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Geometric Properties of Mappings Connected with the Schur-Szegő Composition of Polynomials

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The Schur-Szegő composition (SSC) of two polynomials $P := \sum_{j=0}^{n} a_j x^j$ and $Q := \sum_{j=0}^{n} b_j x^j$ is defined by the formula $P * Q := \sum_{j=0}^{n} a_j b_j x^j / \binom{n}{j}$. The SSC is commutative and associative. It can be defined in a self-evident way for more than two polynomials. Properties of the SSC are exposed in the monographs [9] and [10]. In this paper we consider the presentation of polynomials as SSC of polynomials of a special form and an affine mapping in the space of polynomials which is defined by this presentation. The results are proved in the cited papers.

Definition 1. A polynomial of the form $K_a := (x+1)^{n-1}(x+a)$, $a \in \mathbb{C}$, is called a *composition factor*. We set $K_\infty := (x+1)^{n-1}$.

Notation 1. For n fixed set $b_j := -j/(n-j)$, $j = 0, \dots, n-1$; $b_n := -\infty$.

The following theorem is announced in Remark 7 of [3] and proved in [1]:

Theorem 1. *Every monic polynomial having one of its roots at* (-1) *(i.e. of the form* $P_n := (x+1)(x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-1})$*) is representable as an SSC of* $n-1$ *composition factors* K_{a_i}*, where the numbers* a_i *are unique up to permutation.*

Remark 1. If the polynomial is not necessarily monic, then it can be presented in the form

$$c_0 K_{a_1} * \cdots * K_{a_{n-1}} . \tag{1}$$

Remark 2. If a degree $n-k$ polynomial P is considered as a degree n one with k leading coefficients equal to 0, then k of the numbers a_i equal b_ν, $\nu = n, \dots, n-k+1$. If a polynomial P is divisible by x^s, then s of the numbers a_i equal b_ν, $\nu = 0, \dots, s-1$. Indeed, the coefficient of x^ν in K_a is equal to 0 exactly when $a = b_\nu$. On the other hand, if this coefficient equals 0 in P, then it must be 0 in at least one of the composition factors K_{a_i}.

Proposition 1. *For $l \leq n-1$ the composition $K_{a_1} * \cdots * K_{a_l}$ is a polynomial having a root of multiplicity $\geq n-l$ at (-1). This multiplicity is exactly $n-l$ if all numbers a_i are $\neq 1$.*

The proposition implies the following result (see the proof in [4]):

Corollary 1. *If the polynomial P has a root (-1) of multiplicity $\mu \geq 1$, then among the numbers a_i there are exactly $\mu - 1$ which equal 1.*

Set $\sigma_j := \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n-1} a_{i_1} \cdots a_{i_j}$ and consider the mapping

$$\Phi \ : \ (c_1, \ldots, c_{n-1}) \mapsto (\sigma_1, \ldots, \sigma_{n-1}) \ .$$

Remark 3. (i) It is shown in [1] that the mapping Φ is affine and bijective. The proof of the rest of the facts about the mapping Φ exposed in this paper can be found in [4].

(ii) It is natural to view the numbers $(-a_i)$ as roots of another polynomial. Thus the mapping Φ can be considered as a mapping $Pol_{n-1} \to Pol_{n-1}$, where Pol_{n-1} stands for the space of polynomials of degree $n-1$.

Definition 2. Denote by P^R the reverted of the degree n polynomial P; that is, $P^R = x^n P(1/x)$. The polynomial P is *self-reciprocal* if $P^R = \pm P$. For such a polynomial if x_0 is its root, then $1/x_0$ is also its root.

Theorem 2. *(a) The mapping Φ has $n-1$ distinct rational positive eigenvalues*

$$\lambda_1 = 1 \ , \quad \lambda_2 = \frac{n}{n-1} \ , \quad \lambda_3 = \frac{n^2}{(n-1)(n-2)} \ , \quad \ldots \ , \quad \lambda_{n-1} = \frac{n^{n-2}}{(n-1)!} \ .$$

(b) The corresponding eigenvectors are defined by monic polynomials of the form

$$(x+1)^{n-1} \ , \quad x(x+1)^{n-2} \ , \quad x(x+1)^{n-3}Q_1(x) \ , \quad \ldots \ , \quad x(x+1)Q_{n-3}(x) \ ,$$

where $\deg Q_j = j$, $Q_j(-1) \neq 0$. The coefficients of the polynomials Q_j are rational.

