Markov Type Inequalities for Oscillating Exponential Polynomials

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Given $\bar{\alpha} = (\alpha_0, \ldots, \alpha_n)$ with $0 < \alpha_0 < \cdots < \alpha_n$, let $V_n(\bar{\alpha})$ be the set of all exponential polynomials of the form $v(x) = \sum_{i=0}^{n} b_i e^{-\alpha_i x}$. We denote by $V_n(\bar{\alpha})$ the subset of $V_n(\bar{\alpha})$ consisting of the polynomials $v(x)$ which have $n$ simple zeros in $(0, \infty)$. Let $h_j(v)$, $j = 0, \ldots, n$, be the absolute values of the local extrema of a polynomial $v \in V_n(\bar{\alpha})$. We prove that for every $v \in V_n(\bar{\alpha})$, $k \in \mathbb{N}$ and every convex and strictly increasing on $[0, \infty)$ function $\psi$ such that $\psi(0) = 0$, the quantities $h_j(v^{(k)})$, $j = 0, \ldots, n$, and the integral $\int_{0}^{\infty} \psi(|v^{(k)}(x)|) \, dx$ are increasing functions of $h_0(v), \ldots, h_n(v)$. As a corollary we obtain the following exact Markov-type inequality for polynomials from $V_n(\bar{\alpha})$:

$$\|v^{(k)}\|_{L_p[0,\infty)} \leq \|v_{n,\ast}\|_{L_p[0,\infty)} \|v\|_{C[0,\infty)}, \quad 1 \leq p < \infty, \quad k \in \mathbb{N},$$

where $v_{n,\ast}$ is the Chebyshev polynomial from $V_n(\bar{\alpha})$.

Keywords and Phrases: Markov inequality, exponential polynomials.

1. Introduction

Let us denote by $\pi_n$ the set of all real algebraic polynomials of degree at most $n$. We shall say that a polynomial $f \in \pi_n$ is oscillating in the interval $(a, b)$ if $f$ has $n$ simple zeros in $(a, b)$. Let $P_n$ be the subset of $\pi_n$, which consists of the oscillating polynomials in $(-1, 1)$.

Denote by $\Phi$ the class of all functions $\varphi \in C^4[0, \infty) \cap C^2(0, \infty)$, which are strictly increasing and convex on $[0, \infty)$.

In [4] Bojanov proved the following remarkable result.

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**Theorem A.** Let $\varphi \in \Phi$ and $M > 0$. Then for every $f \in \pi_n$ such that $\|f\|_{C[-1,1]} \leq M$, we have

$$\int_{-1}^{1} \varphi(|f'(x)|) \, dx \leq \int_{-1}^{1} \varphi(M |T'_n(x)|) \, dx,$$

(1)

where $T_n(x) = \cos(n \arccos x)$, $x \in [-1,1]$, is the $n$-th Chebyshev polynomial of the first kind. Moreover, the equality is attained if and only if $f = \pm MT_n$.

The above theorem generalizes two other famous Bojanov’s results.

The particular case $\varphi(x) = \sqrt{1 + x^2}$ was studied in [2], where Bojanov gave a proof of a longstanding conjecture of Erdős [12] about the “longest” polynomial.

Another important case is $\varphi(x) = x^p$, $1 \leq p < \infty$, which leads to the following generalization of the inequality of A. Markov:

$$\|f''\|_{L_p[-1,1]} \leq \|T'_n\|_{L_p[-1,1]} \|f\|_{C[-1,1]}, \quad \text{for all } f \in \pi_n.$$

(2)

The equality in (2) is attained only for polynomials of the form $f = c T_n$, where $c$ is a nonzero constant. This was proved directly in [3].

A problem, which was of special interest to Professor Bojanov, is to extend Theorem A to higher order derivatives. In its full generality the above problem is still open. An elegant solution for the class $P_n$ was obtained by Bojanov and Rahman [10] as a consequence from the following monotonicity results.

