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Markov Type Inequalities for Oscillating Exponential Polynomials*

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Given $\bar{\alpha} = (\alpha_0, \ldots, \alpha_n)$ with $0 < \alpha_0 < \cdots < \alpha_n$, let $V_n(\bar{\alpha})$ be the set of all exponential polynomials of the form $v(x) = \sum_{i=0}^{n} b_i e^{-\alpha_i x}$. We denote by $\mathcal{V}_n(\bar{\alpha})$ the subset of $V_n(\bar{\alpha})$ consisting of the polynomials $v(x)$ which have n simple zeros in $(0, \infty)$. Let $h_j(v)$, $j = 0, \ldots, n$, be the absolute values of the local extrema of a polynomial $v \in \mathcal{V}_n(\bar{\alpha})$. We prove that for every $v \in \mathcal{V}_n(\bar{\alpha})$, $k \in \mathbb{N}$ and every convex and strictly increasing on $[0, \infty)$ function ψ such that $\psi(0) = 0$, the quantities $h_j(v^{(k)})$, $j = 0, \ldots, n$, and the integral $\int_0^\infty \psi(|v^{(k)}(x)|)\, dx$ are increasing functions of $h_0(v), \ldots, h_n(v)$. As a corollary we obtain the following exact Markov-type inequality for polynomials from $\mathcal{V}_n(\bar{\alpha})$:

$$\|v^{(k)}\|_{L_p[0,\infty)} \le \|v_{n,*}^{(k)}\|_{L_p[0,\infty)} \, \|v\|_{C[0,\infty)}, \qquad 1 \le p < \infty, \quad k \in \mathbb{N},$$

where $v_{n,*}$ is the Chebyshev polynomial from $V_n(\bar{\alpha})$.

Keywords and Phrases: Markov inequality, exponential polynomials.

1. Introduction

Let us denote by π_n the set of all real algebraic polynomials of degree at most n. We shall say that a polynomial $f \in \pi_n$ is oscillating in the interval (a, b) if f has n simple zeros in (a, b). Let $\mathcal{P}_n$ be the subset of π_n, which consists of the oscillating polynomials in $(-1, 1)$.

Denote by Φ the class of all functions $\varphi \in C^1[0, \infty) \cap C^2(0, \infty)$, which are strictly increasing and convex on $[0, \infty)$.

In [4] Bojanov proved the following remarkable result.

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Theorem A. *Let $\varphi \in \Phi$ and $M > 0$. Then for every $f \in \pi_n$ such that $\|f\|_{C[-1,1]} \le M$, we have*

$$\int_{-1}^{1} \varphi(|f'(x)|)\,dx \le \int_{-1}^{1} \varphi(M\,|T_n'(x)|)\,dx, \tag{1}$$

where $T_n(x) = \cos(n \arccos x)$, $x \in [-1,1]$, is the n-th Chebyshev polynomial of the first kind. Moreover, the equality is attained if and only if $f = \pm M\,T_n$.

The above theorem generalizes two other famous Bojanov's results.

The particular case $\varphi(x) = \sqrt{1+x^2}$ was studied in [2], where Bojanov gave a proof of a longstanding conjecture of Erdős [12] about the "longest" polynomial.

Another important case is $\varphi(x) = x^p$, $1 \le p < \infty$, which leads to the following generalization of the inequality of A. Markov:

$$\|f'\|_{L_p[-1,1]} \le \|T_n'\|_{L_p[-1,1]}\,\|f\|_{C[-1,1]}, \qquad \text{for all } f \in \pi_n. \tag{2}$$

The equality in (2) is attained only for polynomials of the form $f = c\,T_n$, where c is a nonzero constant. This was proved directly in [3].

A problem, which was of special interest to Professor Bojanov, is to extend Theorem A to higher order derivatives. In its full generality the above problem is still open. An elegant solution for the class $\mathcal{P}_n$ was obtained by Bojanov and Rahman [10] as a consequence from the following monotonicity results.

Let us denote by $h_j(f)$, $j = 0,\dots,n$, the absolute values of the local extrema of a polynomial $f \in \mathcal{P}_n$, including these at the end points of the interval $[-1,1]$. According to a result of Davis [11] (see also [22], [13], [1] and [7]) the values $\{h_j(f)\}_{j=0}^{n}$ determine uniquely (up to multiplication by -1) the oscillating polynomial f. The following theorems were proved in a more general setting in [10] (see also [5] and [7]).

Theorem B. *If f and g are polynomials from $\mathcal{P}_n$ such that*

$$h_j(f) \le h_j(g), \qquad j = 0,\dots,n$$

then for every $k = 1,\dots,n$

$$h_j(f^{(k)}) \le h_j(g^{(k)}), \qquad j = 0,\dots,n-k. \tag{3}$$

Moreover, all the inequalities (3) are strict, unless $f = \pm g$.

