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Interlacing Properties of Certain Tchebycheff Systems*

Lozko Milev and Nikola Naidenov

We present the main results of [8], where a general condition, denoted by (P), for the validity of the Markov interlacing property for extended Tchebycheff systems on the real line was formulated. It was also proved in [8] that (P) is satisfied for some known systems, including exponential and Müntz polynomials.

Here we give various examples, showing that condition (P) is essential for the correctness of the results.

Keywords and Phrases: Exponential polynomials, Müntz polynomials, Markov interlacing property, Tchebycheff systems.

1. Main Results

Denote by π_n the set of all real algebraic polynomials of degree at most n. A classical result for polynomials, which have only real zeros, is the following

Lemma (V. A. Markov). *Suppose that the polynomials p and q from π_n have zeros $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, respectively, which satisfy the interlacing conditions*

$$x_1 \le y_1 \le \cdots \le x_n \le y_n.$$

Then the zeros $t_1 < \cdots < t_{n-1}$ of $p'(x)$ and the zeros $\tau_1 < \cdots < \tau_{n-1}$ of $q'(x)$ interlace too, that is

$$t_1 \le \tau_1 \le \cdots \le t_{n-1} \le \tau_{n-1}.$$

Moreover, the above inequalities are strict, unless $x_i = y_i$, $i = 1, \ldots, n$.

Markov's lemma is often used in the study of extremal problems for algebraic polynomials and also in questions related to the distribution of the zeros of

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derivatives, see [7, 14, 9, 4, 13, 5, 15, 10, 3, 2, 11]. Another problem concerning interlacing properties of Tchebycheff systems was studied in [12].

A natural goal is to extend Markov's interlacing property to more general classes of functions. Results of this type are obtained by Videnskii [16] and Bojanov [1]. However, some important Tchebycheff systems on the real line do not fulfil the requirements from [16] and [1].

In the recent paper [8] we formulated a condition (denoted by (P)), such that if an ET-system (see the definition below) satisfies (P), then it possesses Markov's interlacing property. We gave various examples of ET-systems of exponential polynomials, which have the property (P) and obtained the corresponding results for interlacing. We also showed that for some systems Markov's interlacing property holds even for derivatives of arbitrary order. Next we summarize the main results of [8].

A set of functions $\{u_0, \dots, u_n\}$ is called an *Extended Tchebycheff system (ET-system)* on $\mathbb{R}$, if $u_i \in C^n(\mathbb{R})$ for $i = 0, \dots, n$, and every non-zero polynomial in this system $u = \sum_{i=0}^{n} a_i u_i$, where $(a_0, \dots, a_n) \in \mathbb{R}^{n+1}$, has at most n real zeros, counting multiplicities. Then the linear space

$$U_n := \operatorname{span}\{u_0, \dots, u_n\}$$

is said to be an *Extended Tchebycheff space (ET-space)*. We set

$$X = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 < \cdots < x_n\}.$$

Given a point $\bar{x} \in X$, we define

$$f(\bar{x}; t) := \begin{vmatrix} u_0(t) & \dots & u_n(t) \\ u_0(x_1) & \dots & u_n(x_1) \\ \dots & \dots & \dots \\ u_0(x_n) & \dots & u_n(x_n) \end{vmatrix}.$$

Clearly, $f(\bar{x}; t)$ is a polynomial from U_n, which has zeros $x_1, \dots, x_n$. Note that if $g \in U_n$ is any other polynomial having the same zeros, then there exists a constant C such that $g(t) = Cf(\bar{x}; t)$. In general, we shall say that $f \in U_n$ is an *oscillating polynomial* if it has n distinct real zeros.

