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Infinite-dimensional Generalization of Kolmogorov Widths

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Recently the theory of widths of Kolmogorov-Gelfand has received a great deal of interest due to its close relationship with the newly born area of Compressive Sensing in Signal Processing, cf. [5] and references therein. However fundamental problems of the theory of widths in multidimensional Theory of Functions remain untouched, as well as analogous problems in the theory of multidimensional Signal Analysis. In the present paper we provide a multidimensional generalization of the original result of Kolmogorov about the widths of "ellipsoidal sets" consisting of functions defined on an interval.

1. Introduction

In his seminal paper [8] Kolmogorov has introduced the theory of widths and applied it very successfully to the following set of functions defined in the compact interval:

$$K_p := \Big\{ f \in AC^{p-1}([0,1]) : \int_0^1 |f^{(p)}(t)|^2 \, dt \leq 1 \Big\}. \tag{1}$$

In the present paper we consider a natural multivariate generalization of the set K_p given by

$$K_p^* := \Big\{ u \in H^{2p}(B) : \int_B |\Delta^p u(x)|^2 \, dx \leq 1 \Big\}, \tag{2}$$

where Δ^p is the p-th iterate of the Laplace operator $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ in $\mathbb{R}^n$ and B is the unit ball in $\mathbb{R}^n$. We generalize the notion of width in the framework of the Polyharmonic Paradigm, and obtain analogs to the one-dimensional results of Kolmogorov.

The *Polyharmonic Paradigm* has been announced in [9] as a new approach in Multidimensional Mathematical Analysis (in particular, in the Moment Problem, Approximation and Spline Theory) which is based on solutions of higher order elliptic equations as opposed to the usual concept which is based on algebraic and trigonometric polynomials of several variables. The main result of the present research is a new aspect of the Polyharmonic Paradigm. It provides a new hierarchy of infinite-dimensional spaces of functions which are used for a generalization of the Kolmogorov's theory of widths. This new hierarchy generalizes the hierarchy of finite-dimensional subspaces S_N of the space $C^\infty(I)$ for an interval $I \subset \mathbb{R}$. Let us give a rough idea of this hierarchy in the case of a domain $D \subset \mathbb{R}^n$, where D is a compact domain with sufficiently smooth boundary ∂D. In the new hierarchy in $\mathbb{R}^n$, the N-dimensional subspaces in $C^\infty(I)$ will be generalized by solution spaces

$$S_N = \{u : P_{2N}u(x) = 0 \text{ for } x \in D\} \subset C^\infty(D),$$

where P_{2N} is an elliptic operator of order $2N$ in the domain D; the precise definitions will be specified later on.

2. Kolmogorov's Result – a Reminder

Let us recall the original result of Kolmogorov provided in his seminal paper [8] where he introduced for the first time the theory of widths. Kolmogorov has considered the set K_p defined in (1). He proved that this is an ellipsoid by constructing explicitly its principal axes. Namely, he considered the eigenvalue problem

$$(-1)^p u^{(2p)}(t) = \lambda u(t) \qquad \text{for } t \in (0,1), \tag{3}$$

$$u^{(p+j)}(0) = u^{(p+j)}(1) = 0 \qquad \text{for } j = 0, 1, \ldots, p-1. \tag{4}$$

By the results of M. Krein proved an year earlier [10, 13], Kolmogorov proved that problem (3)–(4) has the following properties, cf. also [12, Chapter 9.6, Theorem 9, p. 146], [15, Section 4.4.4, Theorem 6, p. 244], [14]:

Proposition 1. *Problem* (3)–(4) *has a countable set of non-negative real eigenvalues with finite multiplicity. If we denote them by λ_j in a monotone order, they satisfy $\lambda_j \to \infty$ for $j \to \infty$. They satisfy the following asymptotic $\lambda_j = \pi^{2p} j^{2p}(1 + O(j^{-1}))$. The corresponding orthonormalized eigenfunctions $\{\psi_j\}_{j=1}^\infty$ form a complete orthonormal system in $L_2([0,1])$. The eigenvalue $\lambda = 0$ has multiplicity p and the corresponding eigenfunctions $\{\psi_j\}_{j=1}^p$ are a basis for the solutions to equation $u^{(p)}(t) = 0$ in the interval* $(0,1)$.

Further, Kolmogorov provided a description of the axes of the "cylindrical ellipsoid set" K_p, from which easily follows an approximation theorem of Jackson type.