Theorem C. *Let $\varphi \in \Phi$. Then for every $f \in \mathcal{P}_n$ and $k = 1,\dots,n$ the integral*

$$I(f) = \int_{-1}^{1} \varphi(|f^{(k)}(x)|)\,dx$$

is a strictly increasing function of $h_0(f),\dots,h_n(f)$.

The ideas and methods related to the proofs of Theorems A, B and C were applied and developed to solve various extremal problems for polynomials and other spaces of functions. For example, in [5] Bojanov obtained generalizations of the inequalities of I. Schur and P. Turán for algebraic polynomials. The paper [6] contains the trigonometric variants of Theorems B and C. Markov-type inequalities for weighted polynomials on infinite intervals were proved in [14, 16, 17, 18, 19]. The corresponding algorithmic aspects were studied in [15]. The paper [8] provides Markov-type inequalities for oscillating perfect splines and oscillating splines with fixed knots. Additional properties of oscillating polynomials were revealed in [23, 20, 9].

2. Statement of the Results

The aim of this paper is to establish results of the type of Theorems B and C for exponential polynomials. We begin with some definitions.

Given $\bar{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$ such that $0 < \alpha_0 < \cdots < \alpha_n$, we set

$$V_n(\bar{\alpha}) := \Big\{ v(x) = \sum_{i=0}^{n} b_i e^{-\alpha_i x} : (b_0, \ldots, b_n) \in \mathbb{R}^{n+1} \Big\}$$

and

$$\mathcal{V}_n(\bar{\alpha}) := \{ v \in V_n(\bar{\alpha}) : v \text{ has } n \text{ simple zeros in } (0, \infty) \}.$$

Furthermore, let

$$H := \{ \mathbf{h} = (h_0, \ldots, h_n) : h_0 > 0, \ldots, h_n > 0 \}.$$

Given a vector $\mathbf{h} \in H$, there exists a unique $v = v(\mathbf{h}; \cdot) \in V_n(\bar{\alpha})$ and a unique set of points $0 =: t_0(\mathbf{h}) < t_1(\mathbf{h}) < \cdots < t_n(\mathbf{h})$, such that

$$\begin{aligned} v(\mathbf{h}; t_k(\mathbf{h})) &= (-1)^{n-k} h_k, \qquad k = 0, \ldots, n, \\ v'(\mathbf{h}; t_k(\mathbf{h})) &= 0, \qquad k = 1, \ldots, n. \end{aligned} \tag{4}$$

This can be proved by using the method of Fitzgerald and Schumaker [13]. We shall denote by $h_i(v)$, $i = 0, \ldots, n$, the absolute values of the local extrema of a $v \in \mathcal{V}_n(\bar{\alpha})$ on $[0, \infty)$. Note that if $v \in \mathcal{V}_n(\bar{\alpha})$ then $v^{(k)} \in \mathcal{V}_n(\bar{\alpha})$ for all $k \in \mathbb{N}$.

Theorem 1. *If $v_1, v_2 \in \mathcal{V}_n(\bar{\alpha})$ and $h_j(v_1) \le h_j(v_2)$, $j = 0, \ldots, n$, then for every natural number k,*

$$h_i(v_1^{(k)}) \le h_i(v_2^{(k)}), \qquad i = 0, \ldots, n. \tag{5}$$

Moreover, if at least one of the inequalities $h_j(v_1) \le h_j(v_2)$, $j = 1, \ldots, n$ is strict, then inequalities (5) are strict for every $k \in \mathbb{N}$. If $h_0(v_1) < h_0(v_2)$ then $h_0(v_1^{(k)}) < h_0(v_2^{(k)})$, $k \in \mathbb{N}$.

Let $v_{n,*} := v((1,1,\dots,1);\,\cdot\,)$ be the Chebyshev polynomial from $V_n(\bar{\alpha})$. As an immediate consequence of Theorem 1, we obtain the following analog of V. Markov's inequality for $\mathcal{V}_n(\bar{\alpha})$.

Corollary 1. *For every $v \in \mathcal{V}_n(\bar{\alpha})$ and $k \in \mathbb{N}$, the inequality*

$$\|v^{(k)}\|_{C[0,\infty)} \le \|v_{n,*}^{(k)}\|_{C[0,\infty)}\, \|v\|_{C[0,\infty)} \tag{6}$$

holds true. The equality in (6) is attained if and only if $v = c\,v_{n,*}$, where c is a nonzero constant.

We denote by Ψ the class of all functions $\psi \in C^1[0,\infty) \cap C^2(0,\infty)$, which are strictly increasing and convex on $[0,\infty)$ and satisfy $\psi(0) = 0$.

Theorem 2. *Let $0 < \alpha_0 < \cdots < \alpha_n$ and $\psi \in \Psi$. Then for every $\mathbf{h} \in H$ and every natural number k, the integral*

$$I_k(\mathbf{h}) = \int_0^\infty \psi(|v^{(k)}(\mathbf{h};x)|)\,dx$$

is a strictly increasing function of $h_0,\dots,h_n$.