(c) The polynomials Q_j are self-reciprocal; that is, $(Q_j)^R = (-1)^j Q_j$. For j odd (resp. for j even) one has $Q_j(1) = 0$ (resp. $Q_j(1) \neq 0$). The middle coefficient of $(x+1)^{n-j-2}Q_j$ is 0 when n is even and j is odd.

(d) The roots of every polynomial Q_j, $j \geq 1$, are positive and distinct.

(e) For j fixed and as $n \to \infty$ the polynomial Q_j has a limit which is a hyperbolic monic degree j polynomial Q_j^* with all roots positive, with rational coefficients, satisfying the equality $(Q_j^*)^R = (-1)^j Q_j^*$ and the condition $Q_j^*(1) = 0$ for j odd.

Remark 4. The eigenpolynomials of the mapping Φ are of degree $n-1$ because they span the tangent space to the space of monic polynomials P_n (see Theorem 1). This is the space of polynomials $(x+1)(c_1x^{n-2}+\cdots+c_{n-1})$. One can consider Φ also as a linear mapping, for polynomials of the form $(x+1)(c_0x^{n-1}+c_1x^{n-2}+\cdots+c_{n-1})$. In this case one adds an eigenvalue $\lambda_0=1$ and an eigenpolynomial $(x+1)^n$ and presents the polynomials P_n in the form (1).

Remark 5. Interlacing properties of the zeros of the polynomials Q_j and Q_j^* are proved respectively in papers [7] and [8].

This paper is devoted to some geometric properties of the mapping Φ. In particular, Φ preserves the set of polynomials with positive real parts of the roots (see the proofs in [5]).

Notation 2. Denote by $\Pi_{n-1}\subset\mathbb{R}^{n-1}\cong Oc_1\cdots c_{n-1}=:\mathcal{R}$ the *closed* subset for which P is hyperbolic. Set $\sigma_j=\sum_{1\leq i_1<\cdots<i_j\leq n-1}a_{i_1}\cdots a_{i_j}$. Denote by U_{n-1} (resp. by V_{n-1}) the *open* subsets of $\mathcal{R}$ for which $c_1<0,\ c_2>0,\ldots$, $(-1)^{n-1}c_{n-1}>0$ (resp. for which the real parts of all roots of P are >0). Set $\tilde{c}:=(c_1,\ldots,c_{n-1})$, $\tilde{\sigma}:=(\sigma_1,\ldots,\sigma_{n-1})$. Writing $P\in U_{n-1}$ means $\tilde{c}\in U_{n-1}$ etc. Denote the closure (resp. the boundary) of a set Δ by $\overline{\Delta}$ (resp. by $\partial\Delta$).

Set $\Phi(P)=(x+1)(x^{n-1}+\sigma_1x^{n-1}+\cdots+\sigma_{n-1})=(x+1)(x+a_1)\cdots(x+a_{n-1})$.

Lemma 1. *If $P\in\overline{V_{n-1}}$, then $P^R\in\overline{V_{n-1}}$. One has $\Phi(P^R)=(\Phi(P))^R$.*

Lemma 2. *For $n\geq 2$ one has $V_{n-1}\subseteq U_{n-1}$ (hence $\overline{V_{n-1}}\subseteq\overline{U_{n-1}}$) with equality only for $n=2$ and 3.*

Theorem 3. *(a) One has $\Phi(V_{n-1})\subset V_{n-1}$ and $\Phi(\Pi_{n-1}\cap V_{n-1})\subset(\Pi_{n-1}\cap V_{n-1})$.*

(b) One has $\Phi(U_{n-1})\subset U_{n-1}$.

(c) If $C=(c_1^0,\ldots,c_{n-1}^0)\in\partial U_{n-1}$, then $\Phi(C)\in\partial U_{n-1}$ if and only if $c_{n-1}^0=0$.

(d) For each real polynomial $P\neq 0$ there exists $h(P)\in\mathbb{N}$ such that $\Phi^k(P)\in\Pi_{n-1}$ when $k\geq h(P)$.

(e) There exists $\nu\in\mathbb{N}$ depending only on n such that for each $P\in\overline{U_{n-1}}$ one has $\Phi^\nu(P)\in\Pi_{n-1}$.

Remark 6. (i) Part (a) of the theorem is interesting from the point of view of stability theory. Indeed, one can consider a polynomial as the characteristic polynomial of a linear ordinary differential equation. Its solutions are stable if the real parts of all exponents are negative.

(ii) In part (e) of the theorem the set $\overline{U_{n-1}}$ cannot be replaced by $\mathbb{R}^{n-1}$ for $n\geq 3$, Φ being nondegenerate, this would imply $\Pi_{n-1}=\mathbb{R}^{n-1}$.