Proposition 2. *Let $f \in L_2([a,b])$ has the L_2-expansion*

$$f(t) = \sum_{j=1}^{\infty} f_j \psi_j(t).$$

Then $f \in K_p$ if and only if

$$\sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1.$$

For $N \geq p+1$ and every $f \in K_p$ it holds the following estimate (Jackson type approximation):

$$\Big\| f - \sum_{j=1}^{N} f_j \psi_j(t) \Big\|_{L_2} \leq \frac{1}{\sqrt{\lambda_{N+1}}} = O\Big(\frac{1}{(N+1)^p} \Big).$$

However, Komogorov did not stop at this point but asked further, whether the linear space $\widetilde{X}_N := \{\psi_j\}_{j=1}^N$ provides the "best possible approximation among the linear spaces of dimension N" in the following sense: if we put

$$d_N(K_p) := \inf_{X_N \,:\, \dim(X_N) \leq N} \operatorname{dist}(X_N, K_p)\,, \tag{5}$$

then Kolmogorov has proved in [8] the equality

$$d_N(K_p) = \operatorname{dist}(\widetilde{X}_N, K_p).$$

Hence, the above result reads as

$$d_N(K_p) = \begin{cases} \dfrac{1}{\sqrt{\lambda_{N+1}}} & \text{for } N \geq p, \\ \infty & \text{for } N = 0, 1, \ldots, p-1. \end{cases}$$

Here we have used the notations

$$\operatorname{dist}(X, K_p) := \sup_{y \in K_p} \operatorname{dist}(X, y), \qquad \operatorname{dist}(X, y) = \inf_{x \in X} \|x - y\|.$$

Definition 1. The left-hand side quantity in (5) is called *Kolmogorov N-width*, while the best approximation space $\widetilde{X}_N$ is called *extremal (optimal) subspace*, cf. [12], [15], [14].

Thus the main concept of the theory of widths is closely related to a Jackson type theorem by which a special space $\widetilde{X}_N$ is identified. Then one has to find in which sense is the space $\widetilde{X}_N$ the extremal subspace. We may formulate it in other words: one has to find as wide class of spaces X_N as possible, among which $\widetilde{X}_N$ is the extremal subspace.

Now let us consider the set K_p^* defined as in (2), which is a natural multivariate generalization of the set K_p. For simplicity sake we will restrict ourselves with the unit ball B in $\mathbb{R}^n$. Let us remark that the Sobolev space $H^{2p}(B)$ is the multivariate version of the space of absolutely continuous functions on the interval with a highest derivative in L_2 (as in (1)). An important feature of the set K_p^* is that it contains an infinite-dimensional subspace

$$\left\{u \in H^{2p}(B) : \Delta^p u(x) = 0 \text{ for } x \in B\right\}.$$

Hence, all Kolmogorov widths are equal to infinity,

$$d_N(K_p^*) = \infty \qquad \text{for } N \geq 0$$

and no way is seen to improve this if one remains within the finite-dimensional setting.

The main purpose of the present paper is to find a proper setting in the framework of the Polyharmonic Paradigm which generalizes the above results of Kolmogorov.

3. Elliptic Differential Operators and Elliptic BVP

As we said we restrict ourselves to a simple domain as the unit ball B in $\mathbb{R}^n$. However the results below hold for a much bigger class of domains.

We will make extensive use of the following Green formula for the polyharmonic operator Δ^p, cf. [3, p. 10]:

$$\int_B (\Delta^p u \cdot v - u\,\Delta^p v)\,dx = \sum_{j=0}^{p-1} \int_{\partial B} (\Delta^j u \cdot \partial_n \Delta^{p-1-j} v - \partial_n \Delta^j u \cdot \Delta^{p-1-j} v)\,dx, \tag{6}$$

(here ∂_n denotes the normal derivative to ∂B) for functions u and v in the Sobolev classes $H^{2p}(B)$.

For us the following eigenvalue problem will be important to consider for $U \in H^{2p}(B)$:

$$\Delta^{2p} U(x) = \lambda U(x) \qquad \text{for } x \in B, \tag{7}$$

$$\Delta^{p+j} U(y) = \partial_n \Delta^{p+j} U(y) = 0 \qquad \text{for all } y \in \partial B, \quad j = 0, 1, \ldots, p-1, \tag{8}$$

where ∂_n denotes the normal derivative at $y \in \partial B$. The operator Δ^{2p} is formally self-adjoint, cf. [11], however the *Boundary Value Problem (BVP)* (7)–(8) is not a nice one from the point of view of Elliptic BVPs. Since a direct reference seems not to be available, we need a special consideration of this problem provided in the following theorem.