Setting $\psi(t) = t^p$ $(1 \le p < \infty)$ in Theorem 2, we obtain the following exact Markov-type inequality for polynomials from $\mathcal{V}_n(\bar{\alpha})$.

Corollary 2. *For every $v \in \mathcal{V}_n(\bar{\alpha})$, $k \in \mathbb{N}$ and $p \in [1,\infty)$, the inequality*

$$\|v^{(k)}\|_{L_p[0,\infty)} \le \|v_{n,*}^{(k)}\|_{L_p[0,\infty)}\, \|v\|_{C[0,\infty)} \tag{7}$$

holds true. The equality in (7) is attained if and only if $v = c\,v_{n,*}$, where c is a nonzero constant.

3. Proofs of Theorems 1 and 2

We proved recently in [21] that Markov's interlacing property holds true for various spases of exponential polynomials. The next result, which is a particular case of [21, Theorem 2], is crucial for the proof of Theorem 1.

Lemma 1. *Assume that the oscillating polynomials u and v from $V_n(\bar{\alpha})$ have zeros $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, respectively, which interlace:*

$$x_1 \le y_1 \le x_2 \le y_2 \le \cdots \le x_n \le y_n. \tag{8}$$

Then, the zeros $t_1 < \cdots < t_n$ of u' and the zeros $\tau_1 < \cdots < \tau_n$ of v' interlace too:

$$t_1 \le \tau_1 \le t_2 \le \tau_2 \le \cdots \le t_n \le \tau_n. \tag{9}$$

Moreover, if at least one inequality in (8) is strict, then all the inequalities in (9) are strict.

The following lemma provides a useful formula for the derivative of $v'(\mathbf{h};x)$ with respect to h_j, $j=0,\dots,n$. Recall that (see (4)) the zeros of $v'(\mathbf{h};x)$ are denoted by $t_1(\mathbf{h})<\dots<t_n(\mathbf{h})$.

Lemma 2. *We have*

$$\frac{\partial}{\partial h_j}v'(\mathbf{h};x)=(-1)^{n-j}\,g_j'(x),\qquad j=0,\dots,n, \tag{10}$$

where $g_j(x)=g_j(\mathbf{h};x)$ is the unique polynomial from $V_n(\bar{\alpha})$, which satisfies the conditions $g_j(t_i(\mathbf{h}))=\delta_{ij}$ for $i=0,\dots,n$.

Proof. We set $G_j(x):=\frac{\partial}{\partial h_j}v(\mathbf{h};x)$. Since $v(\mathbf{h};x)=\sum_{k=0}^n b_k(\mathbf{h})e^{-\alpha_k x}$, we have $G_j\in V_n(\bar{\alpha})$. Differentiating with respect to h_j the equality $v(\mathbf{h};t_i(\mathbf{h}))=(-1)^{n-i}\,h_i$, we get

$$\frac{\partial}{\partial h_j}v(\mathbf{h};t)\Big|_{t=t_i(\mathbf{h})}+v'(t_i(\mathbf{h}))\,\frac{\partial t_i(\mathbf{h})}{\partial h_j}=(-1)^{n-i}\,\delta_{ij}.$$

Note that if $i\ge 1$, then $v'(t_i(\mathbf{h}))=0$, while $\frac{\partial t_0(\mathbf{h})}{\partial h_j}=0$. This implies $G_j(t_i(\mathbf{h}))=(-1)^{n-i}\,\delta_{ij}$, $i=0,\dots,n$. Comparing with the definition of g_j, we conclude that $G_j(x)=(-1)^{n-j}\,g_j(x)$. In order to finish the proof, we differentiate the last equality with respect to x interchanging the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial h_j}$. $\square$

Proof of Theorem 1. Suppose first that $k=1$. Let $\xi(\mathbf{h})$ be an extremal point of $v'(\mathbf{h};x)$, i.e. $\xi(\mathbf{h})=0$ or $v''(\mathbf{h};\xi(\mathbf{h}))=0$. We shall show that $|v'(\mathbf{h};\xi(\mathbf{h}))|$ is a strictly increasing function of h_j, $j=1,\dots,n$ in the domain H. To this end, we shall prove that

$$\operatorname{sign}\frac{\partial}{\partial h_j}v'(\mathbf{h};\xi(\mathbf{h}))=\operatorname{sign}v'(\mathbf{h};\xi(\mathbf{h})),\qquad j=1,\dots,n. \tag{11}$$

There are two cases to be considered.

Case 1. $\xi(\mathbf{h})>0$, i.e. $v''(\xi(\mathbf{h}))=0$. Then we have

$$\begin{aligned}\frac{\partial}{\partial h_j}v'(\mathbf{h};\xi(\mathbf{h}))&=\frac{\partial}{\partial h_j}v'(\mathbf{h};x)\Big|_{x=\xi(\mathbf{h})}+v''(\xi(\mathbf{h}))\,\frac{\partial\xi(\mathbf{h})}{\partial h_j}\\&=(-1)^{n-j}\,g_j'(\xi(\mathbf{h})).\end{aligned} \tag{12}$$

(We have used (10) for the last equality.)