Applying Rolle's Theorem to a polynomial $f(\bar{x}; t) \in U_n$ we see that $f'(\bar{x}; t)$ has at least one zero in each of the intervals (x_i, x_{i+1}), $i = 1, \dots, n-1$. We shall suppose that U_n has the following

Property (P): There exist numbers δ_0 and δ_n in $\{0, 1\}$ such that for every oscillating polynomial $f(\bar{x}; t) \in U_n$ with zeros $\bar{x} = (x_1, \dots, x_n) \in X$, $f'(\bar{x}; t)$ has exactly:

- δ_0 zeros in $(-\infty, x_1)$;
- one zero in each interval (x_i, x_{i+1}), $i = 1, \dots, n-1$;
- δ_n zeros in (x_n, ∞).

Next we introduce an index set $J(U_n) \subset \{0, \dots, n\}$, which corresponds to the zeros of $f'(\bar{x}; t)$. The definition of $J(U_n)$ is as follows: the set $\{1, \dots, n-1\}$ is contained in $J(U_n)$ and if $\delta_i = 1$ for some $i \in \{0, n\}$ then $i \in J(U_n)$.

Our first result concerns general ET-systems with the property (P).

Theorem 1. *Suppose that* $\{u_0,\dots,u_n\}$ *is an ET-system on the real line and* $U_n := \operatorname{span}\{u_0,\dots,u_n\}$ *satisfies property (P). Let* $\bar{x} = (x_1,\dots,x_n)$ *and* $\bar{y} = (y_1,\dots,y_n)$ *be two vectors from* X*, whose components interlace, that is,*

$$x_1 \le y_1 \le x_2 \le y_2 \le \cdots \le x_n \le y_n. \tag{1}$$

Then the zeros $\{t_i\}_{i\in J(U_n)}$ *of* $f'(\bar{x};t)$ *and the zeros* $\{\tau_i\}_{i\in J(U_n)}$ *of* $f'(\bar{y};t)$ *interlace in the same order:*

$$t_m \le \tau_m \le t_{m+1} \le \tau_{m+1} \le \cdots \le t_M \le \tau_M, \tag{2}$$

where $m := \min\{i : i \in J(U_n)\}$, $M := \max\{i : i \in J(U_n)\}$. *Moreover, if* $\bar{x} \ne \bar{y}$*, then all the inequalities in* (2) *are strict.*

Given real numbers $\bar{\alpha} = (\alpha_0,\alpha_1,\dots,\alpha_n)$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$, we set

$$V_n(\bar{\alpha}) := \operatorname{span}\{e^{\alpha_0 x}, e^{\alpha_1 x}, \dots, e^{\alpha_n x}\}$$

and

$$J(\bar{\alpha}) := \begin{cases} \{0,\dots,n-1\}, & \text{if } \alpha_0 > 0,\\ \{1,\dots,n\}, & \text{if } \alpha_n < 0,\\ \{1,\dots,n-1\}, & \text{if } \alpha_0 \le 0 \le \alpha_n. \end{cases}$$

It is well-known that the functions $\{e^{\alpha_0 x}, e^{\alpha_1 x}, \dots, e^{\alpha_n x}\}$ form an ET-system on $(-\infty,\infty)$. The zeros of the derivative of any oscillating polynomial from $V_n(\bar{\alpha})$ can be indexed by the set $J(\bar{\alpha})$. In the following theorem we establish Markov's interlacing property for the ET-space $V_n(\bar{\alpha})$.

Theorem 2. *Assume that the oscillating polynomials* f *and* g *from* $V_n(\bar{\alpha})$ *have zeros* $\bar{x} \in X$ *and* $\bar{y} \in X$*, respectively, which satisfy the inequalities* (1). *Then the zeros* $\{t_i\}_{i\in J(\bar{\alpha})}$ *of* f' *and the zeros* $\{\tau_i\}_{i\in J(\bar{\alpha})}$ *of* g' *interlace in the same order:*

$$t_i \le \tau_i \le t_{i+1} \le \tau_{i+1}, \qquad \text{for } i, i+1 \in J(\bar{\alpha}). \tag{3}$$

Moreover, if $\bar{x} \ne \bar{y}$*, then all the inequalities in* (3) *are strict.*

In addition, if $\alpha_0 < \cdots < \alpha_n < 0$*, then for every natural* k*, the zeros* $\{t_i^{(k)}\}_{i=1}^n$ *of* $f^{(k)}$ *and the zeros* $\{\tau_i^{(k)}\}_{i=1}^n$ *of* $g^{(k)}$ *interlace too:*

$$t_1^{(k)} \le \tau_1^{(k)} \le t_2^{(k)} \le \tau_2^{(k)} \le \cdots \le t_n^{(k)} \le \tau_n^{(k)}.$$

A similar statement holds true provided $0 < \alpha_0 < \cdots < \alpha_n$.

