: T_1 T_2. \qquad (14)
\end{aligned}$$

On using (6), (7) and integration by parts we calculate

$$\begin{aligned}
T_1 &= \Big\{\int_0^{\infty} (j-nu)Db_{nj}(u)\,du\Big\}^{1/2} \\
&= \Big\{(j-nu)b_{nj}(u)\big|_0^{\infty} + n\int_0^{\infty} b_{nj}(u)\,du\Big\}^{1/2} = \Big(\frac{n}{n-1}\Big)^{1/2}.
\end{aligned}$$

In order to estimate T_2 we first observe that the second moments of the classical Baskakov operators are given by

$$\sum_{k=0}^{\infty} b_{nk}(u)\Big(\frac{k}{n}-x\Big)^2 = \frac{\varphi^2(u)}{n}.$$

Putting this into T_2 and using again (5) we end up in

$$T_2 = \Big(\frac{n}{n-1}\Big)^{1/2}.$$

Replacement in (14) implies the estimate

$$\sum_{k=0}^{\infty} \Big| \int_0^{\infty} b_{nj}(u) \widetilde{D}^2 b_{nk}(u)\, du \Big| \le \frac{n}{n-1}\,.$$

Putting this into (13) and using (6) leads to

$$\big|\widetilde{D}^2(B_n^2 f)(x)\big| \le n\|f\|_\infty.$$

Thus we have proved our proposition for $l = 2$. From this the case $l \ge 3$ follows easily from the commutativity property(4) and from (2), i. e., we write

$$\big\|\widetilde{D}^2(B_n^l f)\big\|_p = \big\|B_n^{l-2}\widetilde{D}^2(B_n^2 f)\big\|_p \le \big\|\widetilde{D}^2(B_n^2 f)\big\|_p \le n\big\|f\big\|_p\,.$$

□

As an immediate consequence of Lemma 1 we get the following

Corollary 1. *Let $f \in L_p[0,\infty)$, $1 \le p \le \infty$, $l \in \mathbb{N}$, $l \ge 2$. Then*

$$\|(\widetilde{D}^2)^2(B_n^{l+1} f)\|_p \le n\|\widetilde{D}^2(B_n f)\|_p \le 2n^2\|f\|_p.$$

Proof. By using (4) the proposition follows from (12) and (11), respectively, i. e.

$$\|(\widetilde{D}^2)^2(B_n^{l+1} f)\|_p = \|\widetilde{D}^2 B_n^l \widetilde{D}^2(B_n f)\|_p \le n\|\widetilde{D}^2(B_n f)\|_p \le 2n^2\|f\|_p.$$

□

We are now in the position to prove an appropriate strong Voronovskaja type result.

Theorem 1. *Let $h \in L_p[0,\infty)$, $1 \le p < \infty$, be such that $(\widetilde{D}^2)^2 h \in L_p[0,\infty)$. Then*

$$I(n) := \Big\|B_n h - h - \frac{1}{2(n-1)}\widetilde{D}^2(h + B_n h)\Big\|_p \le \psi(n)\big\|(\widetilde{D}^2)^2 h\big\|_p,$$

where for $n \ge 3$

$$\begin{aligned}\psi(n) &= \frac{1}{8}\cdot\frac{8n^2-24n+17}{(n-1)^2(2n-3)^2} + \frac{1}{2}\Big\{\sum_{k=2n-1}^{\infty}\frac{1}{k^2(k-1)^2} - \sum_{k=n}^{2n-3}\frac{1}{k^2(k-1)^2}\Big\}\\ &\le \frac{1}{8}\cdot\frac{8n^2-24n+17}{(n-1)^2(2n-3)^2} - \frac{1}{16}\cdot\frac{8n^3-25n^2+23n-4}{(2n-3)(n-2)(2n-1)n(n^2-1)}\\ &=: \psi_1(n)\,,\end{aligned}$$

and

$$\psi(2) = \frac{1}{6}\,\pi^2 - \frac{3}{2}\,.$$

Proof. We use (9), $\frac{1}{n-1} = \sum_{j=n}^{\infty} \frac{1}{j(j-1)}$, (10), again (9) and (10), interchanging the order of summation to obtain

$$\begin{aligned}
&B_n h - h - \frac{1}{2(n-1)}\widetilde{D}^2(h + B_n h) \\
&= \frac{1}{2}\Big\{ \sum_{j=n}^{\infty} \frac{\widetilde{D}^2(B_j h - h)}{j(j-1)} - \sum_{j=n+1}^{\infty} \frac{\widetilde{D}^2(B_n h - B_j h)}{j(j-1)} \Big\} \\
&= \frac{1}{2}\Big\{ \sum_{j=n}^{\infty} \frac{1}{j(j-1)} \sum_{k=j}^{\infty} \frac{(\widetilde{D}^2)^2(B_k h)}{k(k-1)} - \sum_{j=n+1}^{\infty} \frac{1}{j(j-1)} \sum_{k=n}^{j-1} \frac{(\widetilde{D}^2)^2(B_k h)}{k(k-1)} \Big\} \\
&= \frac{1}{2}\Big\{ \sum_{k=n}^{\infty} \frac{(\widetilde{D}^2)^2(B_k h)}{k(k-1)} \sum_{j=n}^{k} \frac{1}{j(j-1)} - \sum_{k=n}^{\infty} \frac{(\widetilde{D}^2)^2(B_k h)}{k(k-1)} \sum_{j=k+1}^{\infty} \frac{1}{j(j-1)} \Big\}.
\end{aligned}$$