It is seen that the zeros of g_j and v' interlace, hence by Lemma 1, the zeros $\eta_1<\dots<\eta_n$ of g_j' and the zeros $\xi_1(\mathbf{h})<\dots<\xi_n(\mathbf{h})$ of v'' interlace strictly, namely

$$\eta_1<\xi_1(\mathbf{h})<\dots<\eta_n<\xi_n(\mathbf{h}).$$

We set for brevity $t_i:=t_i(\mathbf{h})$ and $\xi_i:=\xi_i(\mathbf{h})$ for all admissible values of i. Let us suppose that $\xi=\xi_i$ for some $i\in\{1,\dots,n\}$. Since $\xi_i\in(t_i,t_{i+1})$

and $v'(x) < 0$ for $x > t_n$, we have sign $v'(\xi) = (-1)^{n-i+1}$. On the other hand, sign $\{g_j'(x) : x \in (\eta_n, \infty)\} = (-1)^{n-j+1}$ and $\xi_i \in (\eta_i, \eta_{i+1})$, which implies sign $g_j'(\xi) = (-1)^{i+j+1}$. Consequently, making use of (12) we obtain sign $\frac{\partial}{\partial h_j} v'(\xi) = (-1)^{n-j}\,(-1)^{i+j+1} = \operatorname{sign} v'(\xi)$, which completes the proof of (11) in Case 1.

Case 2. $\xi(\mathbf{h}) = 0$. Similarly to (12), we get

$$\frac{\partial}{\partial h_j} v'(\mathbf{h}; 0) = (-1)^{n-j}\, g_j'(0).$$

Now we have sign $g_j'(0) = (-1)^{j-1}$, hence sign $\frac{\partial}{\partial h_j} v'(\mathbf{h}; 0) = (-1)^{n-1}$. Since v' changes its sign at the points $t_1, \dots, t_n$ and $v'(x) < 0$ for $x > t_n$, we have sign $v'(0) = (-1)^{n+1}$ and (11) is proved.

Next we shall investigate the dependence of $|v'(\mathbf{h}; \xi(\mathbf{h}))|$ on the parameter h_0, i.e. we shall determine the sign of $\frac{\partial}{\partial h_0} |v'(\mathbf{h}; \xi(\mathbf{h}))|$. Suppose first that $\xi = \xi_i$ for some $i \in \{1, \dots, n\}$. As in (12)

$$\frac{\partial}{\partial h_0} |v'(\mathbf{h}; \xi_i)| = \operatorname{sign}\,(v'(\mathbf{h}; \xi_i)) \cdot (-1)^n\, g_0'(\xi_i).$$

Furthermore, the zeros of g_0 and v' coincide, hence $g_0(x) = cv'(x)$ and $g_0'(\xi_i) = cv''(\xi_i) = 0$, i.e. sign$\frac{\partial}{\partial h_0} |v'(\mathbf{h}; \xi_i)| = 0$. It remains to consider the case $\xi(\mathbf{h}) = 0$. In this case we have

$$\frac{\partial}{\partial h_0} |v'(\mathbf{h}; 0)| = \operatorname{sign}\,(v'(\mathbf{h}; 0)) \cdot (-1)^n\, g_0'(0).$$

Using the fact that $g_0(0) = 1$, we get

$$\operatorname{sign} \frac{\partial}{\partial h_0} |v'(\mathbf{h}; 0)| = (-1)^{n+1}\,(-1)^n\,(-1) = 1. \tag{13}$$

The conclusion is that $|v'(\mathbf{h}; \xi(\mathbf{h}))|$ is a nondecreasing function of h_0. This finishes the proof of (5) for the first derivative.

The validity of (5) for $k \geq 2$ follows by induction.

Finally, let us suppose that $h_j(v_1) < h_j(v_2)$ for some $j \in \{1, \dots, n\}$. It follows from (11) that all the quantities $h_i(v')$, $i = 0, \dots, n$, are strictly increasing functions of h_j, which implies $h_i(v_1') < h_i(v_2')$ for every $i = 0, \dots, n$. By induction, we conclude that (5) are strict for every natural number k. Similarly, if $h_0(v_1) < h_0(v_2)$, then using (13) we obtain $h_0(v_1^{(k)}) < h_0(v_2^{(k)})$ for every $k \in \mathbb{N}$. The theorem is proved. □

Lemma 3. *Let f and g be polynomials from $\mathcal{V}_n(\bar{\alpha})$ with zeros $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, respectively. Suppose that*

$$x_1 \leq y_1 \leq \cdots \leq x_n \leq y_n. \tag{14}$$

Then $R(x) := f'(x)g(x) - f(x)g'(x)$ does not change its sign on $\mathbb{R}$.