The next result concerns the space of Müntz polynomials:

$$M_n(\bar{\gamma}) := \operatorname{span}\{x^{\gamma_0},\dots,x^{\gamma_n}\},$$

where $\bar{\gamma} = (\gamma_0, \ldots, \gamma_n)$ with $\gamma_0 < \gamma_1 < \cdots < \gamma_n$. As it is well-known, $\{x^{\gamma_0}, \ldots, x^{\gamma_n}\}$ is an ET-system on $(0, \infty)$. Note that if $f(x) \in M_n(\bar{\gamma})$ then a change of the variable $x = e^t$ produces a polynomial $F(t) \in V_n(\bar{\gamma})$. Consequently, if f has n distinct zeros in $(0, \infty)$, then F is an oscillating polynomial. The above reasoning and the monotonicity of the exponential function show that the zeros of the derivative of any oscillating Müntz polynomial can be indexed by the set $J(\bar{\gamma})$. For this system we prove the following

Theorem 3. *Let $f \in M_n(\bar{\gamma})$ has zeros $\bar{x} \in X$ with $x_1 > 0$ and $g \in M_n(\bar{\gamma})$ has zeros $\bar{y} \in X$ with $y_1 > 0$. Assume that $\bar{x}$ and $\bar{y}$ interlace as in* (1). *Then the zeros $\{t_i\}_{i \in J(\bar{\gamma})}$ of f' and the zeros $\{\tau_i\}_{i \in J(\bar{\gamma})}$ of g' interlace too:*

$$t_i \le \tau_i \le t_{i+1} \le \tau_{i+1}, \qquad \text{for } \ i, i+1 \in J(\bar{\gamma}).$$

Moreover, if $\bar{x} \ne \bar{y}$, then all the inequalities above are strict.

In addition, if $\gamma_0 < \cdots < \gamma_n < 0$, then for every $k \in \mathbb{N}$, $f^{(k)}$ (resp., $g^{(k)}$) has n simple zeros $t_1^{(k)} < \cdots < t_n^{(k)}$ ($\tau_1^{(k)} < \cdots < \tau_n^{(k)}$) in $(0, \infty)$ and

$$t_1^{(k)} \le \tau_1^{(k)} \le \cdots \le t_n^{(k)} \le \tau_n^{(k)}.$$

It is known that the system of polynomials with Laguerre weight

$$V_n := \left\{ e^{-x} P_n(x) : \, P_n \in \pi_n \right\}$$

can be considered as a limit case of $V_n(\bar{\alpha})$, as $\alpha_0, \alpha_1, \ldots, \alpha_n$ tend to -1. Then, it is quite natural to expect that the Markov interlacing property holds for V_n, too. Note that if $f \in V_n$ is an oscillating polynomial with zeros $x_1 < \cdots < x_n$, then $f' \in V_n$ and it is also oscillating, with zeros $t_i \in (x_i, x_{i+1})$, $i = 1, \ldots, n$, where $x_{n+1} := \infty$. For V_n we have

Theorem 4. *Let f and g be two oscillating polynomials from V_n with zeros $\bar{x}$ and $\bar{y}$, respectively. Assume that $\bar{x}$ and $\bar{y}$ interlace as in* (1). *Then for every natural number k, the zeros $\{t_i^{(k)}\}_{i=1}^n$ and $\{\tau_i^{(k)}\}_{i=1}^n$ of $f^{(k)}$ and $g^{(k)}$, respectively, interlace in the same order:*

$$t_1^{(k)} \le \tau_1^{(k)} \le \cdots \le t_n^{(k)} \le \tau_n^{(k)}.$$

Moreover, if $\bar{x} \ne \bar{y}$, then all the inequalities above are strict.