Theorem 1. *Problem* (7)–(8) *has only real non-negative eigenvalues.*

(a) The eigenvalue $\lambda = 0$ has infinite multiplicity with corresponding eigenfunctions $\{\psi_j'\}_{j=1}^{\infty}$ which represent an orthonormal basis of the space of all solutions to the equation $\Delta^p U(x) = 0$ for $x \in B$.

(b) The positive eigenvalues are countably many and each has finite multiplicity, and if we denote them by λ_j ordered increasingly, they satisfy $\lambda_j \to \infty$ for $j \to \infty$.

(c) Let $\{\psi_j\}_{j=1}^{\infty}$ be the orthonormalized eigenfunctions, corresponding to eigenvalues $\lambda_j > 0$. Then the set of functions $\{\psi_j\}_{j=1}^{\infty} \cup \{\psi_j'\}_{j=1}^{\infty}$ form a complete orthonormal system in $L_2(B)$.

Remark 1. Problem (7)–(8) is widely known to be non-regular elliptic BVP, as well as non-coercive variational, c.f. [1], p. 150 at the end of Section 10, and [11], Remark 9.8 (Chapter 12, Section 9.6, p. 240 in the Russian edition) and Section 9.8 there, p. 242. This problem will give us the eigenfunctions ψ_k in the notations in [12].

The proof is provided in Section 5.

4. The Principal Axes of the Ellipsoid K_p^* and Jackson Type Theorem

Here we will find the principal axes of the ellipsoid K_p^* defined in (2).

We prove the following theorem which generalizes Kolmogorov's one-dimensional [8], about the representation of the ellipsoid K_p in principal axes.

Theorem 2. *Let $f \in K_p^*$. Then f is represented in a L_2-series as*

$$f(x) = \sum_{j=1}^{\infty} f_j' \psi_j'(x) + \sum_{j=1}^{\infty} f_j \psi_j(x)$$

where, by Theorem 1, the eigenfunctions ψ_j' satisfy $\Delta^p \psi_j'(x) = 0$ while the eigenfunctions ψ_j correspond to the eigenvalues $\lambda_j > 0$, and the coefficients $\{f_j\}_{j=1}^{\infty}$ satisfy the inequality

$$\sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1. \tag{9}$$

Vice versa, every sequence $\{f_j'\}_{j=1}^{\infty} \cup \{f_j\}_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty} |f_j'|^2 + \sum_{j=1}^{\infty} |f_j|^2 < \infty$ and $\sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1$ defines a function $f \in L_2(B)$ which is in K_p^*.

Proof. According to Theorem 1, we know that an arbitrary $f \in L_2(B)$ is represented as

$$f(x) = \sum_{j=1}^{\infty} f_j' \psi_j'(x) + \sum_{j=1}^{\infty} f_j \psi_j(x), \qquad \|f\|_{L_2}^2 = \sum_{j=1}^{\infty} |f_j'|^2 + \sum_{j=1}^{\infty} |f_j|^2 < \infty$$

with convergence in the space $L_2(B)$.

From the proof of Theorem 1, we know that if we put

$$\phi_j(x) = \Delta^p \psi_j(x) \qquad \text{for } j \geq 1,$$

then the system of functions

$$\frac{\phi_j(x)}{\sqrt{\lambda_j}} \qquad \text{for } j \geq 1$$

is orthonormal sequence which is complete in $L_2(B)$.

We will prove now that if $f \in L_2(B)$ then $f \in K_p^*$ iff $\sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1$.

Indeed, we have the expansion $f(x) = \sum_{j=1}^{\infty} f'_j \psi'_j(x) + \sum_{j=1}^{\infty} f_j \psi_j(x)$ for every $f \in H^{2p}(B)$. We want to see that it is possible to differentiate termwise this expansion, i.e.

$$\Delta^p f(x) = \sum_{j=1}^{\infty} f_j \Delta^p \psi_j(x) = \sum_{j=1}^{\infty} f_j \phi_j(x).$$

Since $\left\{ \frac{\phi_j}{\sqrt{\lambda_j}} \right\}_{j \geq 1}$ is a complete orthogonal basis of $L_2(B)$ it is sufficient to see that

$$\int_B \Delta^p f(x) \phi_j \, dx = \int_B \Big(\sum_{j=1}^{\infty} f_j \Delta^p \psi_j(x) \Big) \phi_j \, dx.$$