Proof. *Step 1.* Let us suppose first that $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ interlace strictly, i.e.

$$x_1 < y_1 < \cdots < x_n < y_n. \tag{15}$$

We shall prove that $R(x) \neq 0$ for every $x \in \mathbb{R}$. Let us fix a point $\eta \in \mathbb{R}$. We consider $v(x) := f(x)g(\eta) - f(\eta)g(x) \in V_n(\bar{\alpha})$. Clearly $v(\eta) = 0$. Since $v'(\eta) = R(\eta)$, it is sufficient to prove that $v'(\eta) \neq 0$.

Case 1. $\eta = x_k$ for some $k \in \{1, \ldots, n\}$. Then $v(x) = f(x)g(\eta)$ and it has only simple zeros $x_1, \ldots, x_n$, hence $v'(\eta) = v'(x_k) \neq 0$.

Case 2. $\eta = y_k$ for some $k \in \{1, \ldots, n\}$. This case is completely analogous to Case 1.

Case 3. $\eta \neq \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$. Note first that $v(x) \not\equiv 0$. Otherwise, $f(x) = (f(\eta)/g(\eta))g(x)$ which contradicts (15). Let us assume the contrary, i.e. $v'(\eta) = 0$. Thus η is at least double zero of v. We shall show that v has $n+1$ zeros, counting the multiplicities. Since $V_n(\bar{\alpha})$ is a Chebyshev space, this will imply $v \equiv 0$, a contradiction.

We have $v(x_k) = -f(\eta)g(x_k)$. By (15), the numbers $\{v(x_k)\}_{k=1}^n$ have alternating signs, hence there exist at least $n-1$ points $z_1 < \cdots < z_{n-1}$ in (x_1, x_n), where v changes its sign. If $\eta \notin \{z_1, \ldots, z_{n-1}\}$ then v has $n+1$ zeros, as desired. Suppose now that $\eta = z_l$. Since z_l has to be at least triple zero of v, we conclude again that v has $n+1$ zeros.

Step 2. Now we shall consider the general case, where there can be equalities in (14). We introduce the points

$$t_k(\epsilon) = \begin{cases} y_k, & \text{if } x_k < y_k < x_{k+1}, \\ y_k + \epsilon, & \text{if } x_k = y_k, \\ y_k - \epsilon, & \text{if } y_k = x_{k+1}. \end{cases}$$

We have $x_1 < t_1(\epsilon) < \cdots < x_n < t_n(\epsilon)$, provided $\epsilon > 0$ is sufficiently small. Let $g_\epsilon \in V_n(\bar{\alpha})$ be the unique polynomial from $V_n(\bar{\alpha})$ which satisfies the conditions:

$$\begin{aligned} &g_\epsilon(t_k(\epsilon)) = 0, \qquad k = 1, \ldots, n, \\ &g_\epsilon(t^*) = g(t^*), \end{aligned}$$

where t^* is an arbitrary point, different from $\{y_k\}_{k=1}^n$. Applying Step 1 to f and g_ϵ, we conclude that

$$R_\epsilon(x) := f'(x)g_\epsilon(x) - f(x)g'_\epsilon(x)$$

does not vanish on $\mathbb{R}$. In order to determine the sign of $R_\epsilon(x)$, we consider $R_\epsilon(x_n)$. It is not difficult to see that sign $f'(x_n) = \sigma$ and sign $g_\epsilon(x_n) = -\delta$, where $\sigma :=$ sign $\{f(x) : x \to \infty\}$ and $\delta :=$ sign $\{g(x) : x \to \infty\}$. Hence sign $R_\epsilon(x) = -\sigma\delta$ for every sufficiently small ϵ.

It follows from the definition of g_ϵ that $g_\epsilon(x) \to g(x)$ and $g'_\epsilon(x) \to g'(x)$ as $\epsilon \to 0$, for every $x \in \mathbb{R}$. Therefore $R_\epsilon(x) \to R(x)$ as $\epsilon \to 0$, which implies that sign $R(x)$ is equal to $-\sigma\delta$ or 0. Lemma 3 is proved. □

Lemma 4. *Let $\psi \in C^1[0,\infty) \cap C^2(0,\infty)$ be a convex and increasing on $[0,\infty)$ function, such that $\psi(0) = \psi'(0) = 0$. Then the integral*

$$I(\mathbf{h}) := \int_0^\infty \psi(|v'(\mathbf{h};x)|)\,dx$$

is an increasing function of every argument h_j, $j = 0,\dots,n$, in the domain H. Moreover, if ψ is strictly increasing, then $I(\mathbf{h})$ is strictly increasing, too.