Let us denote by $h_j(f)$, $j = 0, \ldots, n$, the absolute values of the local extrema of a polynomial $f \in P_n$, including these at the end points of the interval $[-1,1]$. According to a result of Davis [11] (see also [22], [13], [1] and [7]) the values $\{h_j(f)\}_{j=0}^n$ determine uniquely (up to multiplication by $-1$) the oscillating polynomial $f$. The following theorems were proved in a more general setting in [10] (see also [5] and [7]).

**Theorem B.** If $f$ and $g$ are polynomials from $P_n$ such that

$$h_j(f) \leq h_j(g), \quad j = 0, \ldots, n$$

then for every $k = 1, \ldots, n$

$$h_j(f^{(k)}) \leq h_j(g^{(k)}), \quad j = 0, \ldots, n-k.$$ (3)

Moreover, all the inequalities (3) are strict, unless $f = \pm g$.

**Theorem C.** Let $\varphi \in \Phi$. Then for every $f \in P_n$ and $k = 1, \ldots, n$ the integral

$$I(f) = \int_{-1}^{1} \varphi(|f^{(k)}(x)|) \, dx$$

is a strictly increasing function of $h_0(f), \ldots, h_n(f)$.

The ideas and methods related to the proofs of Theorems A, B and C were applied and developed to solve various extremal problems for polynomials and
other spaces of functions. For example, in [5] Bojanov obtained generalizations of the inequalities of I. Schur and P. Turán for algebraic polynomials. The paper [6] contains the trigonometric variants of Theorems B and C. Markov-type inequalities for weighted polynomials on infinite intervals were proved in [14, 16, 17, 18, 19]. The corresponding algorithmic aspects were studied in [15]. The paper [8] provides Markov-type inequalities for oscillating perfect splines and oscillating splines with fixed knots. Additional properties of oscillating polynomials were revealed in [23, 20, 9].

2. Statement of the Results

The aim of this paper is to establish results of the type of Theorems B and C for exponential polynomials. We begin with some definitions.

Given \( \bar{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \) such that \( 0 < \alpha_0 < \cdots < \alpha_n \), we set

\[
V_n(\bar{\alpha}) := \left\{ v(x) = \sum_{i=0}^{n} b_i e^{-\alpha_i x} : (b_0, \ldots, b_n) \in \mathbb{R}^{n+1} \right\}
\]

and

\[
V_n(\bar{\alpha}) := \{ v \in V_n(\bar{\alpha}) : v \text{ has } n \text{ simple zeros in } (0, \infty) \}.
\]

Furthermore, let

\[
H := \{ h = (h_0, \ldots, h_n) : h_0 > 0, \ldots, h_n > 0 \}.
\]

Given a vector \( h \in H \), there exists a unique \( v = v(h; \cdot) \in V_n(\bar{\alpha}) \) and a unique set of points \( 0 =: t_0(h) < t_1(h) < \cdots < t_n(h) \), such that

\[
v(h; t_k(h)) = (-1)^{n-k} h_k, \quad k = 0, \ldots, n, \\
v'(h; t_k(h)) = 0, \quad k = 1, \ldots, n.
\] (4)

This can be proved by using the method of Fitzgerald and Schumaker [13]. We shall denote by \( h_i(v) \), \( i = 0, \ldots, n \), the absolute values of the local extrema of a \( v \in V_n(\bar{\alpha}) \) on \( [0, \infty) \). Note that if \( v \in V_n(\bar{\alpha}) \) then \( v^{(k)} \in V_n(\bar{\alpha}) \) for all \( k \in \mathbb{N} \).

**Theorem 1.** If \( v_1, v_2 \in V_n(\bar{\alpha}) \) and \( h_j(v_1) \leq h_j(v_2) \), \( j = 0, \ldots, n \), then for every natural number \( k \),

\[
h_i(v^{(k)}_1) \leq h_i(v^{(k)}_2), \quad i = 0, \ldots, n.
\] (5)

Moreover, if at least one of the inequalities \( h_j(v_1) \leq h_j(v_2) \), \( j = 1, \ldots, n \) is strict, then inequalities (5) are strict for every \( k \in \mathbb{N} \). If \( h_0(v_1) < h_0(v_2) \) then \( h_0(v^{(k)}_1) < h_0(v^{(k)}_2) \), \( k \in \mathbb{N} \).
Let $v_{n,*} := v((1, 1, \ldots, 1); \cdot)$ be the Chebyshev polynomial from $V_n(\bar{\alpha})$. As an immediate consequence of Theorem 1, we obtain the following analog of V. Markov’s inequality for $V_n(\bar{\alpha})$.