Due to the boundary properties of ϕ_j and since $\phi_j = \Delta^p \psi_j$, we obtain

$$\int_B \Delta^p f(x) \phi_j \, dx = \int_B f(x) \Delta^p \phi_j \, dx = \lambda_j \int_B f \psi_j \, dx = \lambda_j f_j.$$

On the other hand

$$\int_B \Big(\sum_{k=1}^{\infty} f_k \phi_k(x) \Big) \phi_j \, dx = \lambda_j f_j.$$

Hence

$$\Delta^p f(x) = \sum_{j=1}^{\infty} f_j \Delta^p \psi_j(x) = \sum_{j=1}^{\infty} f_j \phi_j(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \, f_j \, \frac{\phi_j(x)}{\sqrt{\lambda_j}}$$

and since $\left\{ \frac{\phi_j}{\sqrt{\lambda_j}} \right\}_{j \geq 1}$ is an orthonormal system, it follows

$$\|\Delta^p f\|_{L_2}^2 = \sum_{j=1}^{\infty} \lambda_j f_j^2.$$

Thus if $f \in K_p$ it follows that $\sum_{j=1}^{\infty} \lambda_j f_j^2 \leq 1$.

Now, assume vice versa, that $\sum_{j=1}^{\infty} f_j^2 \lambda_j \leq 1$ holds together with the inequality $\sum_{j=1}^{\infty} |f_j'|^2 + \sum_{j=1}^{\infty} |f_j|^2 < \infty$. We have to prove that the function

$$f(x) = \sum_{j=1}^{\infty} f_j' \psi_j'(x) + \sum_{j=1}^{\infty} f_j \psi_j(x)$$

belongs to the space $H^{2p}(B)$. Based on the completeness and orthonormality of the system $\left\{ \frac{\phi_j(x)}{\sqrt{\lambda_j}} \right\}_{j=1}^{\infty}$ we may define the function $g \in L_2$ by putting

$$g(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, f_j \, \frac{\phi_j(x)}{\sqrt{\lambda_j}} = \sum_{j=1}^{\infty} f_j \phi_j(x);$$

it obviously satisfies $\|g\|_{L_2} \leq 1$.

As is well-known from the theory of Elliptic BVPs we may find a function $F \in H^{2p}(B)$ which is a solution to equation $\Delta^p F = g$ (see [11, Chapter 2, Section 5.3, Theorem 5.3]). Let its representation be

$$F(x) = \sum_{j=1}^{\infty} f_j' \psi_j'(x) + \sum_{j=1}^{\infty} F_j \psi_j(x)$$

with some F_j satisfying $\sum_j |F_j|^2 < \infty$. As above we obtain

$$\lambda_j \int_B F \psi_j \, dx = \int_B F \Delta^{2p} \psi_j \, dx = \int_B \Delta^p F \cdot \Delta^p \psi_j \, dx = \int_B g \cdot \phi_j \, dx,$$

which implies $F_j = f_j$. Hence, $F = f$ and $f \in H^{2p}(B)$. This ends the proof. □

We are able to prove finally a Jackson type result analogous to Proposition 2.

Theorem 3. *Let $N \geq 1$. Then for every $N \geq 1$ and every $f \in K_p^*$ it holds the following estimate:*

$$\left\| f - \sum_{j=1}^{\infty} f_j' \psi_j'(x) - \sum_{j=1}^{N} f_j \psi_j(x) \right\|_{L_2} \leq \frac{1}{\sqrt{\lambda_{N+1}}}.$$

Proof. Due to the monotonicity of λ_j, and inequality (9), we obtain

$$\left\| f - \sum_{j=1}^{\infty} f_j' \psi_j'(x) - \sum_{j=1}^{N} f_j \psi_j(x) \right\|_{L_2}^2 = \sum_{j=N+1}^{\infty} f_j^2 \leq \frac{1}{\lambda_{N+1}} \sum_{j=N+1}^{\infty} f_j^2 \lambda_j \leq \frac{1}{\lambda_{N+1}}.$$

This ends the proof. □

Now we are able to prove a generalization of Kolmogorov's result about widths [8]. It is important which classes of spaces we are going to choose for generalizing the widths. We introduce the following subspaces in $L_2(B)$: For integers $M \geq 1$ we define

$$S_M := \{u \in H^{2M}(B) : Q_{2M}u(x) = 0 \text{ for } x \in B\} \tag{10}$$

where Q_{2M} is a uniformly strongly elliptic operator of order $2M$, cf. [2], [11], or [9, p. 473]. We denote by F_N a finite-dimensional subspace of $L_2(B)$ of dimension N. The special subspaces for $P_{2M} = \Delta^M$ are denoted by $\widetilde{S}_M$,

$$\widetilde{S}_M := \{u \in H^{2M}(B) : \Delta^M u(x) = 0 \text{ for } x \in B\},$$

and $\widetilde{F}_N$ are the special finite-dimensional subspaces

$$\widetilde{F}_N := \operatorname{span}\{\psi_j : j \leq N\},$$

with ψ_j being the eigenfunctions from Theorem 1.