Proof. We fix an index $j \in \{0,\dots,n\}$. Differentiating I with respect to h_j we get

$$\frac{\partial I}{\partial h_j} = \int_0^\infty \psi'(|v'(\mathbf{h};x)|)\,\mathrm{sign}\,v'(\mathbf{h};x)\,\frac{\partial}{\partial h_j} v'(\mathbf{h};x)\,dx. \tag{16}$$

According to Lemma 2,

$$\frac{\partial}{\partial h_j} v'(\mathbf{h};x) = (-1)^{n-j}\,g_j'(x), \tag{17}$$

where $g_j(x) = g_j(\mathbf{h};x)$ is the unique polynomial from $V_n(\bar{\alpha})$, which satisfies the conditions $g_j(t_i) = \delta_{ij}$ for $i = 0,\dots,n$.

Substituting (17) in (16), we obtain

$$\frac{\partial I}{\partial h_j} = \int_0^\infty \chi_j(x)\,dx, \tag{18}$$

where

$$\chi_j(x) := (-1)^{n-j}\,\psi'(|v'(\mathbf{h};x)|)\,\mathrm{sign}\,v'(\mathbf{h};x)\,g_j'(x).$$

The condition $\psi'(0) = 0$ implies that χ_j is a continuous function. Next we introduce the set $E_j(\delta) := \mathbb{R}^+ \setminus (t_j - \delta, t_j + \delta)$. If $I_j(\delta) := \int_{E_j(\delta)} \chi_j(x)\,dx$ then by the continuity of χ_j

$$\lim_{\delta\to 0} I_j(\delta) = \int_0^\infty \chi_j(x)\,dx. \tag{19}$$

We transform $I_j(\delta)$ as follows:

$$\begin{aligned}
I_j(\delta) &= \int_{E_j(\delta)} (-1)^{n-j}\,\psi'(|v'(x)|)\,\mathrm{sign}\,v'(x)\Big\{v'(x)\,\frac{g_j(x)}{v'(x)}\Big\}'\,dx\\
&= \int_{E_j(\delta)} (-1)^{n-j}\,\psi'(|v'(x)|)\,\mathrm{sign}\,v'(x)\Big\{v''(x)\frac{g_j(x)}{v'(x)} + v'(x)\Big(\frac{g_j(x)}{v'(x)}\Big)'\Big\}dx\\
&= \int_{E_j(\delta)} (-1)^{n-j}\,\frac{g_j(x)}{v'(x)}\,d\psi(|v'(x)|)\\
&\quad + \int_{E_j(\delta)} (-1)^{n-j}\,\psi'(|v'(x)|)\,|v'(x)|\Big(\frac{g_j(x)}{v'(x)}\Big)'dx\\
&=: A_j(\delta) + B_j(\delta).
\end{aligned}$$

Let us suppose that $j \geq 1$. We integrate by parts $A_j(\delta)$ and obtain

$$A_j(\delta) = C_j(\delta) - \int_{E_j(\delta)} (-1)^{n-j}\, \psi(|v'(x)|) \Big(\frac{g_j(x)}{v'(x)}\Big)' \, dx, \tag{20}$$

where

$$C_j(\delta) := (-1)^{n-j} \Big[\frac{g_j(x)}{v'(x)}\, \psi(|v'(x)|) \Big|_0^{t_j-\delta} + \frac{g_j(x)}{v'(x)}\, \psi(|v'(x)|) \Big|_{t_j+\delta}^{\infty} \Big].$$

Bringing together $B_j(\delta)$ and the integral in (20), we get

$$\begin{aligned} I_j(\delta) = {} & C_j(\delta) \\ & + \int_{E_j(\delta)} (-1)^{n-j} \big[\psi'(|v'(x)|)|v'(x)| - \psi(|v'(x)|)\big] \Big(\frac{g_j(x)}{v'(x)}\Big)' \, dx. \end{aligned} \tag{21}$$

From convexity of ψ and conditions $\psi(0) = \psi'(0) = 0$ we infer that the term in the square brackets in the last integral is nonnegative for every x. We set

$$H(x) := (-1)^{n-j} \Big(\frac{g_j(x)}{v'(x)}\Big)' = (-1)^{n-j}\, \frac{h(x)}{(v'(x))^2},$$

where $h(x) := g_j'(x)v'(x) - g_j(x)v''(x)$. We shall show that $H(x)$ is nonnegative for every $x \in E_j(\delta)$. It is seen that the zeros of $g_j(x)$ and $v'(x)$ interlace. According to Lemma 3, $h(x)$ does not change its sign on $\mathbb{R}$. But $\operatorname{sign} v''(t_j) = (-1)^{n-j+1}$, hence $\operatorname{sign} h(t_j) = (-1)^{n-j}$. This implies $H(x) \geq 0$ for $x \in E_j(\delta)$.