A linear change of the variable shows that Markov's interlacing property remains valid even for the space $\left\{ e^{\lambda x} P_n(x) : \, P_n \in \pi_n \right\}$, $\lambda \ne 0$.

The next theorem extends the Markov interlacing property to a more general than $V_n(\bar{\alpha})$ space of functions. Let $\mu \in C^\infty(\mathbb{R})$ be a positive function such that the ratio μ'/μ is non-increasing on the real line. We introduce the set

$$W_n(\bar{\alpha}; \mu) := \left\{ \mu(x) v(x) : \, v \in V_n(\bar{\alpha}) \right\}.$$

Clearly, $\{\mu(x)e^{\alpha_0 x}, \mu(x)e^{\alpha_1 x}, \dots, \mu(x)e^{\alpha_n x}\}$ is an ET-system on $\mathbb{R}$. Furthermore, we define an index set $J(\bar{\alpha};\mu) \subset \{0,1,\dots,n\}$ as follows:

- $\{1,\dots,n-1\} \subset J(\bar{\alpha};\mu)$;
- $0 \in J(\bar{\alpha};\mu)$ if and only if $\alpha_0 > -A$, where $A := \lim_{x\to-\infty} \mu'(x)/\mu(x)$;
- $n \in J(\bar{\alpha};\mu)$ if and only if $\alpha_n < -B$, where $B := \lim_{x\to+\infty} \mu'(x)/\mu(x)$.

We prove that $W_n(\bar{\alpha};\mu)$ has property (P) and $J(W_n(\bar{\alpha};\mu)) = J(\bar{\alpha};\mu)$. For this space we have the following result.

Theorem 5. *Let f and g be two oscillating polynomials from $W_n(\bar{\alpha};\mu)$ with zeros $\bar{x}$ and $\bar{y}$, respectively. Assume that the interlacing condition* (1) *holds true. Then the zeros $\{t_i\}_{i\in J(\bar{\alpha};\mu)}$ of f' and the zeros $\{\tau_i\}_{i\in J(\bar{\alpha};\mu)}$ of g' satisfy the inequalities:*

$$t_i \le \tau_i \le t_{i+1} \le \tau_{i+1}, \qquad \textit{for}\ \ i, i+1 \in J(\bar{\alpha};\mu). \tag{4}$$

Moreover, if $\bar{x} \neq \bar{y}$, then all the inequalities in (4) *are strict.*

The next corollary describes the interlacing properties of the zeros of linear combinations of Gaussian kernels.

Corollary 1. *Let f and g be two oscillating polynomials of the form $\sum_{i=0}^{n} b_i e^{-(x-\beta_i)^2}$ $(\beta_0 < \cdots < \beta_n)$ which have zeros $\bar{x}$ and $\bar{y}$, respectively. Assume that $\bar{x}$ and $\bar{y}$ interlace in the order* (1). *Then the zeros $\{t_i\}_{i=0}^{n}$ of f' and the zeros $\{\tau_i\}_{i=0}^{n}$ of g' interlace in the same order. Moreover, if $\bar{x} \neq \bar{y}$, then the interlacing is strict.*

2. Examples

We give some examples, which show that the assumptions in the theorems presented in Section 1, are essential.