Example 1. For $n = 2$ one has Φ =id and all statements of the theorem are evident (one has $P = (x+1)(x-a) = K_{-a}$, i.e. $a_1 = -a$ and P is hyperbolic).

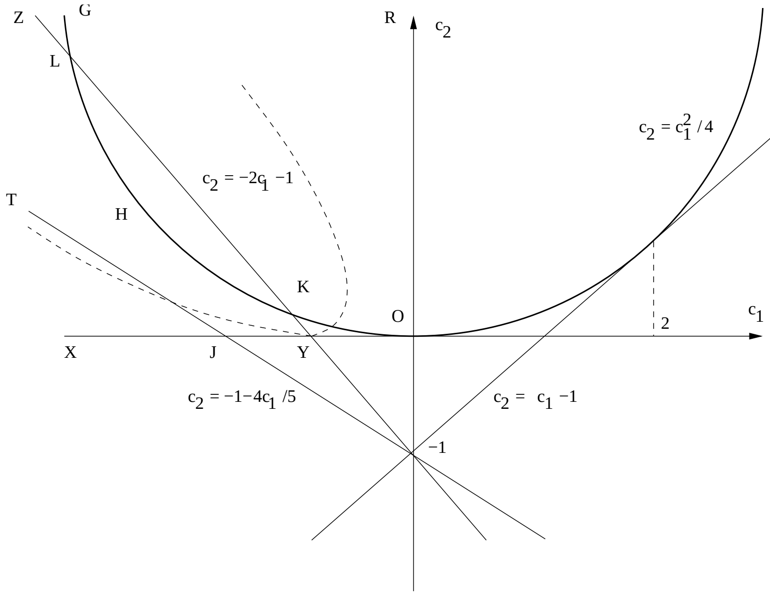

Figure 1. The mapping Φ for $n = 3$.

Example 2. For $n = 3$ one has (see [4]) $\Phi : (c_1, c_2) \mapsto ((3c_1 - c_2 - 1)/2, c_2)$ or equivalently $\Phi : (c_1 - 1, c_2) \mapsto ((3(c_1 - 1) - c_2)/2, c_2)$. The sector XOR represents the sets $U_2 = V_2 = \{ c_1 \leq 0 \leq c_2 \}$. One has

$$\Pi_2 \cap U_2 = \{ c_1 \leq 0 \ , \ 0 \leq c_2 \leq c_1^2/4 \} \ , \qquad \Phi(U_2) = \{ 0 \leq c_2 \leq -2c_1 - 1 \} \ .$$

The last two sets are respectively the curvilinear sector $XOKHLG$ and the sector $XYKLZ$. Thus parts (a), (b) and (c) of Theorem 3 are true. One can see all sets on Fig. 1.

The sector XJT is the set $\Phi^2(U_2) = \{ 0 \leq c_2 \leq -1 - 4c_1/5 \}$. It belongs to the curvilinear sector $XOKHLG = (\Pi_2 \cap U_2)$.

Hence there holds part (e) of Theorem 3 with $\nu = 2$. The mapping Φ has fixed points along the line $c_2 = c_1 - 1$ which define hyperbolic polynomials $(x+1)^2(x + c_1)$. For every other point (c_1^0, c_2^0) the point $\Phi^k(c_1^0, c_2^0)$ defines hyperbolic polynomials for k sufficiently large (the eigenvalue 3/2 makes the module of the first component of $\Phi^k(c_1^0, c_2^0)$ tend to ∞, the second component remains fixed). For large values of k such a quadratic polynomial is hyperbolic.

Thus for $n = 3$ one can set $\nu = 2$ (but not $\nu = 1$ because it is not true that $\Phi(U_2) \subset (\Pi_2 \cap U_2)$ – observe that the line $YZ : c_2 = -2c_1 - 1$ intersects the parabola $c_2 = c_1^2/4$, see Fig. 1).

Remark 7. If the real polynomial P_n (see Theorem 1) has m real positive roots, then at least m of the numbers a_i defined by the mapping Φ are distinct, negative and belonging to different intervals of the form $[b_{j+1}, b_j]$, see Notation 1. In particular, if $P_n/(x+1)$ has all its roots positive, then all numbers a_i are negative and belonging to different intervals of the aforementioned form. Indeed, by the Descartes rule there must be at least m sign changes in the sequence of coefficients of the polynomial P_n. The sequence of coefficients of each composition factor K_{a_i} has at most one sign change. These sign changes must take place at the coefficients of different monomials x^k.