Note that $g_j(0) = 0$ and $\lim_{t\to 0} \frac{\psi(t)}{t} = \psi'(0) = 0$. We also have $v'(t_j) = 0$ and $\lim_{x\to\infty} v'(x) = 0$. Therefore $\lim_{\delta\to 0} C_j(\delta) = 0$. Letting $\delta \to 0$ in (21), by using (18) and (19), we obtain

$$\frac{\partial I}{\partial h_j} = \int_0^\infty \big[\psi'(|v'(x)|)|v'(x)| - \psi(|v'(x)|)\big] H(x)\, dx \geq 0. \tag{22}$$

It remains to prove that if $\psi(t)$ is strictly increasing, then (22) holds true as a strict inequality. Since $v(x) \not\equiv 0$ and $H(x) \not\equiv 0$, it is sufficient to show that $f(t) := \psi'(t)t - \psi(t) > 0$ for every $t > 0$. Indeed, $f(0) = 0$ and $f'(t) \geq 0$ for every $t \geq 0$, hence $f(t) \geq 0$ for $t > 0$. Suppose that there exists a point $t_0 > 0$ such that $f(t_0) = 0$. Then $f(t) \equiv 0$ on $(0, t_0)$, which implies $\psi''(t) = 0$ for every $t \in (0, t_0)$. But $\psi(0) = \psi'(0) = 0$, hence $\psi(t) \equiv 0$ on $(0, t_0)$, which contradicts the strict monotonicity of ψ. This completes the case $j \geq 1$.

Suppose now that $j = 0$. It follows from (16) and (17) that

$$\frac{\partial I}{\partial h_0} = \int_0^\infty \psi'(|v'(x)|) \operatorname{sign} v'(x) \cdot (-1)^n\, g_0'(x)\, dx.$$

On the other hand, the zeros of g_0 and v' coincide, hence $g_0(x) = cv'(x)$. The condition $g_0(0) = 1$ gives $c = 1/v'(0) = (-1)^{n+1}/|v'(0)|$. Therefore

$$\begin{aligned}\frac{\partial I}{\partial h_0} &= -\frac{1}{|v'(0)|}\int_0^\infty \psi'(|v'(x)|)\,\mathrm{sign}\,v'(x)\cdot v''(x)\,dx \\ &= -\frac{1}{|v'(0)|}\,\psi(|v'(x)|)\Big|_0^\infty = \frac{\psi(|v'(0)|)}{|v'(0)|} \ge 0.\end{aligned}$$

Note that $v'(0) \ne 0$ since v is an oscillating polynomial. Consequently, $\frac{\partial I}{\partial h_0} > 0$ provided $\psi(t)$ is a strictly increasing function. □

Lemma 5. *If $\psi \in \Psi$, then the integral $I(\mathbf{h})$ is a strictly increasing function of every argument h_j, $j = 0, \ldots, n$, in the domain H.*

Proof. Let us suppose first that $\psi(t) = t$. Recall that $0 =: t_0 < t_1 < \cdots < t_n$ are the extremal points of $v(\mathbf{h}; x)$. Let us set $t_{n+1} := \infty$. In this case $I(\mathbf{h})$ can be computed in explicit form as follows:

$$I = \int_0^\infty |v'(\mathbf{h}; x)|\,dx = \sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} |v'(\mathbf{h}; x)|\,dx = h_0 + 2\sum_{k=1}^{n} h_k.$$

Clearly, this is a strictly increasing function of $h_0, \ldots, h_n$.

Next we consider the general case, i.e. ψ is an arbitrary function from Ψ. Since $\psi(0) = 0$, $\psi(t)$ can be represented as

$$\psi(t) = \psi'(0)t + \tilde{\psi}(t),$$

where $\tilde{\psi}(t) := \psi(t) - \psi'(0)t$ is an increasing and convex function, such that $\tilde{\psi}(0) = \tilde{\psi}'(0) = 0$. Consequently,

$$\begin{aligned}\int_0^\infty \psi(|v'(\mathbf{h}; x)|)\,dx &= \psi'(0)\int_0^\infty |v'(\mathbf{h}; x)|\,dx + \int_0^\infty \tilde{\psi}(|v'(\mathbf{h}; x)|)\,dx \\ &= \psi'(0)\Big(h_0 + 2\sum_{k=1}^{n} h_k\Big) + \int_0^\infty \tilde{\psi}(|v'(\mathbf{h}; x)|)\,dx.\end{aligned}$$

If $\psi'(0) = 0$, then $\tilde{\psi}(t) = \psi(t)$ is strictly increasing and the statement follows from Lemma 4. Otherwise, the first summand is strictly increasing, while according to Lemma 4, the second summand is increasing. Therefore $I(\mathbf{h})$ is strictly increasing. Lemma 5 is proved. □

Proof of Theorem 2. For a fixed $j \in \{0, \ldots, n\}$, let $\mathbf{h}^{(1)} = (h_0^{(1)}, \ldots, h_n^{(1)})$ and $\mathbf{h}^{(2)} = (h_0^{(2)}, \ldots, h_n^{(2)})$ be two vectors from H, whose components satisfy the conditions: $h_j^{(1)} < h_j^{(2)}$ and $h_i^{(1)} = h_i^{(2)}$ for all $i \ne j$. For $k \ge 2$ Theorem 1 gives

$$h_i(v^{(k-1)}(\mathbf{h}^{(1)}; \cdot)) \le h_i(v^{(k-1)}(\mathbf{h}^{(2)}; \cdot)), \qquad i = 0, \ldots, n. \tag{23}$$

Moreover, at least the inequality (23) for $i = j$ is strict. Note that the same is true for $k = 1$ according to the assumptions for $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$.