Example 1. Let us consider the Tchebycheff system $\{\mu(x), \mu(x)x, \mu(x)x^2\}$ and the corresponding space

$$Q_2 := \{\mu(x)p(x) :\ p \in \pi_2\} = \operatorname{span}\{\mu(x), \mu(x)x, \mu(x)x^2\},$$

where

$$\mu(x) := 2 + \frac{\sin x}{\cosh(x/2)}.$$

We take the following polynomials from Q_2:

$$\begin{aligned} f(x) &= \mu(x)(x+1)(x-8),\\ g(x) &= \mu(x)(x+0.5)(x-9),\\ h(x) &= \mu(x)x(x-10). \end{aligned}$$

Clearly, the zeros of every of these polynomials interlace with the zeros of the others. We claim that:

- the only real zero of f' is $t_1 = 1.9262...$;
- the zeros of g' are $\tau_1 = 2.2120...$, $\tau_2 = 4.2752...$, and $\tau_3 = 4.9786...$;
- the zeros of h' are $\theta_1 = 2.6490...$, $\theta_2 = 3.5731...$, and $\theta_3 = 5.7762...$

The above statements can be verified in the following way. One proves analytically that each of f', g' and h' have no zeros outside the interval $[-10, 10]$, and then uses a computer to find numerically all their zeros in $[-10, 10]$ (see Figure 1).

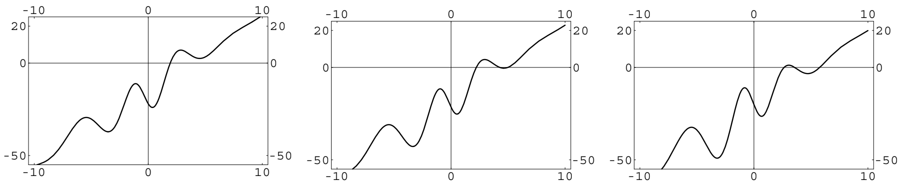

Figure 1. Graphs of f' (left), g' (center) and h' (right).

The space Q_2 does not satisfy property (P), since g' has three zeros in the interval between the zeros of g.

The conclusions of Theorem 1 fail to hold because of two reasons. First, f' and g' have different number of zeros, hence these zeros cannot interlace. On the other hand, g' and h' have the same number of zeros, but they satisfy the inequalities

$$\tau_1 < \theta_1 < \theta_2 < \tau_2 < \tau_3 < \theta_3.$$

Example 2. Let us consider the ET-space $V_n(\bar{\alpha})$, where $-1 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$. Clearly, $\bar{\alpha}$ does not satisfy the requirements of Theorem 2, concerning higher order derivatives. We shall prove that for every oscillating polynomial from $V_n(\bar{\alpha})$, its k-th derivative is different from zero for every $x \in \mathbb{R}$, provided k is sufficiently large and has the same parity as n. This shows that in the general case Theorem 2 is not true for k-th derivative.

Let $f(x) := \sum_{i=0}^{n} a_i e^{\alpha_i x}$ be an oscillating polynomial from $V_n(\bar{\alpha})$. This implies $a_i \neq 0$ for every $i = 0, \ldots, n$. In addition, $\operatorname{sign} a_0 = (-1)^n \operatorname{sign} a_n$, which follows from the asymptotic behavior of f for $x \to \pm\infty$. Since $\alpha_0 = -1$ and $\alpha_n = 1$, we have

$$f^{(k)}(x) = a_0(-1)^k e^{-x} + \sum_{i=1}^{n-1} a_i \alpha_i^k e^{\alpha_i x} + a_n e^x.$$

Furthermore, we shall suppose that $k \equiv n \pmod 2$. This implies

$$\operatorname{sign} a_n = (-1)^k \operatorname{sign} a_0. \tag{5}$$

Case 1. $x \geq 0$. We represent the k-th derivative of f in the form

$$f^{(k)}(x) = e^x[A_k(x) + B_k(x)],$$

where

$$A_k(x) := a_n + a_0(-1)^k e^{-2x}, \qquad B_k(x) := \sum_{i=1}^{n-1} a_i \alpha_i^k e^{(\alpha_i - 1)x}.$$

Since $\alpha_i < 1$ for $i = 1, \ldots, n-1$, we obtain the estimate

$$|B_k(x)| \leq \sum_{i=1}^{n-1} |a_i|\,|\alpha_i|^k e^{(\alpha_i-1)x} \leq \sum_{i=1}^{n-1} |a_i|\,|\alpha_i|^k$$

for every $x \geq 0$. Consequently, $B_k(x)$ tends to zero uniformly on $[0, \infty)$. It follows from (5) that $|A_k(x)| \geq |a_n|$ for every $x \geq 0$. Therefore $\operatorname{sign} f^{(k)}(x) = \operatorname{sign} A_k(x) = \operatorname{sign} a_n$ for every $x \geq 0$, provided k is sufficiently large.