From this we derive by using (2), $\frac{1}{k} = \sum_{j=k+1}^{\infty} \frac{1}{j(j-1)}$, $-\frac{1}{k} + \frac{1}{n-1} = \sum_{j=n}^{k} \frac{1}{j(j-1)}$

$$I(n) \le \frac{1}{2}\big\|(\widetilde{D}^2)^2 h\big\|_p \sum_{k=n}^{\infty} \frac{1}{k(k-1)} \Big| \frac{1}{n-1} - \frac{2}{k} \Big| = \big\|(\widetilde{D}^2)^2 h\big\|_p \psi(n).$$

Thus

$$\psi(2) = \frac{1}{2} \sum_{k=3}^{\infty} \frac{1}{k(k-1)} \Big(1 - \frac{2}{k}\Big) = \Psi(1,2) - \frac{1}{2} = \frac{1}{6}\pi^2 - \frac{3}{2},$$

where $\Psi(1,x)$ denotes the trigamma function (see e. g. [1, p. 260]). In order to calculate $\psi(n)$ for $n \ge 3$, we make use of $\frac{2}{k^2(k-1)} = \frac{1}{(k-1)^2} - \frac{1}{k^2} - \frac{1}{k^2(k-1)^2}$ to obtain

$$\begin{aligned}
2\psi(n) &= \Big\{ \sum_{k=n}^{2n-3} - \sum_{k=2n-1}^{\infty} \Big\} \Big[\frac{1}{(k-1)^2} - \frac{1}{k^2} - \frac{1}{k^2(k-1)^2} - \frac{1}{(n-1)k(k-1)} \Big] \\
&= \frac{1}{4(n-1)^2} + \frac{1}{(n-1)(2n-3)} - \frac{1}{(2n-3)^2} + \Big\{ \sum_{k=2n-1}^{\infty} - \sum_{k=n}^{2n-3} \Big\} \frac{1}{k^2(k-1)^2} \\
&= \frac{1}{4} \cdot \frac{8n^2 - 24n + 17}{(n-1)^2(2n-3)^2} + \sum_{k=2n-1}^{\infty} \frac{1}{k^2(k-1)^2} - \sum_{k=n}^{2n-3} \frac{1}{k^2(k-1)^2}\,. \qquad (15)
\end{aligned}$$

For the last two sums in (15) we obtain the estimates

$$\begin{aligned}
\sum_{k=2n-1}^{\infty} \frac{1}{k^2(k-1)^2} &\le \sum_{k=2n-1}^{\infty} \frac{1}{(k-2)(k-1)k(k+1)} \\
&= \sum_{k=2n-1}^{\infty} \Big\{ \frac{1}{6}\Big(\frac{1}{k-2} - \frac{1}{k+1}\Big) + \frac{1}{2}\Big(\frac{1}{k} - \frac{1}{k-1}\Big) \Big\} \\
&= \frac{1}{6} \cdot \frac{1}{(2n-1)(n-1)(2n-3)}
\end{aligned}$$

and

$$\sum_{k=n}^{2n-3} \frac{1}{k^2(k-1)^2} \geq \sum_{k=n}^{2n-3} \frac{1}{(k-2)(k-1)(k+1)(k+2)}$$
$$= \sum_{k=n}^{2n-3} \left\{ \frac{1}{12}\Big(\frac{1}{k-2} - \frac{1}{k+2}\Big) + \frac{1}{6}\Big(\frac{1}{k+1} - \frac{1}{k-1}\Big) \right\}$$
$$= \frac{1}{24} \cdot \frac{28n^3 - 79n^2 + 61n - 12}{n(n+1)(2n-1)(n-1)(2n-3)(n-2)}.$$

Replacement in (15) implies

$$2\psi(n) \leq \frac{1}{4} \cdot \frac{8n^2 - 24n + 17}{(n-1)^2(2n-3)^2} - \frac{1}{8} \cdot \frac{8n^3 - 25n^2 + 23n - 4}{(2n-3)(n-2)(2n-1)n(n^2-1)} = 2\psi_1(n).$$

□

3. Main Result

We now apply the strong Voronovskaja-type result to prove a strong converse theorem of type A.