Applying Lemma 5 for $v^{(k-1)}(\mathbf{h}^{(1)};\,\cdot\,)$ and $v^{(k-1)}(\mathbf{h}^{(2)};\,\cdot\,)$, we conclude that $I_k(\mathbf{h}^{(1)}) < I_k(\mathbf{h}^{(2)})$. Theorem 2 is proved. □

References

[1] B. D. Bojanov, A generalization of Chebyshev polynomials, *J. Approx. Theory* **26** (1979), 293–300.

[2] B. D. Bojanov, Proof of a conjecture of Erdős about the longest polynomial, *Proc. Amer. Math. Soc.* **84** (1982), 99–103.

[3] B. D. Bojanov, An extension of the Markov inequality, *J. Approx. Theory* **35** (1982), 181–190.

[4] B. D. Bojanov, An extremal problem for polynomials, *in* "Constructive Function Theory'81", pp. 229–233, BAN, Sofia, 1983.

[5] B. Bojanov, Polynomial inequalities, *in* "Open Problems in Approximation Theory" (B. Bojanov, Ed.), pp. 25–42, SCT Publishing, Singapore, 1994.

[6] B. D. Bojanov, Turan's inequalities for trigonometric polynomials, *J. London Math. Soc.* **53** (1996), 539–550.

[7] B. Bojanov, Markov-type inequalities for polynomials and splines, *in* "Approximation Theory X: Abstract and Classical Analysis" (C. K. Chui, L. L. Schumaker, and J. Stökler, Eds.), pp. 31–90, Vanderbilt University Press, Nashville, TN, 2002.

[8] B. Bojanov and N. Naidenov, Exact Markov-type inequalities for oscillating perfect splines, *Constr. Approx.* **18** (2002), 37–59.

[9] B. D. Bojanov and N. G. Naidenov, On oscillating polynomials, *J. Approx. Theory* **162** (2010), 1766–1787.

[10] B. D. Bojanov and Q. I. Rahman, On certain extremal problems for polynomials, *J. Math. Anal. Appl.* **189** (1995), 781–800.

[11] C. Davis, Problem 4714, *Amer. Math. Monthly* **63** (1956), 729; Solution, *Amer. Math. Monthly* **64** (1957), 679–680.

[12] P. Erdős, An extremum-problem concerning trigonometric polynomials, *Acta Sci. Math. (Szeged)* **9** (1939), 113–115.

[13] C. H. Fitzgerald and L. L. Schumaker, A differential equation approach to interpolation at extremal points, *J. Analyse Math.* **22** (1969), 117–134.

[14] L. Milev, Weighted polynomial inequalities on infinite intervals, *East J. Approx.* **5** (1999), 449–465.

[15] L. Milev, Numerical computation of the Markov factors for the systems of polynomials with the Hermite and Laguerre weights, *in* "Numerical Methods and Applications" (T. Boyanov, S. Dimova, K. Georgiev, and G. Nikolov, Eds.), pp. 386–393, Lecture Notes in Computer Science, Vol. 4310, Springer-Verlag, Berlin Heidelberg, 2007.

[16] L. Milev and N. Naidenov, Markov's inequalities in integral norm for oscillating weighted polynomials, *in* "Approximation Theory: A Volume Dedicated to Borislav Bojanov", (D. K. Dimitrov, G. Nikolov, and R. Uluchev, Eds.), pp. 176–185, Marin Drinov Academic Publishing House, Sofia, 2004.

[17] L. Milev and N. Naidenov, An extension of the Markov inequality for the Laguerre weight, *East J. Approx.* **11** (2005), 109–118.

[18] L. Milev and N. Naidenov, Exact Markov inequalities for the Hermite and Laguerre weights, *J. Approx. Theory* **138** (2006), 87–96.

[19] L. Milev and N. Naidenov, An extension of the Markov inequality for the Hermite weight, *East J. Approx.* **13** (2007), 77–103.

[20] L. Milev and N. Naidenov, Monotonicity of the zeros and the extremal points of certain oscillating polynomials, *C. R. Acad. Bulg. Sci.* **60** (2007), 111–116.

[21] L. Milev and N. Naidenov, Markov interlacing property for exponential polynomials, *J. Math. Anal. Appl.* **367** (2010), 669–676.

[22] J. Mycielski and S. Paszckowski, A generalization of Chebyshev polynomials, *Bull. Acad. Polonaise Sci. Série Math. Astr. et Phys.* **8** (1960), 433–438.

[23] N. Naidenov, Estimates for the derivatives of oscillating polynomials, *East J. Approx.* **11** (2005), 301–336.

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