Case 2. $x \leq 0$. Now we have

$$f^{(k)}(x) = e^{-x}[A_k(x) + B_k(x)],$$

where

$$A_k(x) := a_0(-1)^k + a_n e^{2x}, \qquad B_k(x) := \sum_{i=1}^{n-1} a_i \alpha_i^k e^{(\alpha_i + 1)x}.$$

It can be proved, as in Case 1, that $B_k(x)$ tends to zero uniformly on $(-\infty, 0]$ and $|A_k(x)| \geq |a_0|$ for every $x \leq 0$. This implies $\operatorname{sign} f^{(k)}(x) = (-1)^k \operatorname{sign} a_0 = \operatorname{sign} a_n$ for every $x \leq 0$, provided k is sufficiently large.

The conclusion is that if k is a sufficiently large natural number satisfying $k \equiv n \pmod 2$, then $f^{(k)}(x) \neq 0$ for every $x \in \mathbb{R}$.

Example 3. The claim of Theorem 3 concerning higher order derivatives is not true if we replace the assumption $\gamma_0 < \cdots < \gamma_n < 0$ by $0 < \gamma_0 < \cdots < \gamma_n$. (Compare with $V_n(\bar{\alpha})$.) Indeed, take $f(x) := x^{1/2} - x^{3/2} \in M_1(\frac{1}{2}, \frac{3}{2})$, then $f''(x) = -\frac{1}{4}\,x^{-3/2}(1 + 3x) \neq 0$ for every positive x. Consequently, f'' can be different from zero for every $x \in (0, \infty)$, and Theorem 3 cannot be extended to the second derivative.

Example 4. Here we show that the assumptions for the weight function μ from the definition of $W_n(\bar{\alpha}; \mu)$ are essential for the conclusion of Theorem 5.

Let us consider $\mu(x) = e^{x^2}$. It is easy to check that $\mu'(x)/\mu(x)$ increases on $\mathbb{R}$. We set $\bar{\alpha} = (1, 2, 3)$, and claim that the space $W_2(\bar{\alpha}; \mu)$ does not satisfy property (P). To prove this, we take the polynomial $f \in V_2(\bar{\alpha})$ which has zeros $x_1 = 1$, $x_2 = 3$, and let $F(x) := \mu(x) f(x) \in W_2(\bar{\alpha}; \mu)$. The zeros of F' are $t_1 = -0.2910...$, $t_2 = 0.6148...$, and $t_3 = 2.8814...$. The interval $(-\infty, x_1)$ contains two zeros of F', which proves the claim.

Moreover, the interlacing property for the derivatives does not hold true for $W_2(\bar{\alpha};\mu)$. Indeed, let $g \in V_2(\bar{\alpha})$ has zeros $y_1 = 2$, $y_2 = 4$. If $G(x) := \mu(x)g(x)$, then the zeros of F and G interlace. The zeros of G' are $\tau_1 = -0.4466...$, $\tau_2 = 1.7981...$, $\tau_3 = 3.9045....$ Clearly $\tau_1 < t_1 < t_2 < \tau_2 < t_3 < \tau_3$, so the interlacing property is not fulfilled.

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Lozko Milev

Department of Mathematics and Informatics
University of Sofia
Boulevard James Bourchier 5
1164 Sofia
BULGARIA
E-mail: milev@fmi.uni-sofia.bg

Nikola Naidenov

Department of Mathematics and Informatics
University of Sofia
Boulevard James Bourchier 5
1164 Sofia
BULGARIA
E-mail: nikola@fmi.uni-sofia.bg