Theorem 2. *Let $f \in L_p[0,\infty)$, $1 \leq p < \infty$. Then*

$$\widetilde{K}_\varphi^2\big(f, (2(n-1))^{-1}\big)_p \leq \big(4.75 + \varepsilon(n)\big)\,\|B_n f - f\|_p\,,$$

where $\varepsilon(n)$ is monotone decreasing with $\lim_{n\to\infty} \varepsilon(n) = 0$.

Proof. We choose $g = \frac{1}{2}(B_n^4 f + B_n^3 f)$. Thus

$$\|f - g\|_p \leq 3.5\,\|B_n f - f\|_p\,. \tag{16}$$

We now apply the strong Voronovskaja-type result in Theorem 1 to the function $h = B_n^3 f$ and use the inverse triangle inequality to derive

$$\frac{1}{n-1}\|\widetilde{D}^2 g\|_p = \frac{1}{2(n-1)}\|\widetilde{D}^2(B_n^4 f + B_n^3 f)\|_p$$
$$\leq \|B_n^4 f - B_n^3 f\|_p + \psi(n)\|(\widetilde{D}^2)^2 B_n^3 f\|_p$$
$$\leq \|B_n f - f\|_p + \psi(n)\|(\widetilde{D}^2)^2 B_n^3 f\|_p\,. \tag{17}$$

In order to estimate the second term on the right-hand side we use the first inequality in Corollary 1 and proceed further by applying the triangle inequality, (11), (12) and (2) to derive

$$\begin{aligned}\|(\widetilde{D}^2)^2(B_n^3 f)\|_p \le & n\|\widetilde{D}^2(B_n f)\|_p \\ \le & n\Big\{\big\|\widetilde{D}^2 g\big\|_p + \big\|\widetilde{D}^2(B_n[f-B_n f])\big\|_p \\ & + \big\|\widetilde{D}^2(B_n^2[f-B_n f])\big\|_p + \frac{1}{2}\big\|\widetilde{D}^2(B_n^3[f-B_n f])\big\|_p\Big\} \\ \le & n\Big\{\big\|\widetilde{D}^2 g\big\|_p + 3.5n\|f-B_n f\|_p\Big\}.\end{aligned}$$

Inserting this estimate into (17) we get

$$\begin{aligned}& \frac{1-n(n-1)\psi(n)}{n-1}\big\|\widetilde{D}^2 g\big\|_p \le \big[1+3.5n^2\psi(n)\big]\big\|B_n f - f\big\|_p \\ \iff\ & \frac{1}{2(n-1)}\big\|\widetilde{D}^2 g\big\|_p \le \frac{1}{2}\cdot\frac{1+3.5n^2\psi(n)}{1-n(n-1)\psi(n)}\big\|B_n f - f\big\|_p\,. \end{aligned} \tag{18}$$

Now (16) and (18) lead to

$$\begin{aligned}\widetilde{K}_\varphi^2\left(f,(2(n-1))^{-1}\right)_p \le & \big\|f-g\big\|_p + \frac{1}{2(n-1)}\big\|\widetilde{D}^2 g\big\|_p \\ \le & \Big[3.5+\frac{1}{2}\cdot\frac{1+3.5n^2\psi(n)}{1-n(n-1)\psi(n)}\Big]\|B_n f - f\|_p \\ =: & \, C(n)\|B_n f - f\|_p\,.\end{aligned}$$

Next we prove that $C(n) = 4.75+\varepsilon(n)$, where $\varepsilon(n)$ is monotone decreasing and tends to 0 for n tending to infinity. First we note that

$$C(n) \le C_1(n) := 3.5 + \frac{1}{2}\cdot\frac{1+3.5n^2\psi_1(n)}{1-n(n-1)\psi_1(n)}\,.$$

With some long and tedious but elementary calculations one can prove that $1+3.5n^2\psi_1(n)$ is monotone decreasing and $1-n(n-1)\psi_1(n)$ is monotone increasing in n for $n \ge 3$.

Thus $\varepsilon(n) := \frac{1}{2}\cdot\frac{1+3.5n^2\psi_1(n)}{1-n(n-1)\psi_1(n)} - 1.25$ is monotone decreasing. Inserting the representation for $\psi_1(n)$ into $\varepsilon(n)$ shows easily that $\lim_{n\to\infty}\varepsilon(n) = 0$. □

Since $C_1(3) = \frac{7183}{1352} < B(2) = \frac{12}{12-\pi^2}$, we get as a consequence from the direct result (1) and Theorem 2 the equivalence of the error of approximation and the appropriate K-functional:

Corollary 2. *Let $f \in L_p[0,\infty)$, $1 \le p < \infty$. Then we have for each $n \in \mathbb{N}$, $n \ge 2$,*

$$\Big(1-\frac{\pi^2}{12}\Big)\widetilde{K}_\varphi^2\big(f,(2(n-1))^{-1}\big)_p \le \|B_n f - f\|_p \le 2\widetilde{K}_\varphi^2\big(f,(2(n-1))^{-1}\big)_p.$$

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