The following conjecture gives more precisions than the above remark:*

Conjecture. *(a) If the polynomial P_n has m positive roots counted with multiplicity ($m \geq 0$) and a k-fold root at 0 ($k \geq 0$), then there are at least $m + \max(0, k-1)$ negative and distinct among the numbers a_i out of which $\max(0, k-1)$ equal $b_1, \ldots, b_{k-1}$, see Notation 1; if $k \geq 1$, then one of the numbers a_i equals 0.*

(b) If there are q numbers a_i equal to 0 and q_1 ones positive, then the polynomial P_n has at least $q_1 + \max(0, q-1)$ negative roots counted with multiplicity; for $q \geq 1$ it has a root at 0.

An analog of the mapping Φ can be defined for entire functions. The following proposition is used to define below the mappings $\Phi_{n,k}$, $k \geq 1$ (see details in [6]):

Proposition 2. *Each polynomial $P := (x+1)^k(x^n + c_1x^{n-1} + \cdots + c_n)$ is representable as SSC*

$$P = K_{n,k;a_1} * \cdots * K_{n,k;a_n} \quad \text{with} \quad K_{n,k;a_i} := (x+1)^{n+k-1}(x+a_i) \ , \qquad (2)$$

where the complex numbers a_i are unique up to permutation.

The second factor of P is of degree n and not $n-1$ just for convenience. With c_i and a_i as in Proposition 2, the mapping $\Phi_{n,k}$ is defined like this:

$$\Phi_{n,k} \ : \ (c_1, \ldots, c_n) \mapsto (\sigma_1, \ldots, \sigma_n) \ , \qquad \text{where} \ \ \sigma_j := \sum_{1 \leq i_1 < \cdots < i_j \leq n} a_{i_1} \cdots a_{i_j} \ .$$

The SSC of the entire functions $f := \sum_{j=0}^{\infty} \gamma_j x^j/j!$ and $g := \sum_{j=0}^{\infty} \delta_j x^j/j!$ is defined by the formula $f * g = \sum_{j=0}^{\infty} \gamma_j \delta_j x^j/j!$. Set $P_m := 1 + c_1x + \cdots + c_mx^m$, $\tilde{\sigma}_k := \sum_{1 \leq j_1 < \cdots < j_k \leq m} 1/(a_{i_1} \cdots a_{i_k})$. The following proposition allows to define an analog of the mappings $\Phi_{n,k}$ for entire functions:

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*A more general statement is proved in: V. P. Kostov, A refined realization theorem in the context of the Schur–Szegő composition, to appear in *Bulletin des Sciences Mathématiques.*

Proposition 3 (Theorem 3 in [2]). *Each function $e^x P_m$, where P_m is a degree m polynomial such that $P_m(0) = 1$, is representable in the form*

$$e^x P_m = \kappa_{a_1} * \cdots * \kappa_{a_m} \ , \qquad \textit{where} \ \ \kappa_{a_j} = e^x(1 + x/a_j) \ . \tag{3}$$

The numbers a_j are unique up to permutation.

Define the mapping $\tilde{\Phi}$ as follows: $\tilde{\Phi} : (c_1, \ldots, c_m) \mapsto (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m)$. This mapping is in a sense a limit as $k \to \infty$ of the mappings $\Phi_{m,k}$ (use the fact that $\lim_{k\to\infty}(1 + x/k)^k = e^x$).

Some properties of the mapping Φ carry on directly to $\Phi_{n,k}$ and $\tilde{\Phi}$:

Theorem 4. *For each $n \geq 1$ and for each $k \geq 1$ one has $\Phi_{n,k}(U_n) \subset U_n$.*

Corollary 2. *For the mapping $\tilde{\Phi}$ one has $\tilde{\Phi}(U_n) \subset U_n$.*

Remark 8. It is also true that if $A \in \partial U_n$, then $\Phi_{n,k}(A) \in \partial U_n$ if and only if $A \in \{c_n = 0\}$.

However, not all of the statements of Theorem 3 have analogs for the mapping $\tilde{\Phi}$:

Proposition 4. *For $m = 3$ the mapping Φ does not send the set V_m into itself.*

See more about the mappings Φ, $\Phi_{n,k}$ and $\tilde{\Phi}$ in paper [6]. The following theorem (see [6]) is an interesting byproduct of their geometric properties. Denote by $T[f]$ the Taylor series at 0 of the entire function f.

Theorem 5. *If the real polynomial P of degree m has k positive roots, $1 \leq k \leq m$, then there are at least k sign changes in the sequence of the coefficients of $T[e^{\lambda x}P]$ for any $\lambda > 0$.*

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