The following results are analogs to the original Kolmogorov's results about widths, cf. [8], or the more detailed exposition in [12, Theorem 9, p. 146], [15] and [14].

Theorem 4. *Let Q_{2M} be a strongly elliptic differential operator of order $2M$ in B, and let $N \geq 0$ be arbitrary integer.*

(a) If $M < p$ then $\operatorname{dist}\left(S_M \bigoplus F_N, K_p^*\right) = \infty$. *Hence,*

$$\inf_{Q_{2M}} \operatorname{dist}\left(S_M \bigoplus F_N, K_p^*\right) = \infty.$$

(b) If $M = p$ then

$$\inf_{S_p, F_N} \operatorname{dist}\left(S_p \bigoplus F_N, K_p^*\right) = \operatorname{dist}\left(\widetilde{S}_p \bigoplus \widetilde{F}_N, K_p^*\right).$$

Proof. (a) If we assume that S_M and $\widetilde{S}_p$ are transversal, the proof is clear since $\widetilde{S}_p \subset K_p^*$ and there will be an infinite-dimensional space in $\widetilde{S}_p \subset K_p^*$ containing infinite axes with direction $y \in \widetilde{S}_p$, such that $\operatorname{dist}\left(S_M \bigoplus F_N, y\right) > 0$ which implies

$$\operatorname{dist}\left(S_M \bigoplus F_N, K_p^*\right) = \infty.$$

If they are not transversal, we apply Lemma 1 below; it is clear that the finite-dimensional subspaces do not disturb the result, and the proof is finished.

(b) For proving the second item, let us first note that $\widetilde{S}_p \subset S_p \bigoplus F_N$. Indeed, since $\widetilde{S}_p \subset K_p^*$, the violation of $\widetilde{S}_p \subset S_p \bigoplus F_N$ would imply that there exists an infinite axis y in K_p^* not contained in $S_p \bigoplus F_N$ which would immediately give

$$\operatorname{dist}\left(S_p \bigoplus F_N, K_p^*\right) = \infty.$$

But by Lemma 2 it follows that $P_{2p} = C(x)\Delta^p$ for some function $C(x)$. Hence $S_p = \widetilde{S}_p$.

Further we follow the usual way as in [12] to see that $\widetilde{F}_N$ is extremal among all spaces F_N, i.e.

$$\inf_{F_N} \operatorname{dist}\left(\widetilde{S}_p \bigoplus F_N, K_p^*\right) = \operatorname{dist}\left(\widetilde{S}_p \bigoplus \widetilde{F}_N, K_p^*\right).$$

This ends the proof. □

The following result shows the mutual position of two subspaces:

Lemma 1. *Let M, N and M_1 be integers satisfying $M < N$ and $M_1 \geq 0$. Then for the corresponding S_M and S_N defined in* (10) *by the operators P_{2M} and $Q_{2N} = \Delta^N$ respectively,*

$$\operatorname{dist}(S_M \bigoplus F_{M_1}, S_N) = \infty$$

holds. There is a linear subspace $Y_{N-M} \subset S_N$ with $Y_{N-M} \perp S_M$ and it is an infinite-dimensional space of solutions to an Elliptic BVP.

Proof. Let us consider the case $M_1 = 0$. For the uniformly strongly elliptic operator P_{2M} we choose the Dirichlet system of boundary operators $B_j = \partial^{j-1}/\partial n^{j-1}$. It is a classical fact (cf. [11, Chapter 2, Section 1.4, Remark 1.3]) that this system satisfies conditions (iii) in [11, Chapter 2, Section 5.1], or, in other words, the system of operators $\{P_{2M}; \partial^j/\partial n^j : j = 0, 1, ..., M-1\}$ forms a regular Elliptic Boundary Value Problem (this is the so-called Dirichlet BVP associated with the operator P_{2M}). Hence, we may apply the existence Theorems 5.2 and 5.3 in [11]. As in Theorem 2.1 ([11, Chapter 2, Section 2.2]) we complete the system $\{B_j\}_{j=1}^M$ by the system of boundary operators $S_j = \partial^{M-1+j}/\partial n^{M-1+j}$. Hence, the composed system $\{B_j\}_{j=1}^M \cup \{S_j\}_{j=1}^M$ is a Dirichlet system of order $2M$ (cf. e.g., [9, Definition 23.12, p. 474]). Further, by Theorem 2.1 in [11] quoted above, there exists a unique Dirichlet system of order $2M$ of boundary operators $\{C_j, T_j\}_{j=1}^M$ which is uniquely determined as the adjoint to the system $\{B_j, S_j\}_{j=1}^M$, and the following Green formula holds:

$$\int_B (P_{2M}u \cdot v - u \cdot P_{2M}^* v) dx = \sum_{j=1}^M \int_{\partial B} (S_j u \cdot C_j v - B_j u \cdot T_j v)\, d\sigma_y, \tag{11}$$

for all $u, v \in H^{2M}(B)$; here $d\sigma_y$ denotes the surface element on the sphere ∂B.

We consider the elliptic operator $\Delta^N P_{2M}$. As a product of two uniformly strongly elliptic operators it is such again. By using a standard construction from Theorem 2.1 in [11], we complete the Dirichlet system of operators $\{B_j, S_j\}_{j=1}^M$ with $N-M$ boundary operators $R_j = \partial^{2M-1+j}/\partial n^{2M-1+j}$, $j = 1, 2, \ldots, N-M$. Again by the above cited theorem, the Dirichlet system of boundary operators $\{B_j, S_j\}_{j=1}^M \cup \{R_j\}_{j=1}^{N-M}$ covers the operator $\Delta^N P_{2M}$.

Finally, we consider the solutions $g \in H^{2N+2M}(B)$ to the following Elliptic BVP:

$$\begin{aligned} \Delta^N P_{2M}^* g(x) &= 0 && \text{for } x \in B, && (12)\\ B_j g(y) = S_j g(y) &= 0 && \text{for } j = 0, 1, \ldots, N-1, \text{ for } y \in \partial B, && (13)\\ R_j g(y) &= h_j(y) && \text{for } j = 1, 2, \ldots, N-M, \text{ for } y \in \partial B. && (14) \end{aligned}$$

We may apply the existence Theorems 5.2 and 5.3 in [11, Chapter 2] to justify solvability of problem (12)–(14) in the space $H^{2M+2N}(B)$.

First of all, it is clear from (12) that $P_{2M}^* g \in S_N$.

Let us check the properties of the function $P_{2M}^* g$. By the Green formula (11), the function $P_{2M}^* g$ satisfies $P_{2M}^* g \perp S_M$, or equivalently,

$$\int_B P_{2M}^* g \cdot v \, dx = 0 \qquad \text{for all } v \text{ with } P_{2M} v = 0.$$

By the general existence Theorem 5.3 (the Fredholmness property) in [11] mentioned above, we know that a solution g to problem (12)–(14) exists for those boundary data $\{h_j\}_{j=1}^{N-M}$ which satisfy only a finite number of linear restrictions, provided by conditions (5.18) there; these are determined by the solutions to the homogeneous adjoint Elliptic BVP. Hence, it follows that the set Y_{N-M} of the functions $P_{2M}^* g$ where g is a solution to (12)–(14) is infinite-dimensional. It follows that the space $S_N \setminus S_M$ is infinite-dimensional as well, hence

$$\operatorname{dist}\left(S_M \bigoplus F_{M_1}, S_N\right) = \infty.$$

Since obviously a finite-dimensional subspace F_{M_1} would not disturb the above argumentation, this ends the proof. □

Remark 2. Lemma 1 may be considered as a generalization in our setting of a theorem of Gohberg-Krein of 1957 (cf. [12, Theorem 2, p. 137]) in a Hilbert space.

We need the following intuitive result which is however not trivial.

Lemma 2. *Let for some elliptic differential operator P_{2N} of order $2N$ the following inclusion hold $S_N \subset \widetilde{S}_N \setminus F$, i.e.*

$$\left\{u \in H^{2N}(B) : P_{2N} u(x) = 0, \ x \in B\right\} \subset \left\{u \in H^{2N}(B) : \Delta^N u(x) = 0, \ x \in B\right\} \setminus F,$$

where $F \subset L_2(B)$ is a finite-dimensional space. Then

$$P_{2N}(x, D_x) = c(x) \Delta^N \tag{15}$$

for some function $c(x)$.

Proof. Since the general case is rather technical we will consider only $N = 1$ in $B \subset \mathbb{R}^2$. It is clear that the arguments are purely local so we will prove that equality (15) holds at $(x_1, x_2) = x = 0 \in B$. Assume that

$$P_{2N}(x, D_x)u(x) = a(x)u_{x_1,x_1} + 2b(x)u_{x_1,x_2} + c(x)u_{x_2,x_2} + d(x)u_{x_1} + e(x)u_{x_2} + f(x)u;$$

here w_{x_j} denotes the partial derivative $\partial w/\partial x_j$. By assumption, for the function $u \in \widetilde{S}_1 \setminus F$ holds also

$$\big(a(x) - c(x)\big)u_{x_1,x_1} + 2b(x)u_{x_1,x_2} + d(x)u_{x_1} + e(x)u_{x_2} + f(x)u = 0.$$

Let us denote the harmonic functions $1,\ x_1,\ x_2,\ x_1^2 - x_2^2,\ x_1x_2$ by u^j for $j = 1, 2, \dots, 5$. Let us assume that they do not belong to F. We see that the Jacobi matrix of these functions at $x_1 = x_2 = 0$, is

$$\begin{pmatrix} u^j_{x_1,x_1} & u^j_{x_1,x_2} & u^j_{x_1} & u^j_{x_2} & u^j \end{pmatrix}_{j=1}^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

which is obviously non-degenerate. Hence, $a(0) - c(0) = b(0) = d(0) = e(0) = f(0) = 0$.

In the case if some of the above functions u^j belongs to the space F, it is possible to approximate it by other harmonic functions also including up to their second derivatives at 0 (one may apply approximation arguments as in [6]). The respective Jacobian will be non-zero and the conclusion of the theorem will follow. This ends the proof. $\square$

The proof of Theorem 4 above permits a much bigger generalization which will be provided in a forthcoming paper.

5. Appendix on Elliptic Boundary Value Problems

Proof of Theorem 1.
(a) We consider the following auxiliary elliptic eigenvalue problem

$$\Delta^{2p}\phi(x) = \lambda\phi(x) \qquad \text{on } B, \tag{16}$$

$$\partial\Delta^j\phi(y) = \Delta^j\phi(y) = 0 \quad \text{for } \ j = 0, 1, \dots, p-1, \ \text{for } y \in \partial B. \tag{17}$$

It is straigthforward to check that this is a regular Elliptic BVP considered in the Sobolev space $H^{2p}(B)$ since it satisfies all conditions (i)–(iii) in [11, Chapter 2, Section 5.1], cf. also [7]. Hence, we are able to apply the existence theorems in Section 5.3 there. Further, it is straightforward to check that it is a self-adjoint problem (cf. [11, Chapter 2, Section 2.5]): in the polyharmonic

Green formula (6) we put $\{B_j\}_{j=1}^{2p} = \{\partial\Delta^j, \Delta^j\}_{j=0}^{p-1}$ and we see that in the context of the general Green formula (11) the adjoint system of operators $\{C_j\}_{j=1}^{2p} = \{\partial\Delta^j, \Delta^j\}_{j=0}^{p-1}$ which proves the self-adjointness of problem (16)–(17). Hence, we may apply the main results about the Spectral theory of regular self-adjoint Elliptic BVP. We refer to [7, Chapter 2, Section 3, Theorem 2.52, p. 122], and to references therein (cf. in particular the monograph of Berezanskii devoted to expansions in eigenfunctions [4, Chapter 6, Section 2]).

By the uniqueness Lemma 3 the eigenvalue problem (16)–(17) has only zero solution for $\lambda = 0$. It has eigenfunctions $\phi_k \in H^{2p}(B)$ with eigenvalues $\lambda_k > 0$ for $k = 1, 2, 3, \ldots$, for which $\lambda_k \to \infty$ as $k \to \infty$.

(b) Next we consider the problem

$$\Delta^{2p}\varphi = \phi_k \tag{18}$$

$$\partial\Delta^j\varphi(y) = \Delta^j\varphi(y) = 0 \qquad \text{for } j = 0, 1, \ldots, p-1, \text{ for } y \in \partial B, \tag{19}$$

in the Sobolev space $H^{2p}(B)$. Obviously, the Elliptic BVP defined by (18)–(19) coincides with the Elliptic BVP defined by (16)–(17) and all remarks there apply here, too. Hence, problem (18)–(19) has unique solution $\varphi_k \in H^{2p}(B)$. We put

$$\psi_k = \Delta^p\varphi_k.$$

Hence, $\Delta^p\psi_k = \phi_k$. We infer that on the boundary ∂B hold the equalities $\Delta^{p+j}\psi_k = \Delta^j\phi_k$ and $\partial\Delta^{p+j}\psi_k = \partial\Delta^j\phi_k$; since ϕ_k are solutions to (16)–(17) it follows

$$\Delta^{p+j}\psi_k(y) = \partial\Delta^{p+j}\psi_k(y) = 0 \qquad \text{for } j = 0, 1, \ldots, p-1, \text{ for } y \in \partial B. \tag{20}$$

We will prove that ψ_k are solutions to problem (7)–(8), they are mutulally orthogonal, and they are also orthogonal to the space $\{v \in H^{2p} : \Delta^p v = 0\}$.

Let us see that $\Delta^{2p}\psi_k = \lambda_k\psi_k$. By the definition of ψ_k this is equivalent to $\Delta^{3p}\varphi_k = \lambda_k\Delta^p\varphi_k$; from $\Delta^{2p}\varphi_k = \phi_k$ this is equivalent to $\Delta^p\phi_k = \lambda_k\Delta^p\varphi_k$. On the other hand, we have obviously $\Delta^{2p}\phi_k = \lambda_k\Delta^{2p}\varphi_k$ by the basic properties of ϕ_k and φ_k, hence

$$\Delta^{2p}(\phi_k - \lambda_k\varphi_k) = 0.$$

Note that both ϕ_k and φ_k sastisfy the same zero boundary conditions, namely (17) and (19). Hence, by the uniqueness Lemma 3 it follows that $\phi_k - \lambda_k\varphi_k = 0$ which implies $\Delta^{2p}\psi_k = \lambda_k\psi_k$. Thus we see that ψ_k is a solution to problem (7)–(8) and does not satisfy $\Delta^p\psi = 0$.

The orthogonality to the subspace $\{v \in H^{2p} : \Delta^p v = 0\}$ follows easily from the Green formula (6) and the zero boundary conditions (20) of ψ_k, by the following:

$$\int_D (\Delta^{2p}\psi_k \cdot v - \psi_k \cdot \Delta^{2p} v)\, dx = \sum_{j=0}^{2p-1} \int_{\partial D} (\Delta^j\psi_k \cdot \partial_n\Delta^{2p-1-j}v - \partial_n\Delta^j\psi_k \cdot \Delta^{2p-1-j}v)$$

and since $\int_D \Delta^{2p}\psi_k \cdot v\, dx = \lambda_k \int_D \psi_k \cdot v\, dx$.

The orthonormality of the system $\{\psi_k\}_{k=1}^{\infty}$ follows now easily by the equality

$$\lambda_k \int \psi_k \psi_j \, dx = \int \Delta^{2p} \psi_k \psi_j \, dx = \int \Delta^p \psi_k \Delta^p \psi_j \, dx = \int \phi_k \phi_j \, dx$$

and the orthogonality of the system $\{\phi_k\}_{k=1}^{\infty}$. For the completeness of the system $\{\psi_k\}_{k=1}^{\infty}$, let us assume that for some $f \in L_2(B)$ it holds

$$\int_B f \cdot \psi_k \, dx = \int_B f \cdot \psi_k' \, dx = 0 \qquad \text{for all } k \geq 1. \tag{21}$$

Then the Green formula (6) implies

$$\begin{aligned} 0 &= \lambda_k \int_B f \cdot \psi_k \, dx = \int_B f \cdot \Delta^{2p} \psi_k \, dx = \int_B \Delta^p f \cdot \Delta^p \psi_k \, dx \\ &= \int_B \Delta^p f \cdot \phi_k \, dx \qquad \text{for all } k \geq 1. \end{aligned}$$

By the completeness of the system $\{\phi_k\}_{k \geq 1}$ this implies that $\Delta^p f = 0$. From the second orthogonality in (21) it follows that $f \equiv 0$, and this ends the proof of the completeness of the system $\{\psi_j'\}_{j=1}^{\infty} \cup \{\psi_j\}_{j=1}^{\infty}$. □

We have used above the following simple result.

Lemma 3. *The solution to problem* (16)–(17) *for* $\lambda = 0$ *is unique.*

Proof. From Green formula (6) we obtain $\int_B [\Delta^p \phi]^2 \, dx = \int \phi \cdot \Delta^{2p} \phi \, dx = 0$, hence $\Delta^p \phi = 0$. Now we apply the second Green formula (2.11) in [3] which infers immediately $\phi \equiv 0$. □

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