Corollary 1. For every $v \in V_n(\bar{\alpha})$ and $k \in \mathbb{N}$, the inequality
\[
\|v^{(k)}\|_{C[0, \infty)} \leq \|v_{n,*}\|_{C[0, \infty)} \|v\|_{C[0, \infty)}
\]  
holds true. The equality in (6) is attained if and only if $v = cv_{n,*}$, where $c$ is a nonzero constant.

We denote by $\Psi$ the class of all functions $\psi \in C^1[0, \infty) \cap C^2(0, \infty)$, which are strictly increasing and convex on $[0, \infty)$ and satisfy $\psi(0) = 0$.

Theorem 2. Let $0 < \alpha_0 < \cdots < \alpha_n$ and $\psi \in \Psi$. Then for every $h \in H$ and every natural number $k$, the integral
\[
I_k(h) = \int_0^{\infty} \psi(|v^{(k)}(h; x)|) \, dx
\]
is a strictly increasing function of $h_0, \ldots, h_n$.

Setting $\psi(t) = t^p$ ($1 \leq p < \infty$) in Theorem 2, we obtain the following exact Markov-type inequality for polynomials from $V_n(\bar{\alpha})$.

Corollary 2. For every $v \in V_n(\bar{\alpha})$, $k \in \mathbb{N}$ and $p \in [1, \infty)$, the inequality
\[
\|v^{(k)}\|_{L_p[0, \infty)} \leq \|v_{n,*}\|_{L_p[0, \infty)} \|v\|_{C[0, \infty)}
\]  
holds true. The equality in (7) is attained if and only if $v = cv_{n,*}$, where $c$ is a nonzero constant.

3. Proofs of Theorems 1 and 2

We proved recently in [21] that Markov’s interlacing property holds true for various spaces of exponential polynomials. The next result, which is a particular case of [21, Theorem 2], is crucial for the proof of Theorem 1.

Lemma 1. Assume that the oscillating polynomials $u$ and $v$ from $V_n(\bar{\alpha})$ have zeros $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, respectively, which interlace:
\[
x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_n \leq y_n.
\]  
Then, the zeros $t_1 < \cdots < t_n$ of $u'$ and the zeros $\tau_1 < \cdots < \tau_n$ of $v'$ interlace too:
\[
t_1 \leq \tau_1 \leq t_2 \leq \tau_2 \leq \cdots \leq t_n \leq \tau_n.
\]  
Moreover, if at least one inequality in (8) is strict, then all the inequalities in (9) are strict.
The following lemma provides a useful formula for the derivative of $v'(h; x)$ with respect to $h_j$, $j = 0, \ldots, n$. Recall that (see (4)) the zeros of $v'(h; x)$ are denoted by $t_1(h) < \cdots < t_n(h)$.

**Lemma 2.** We have

$$
\frac{\partial}{\partial h_j} v'(h; x) = (-1)^{n-j} g_j'(x), \quad j = 0, \ldots, n,
$$

where $g_j(x) = g_j(h; x)$ is the unique polynomial from $V_n(\alpha)$, which satisfies the conditions $g_j(t_i(h)) = \delta_{ij}$ for $i = 0, \ldots, n$.

**Proof.** We set $G_j(x) := \frac{\partial}{\partial h_j} v(h; x)$. Since $v(h; x) = \sum_{k=0}^n b_k(h)e^{-\alpha_k x}$, we have $G_j \in V_n(\alpha)$. Differentiating with respect to $h_j$ the equality $v(h; t_i(h)) = (-1)^{n-i} h_i$, we get

$$
\frac{\partial}{\partial h_j} v(h; t_i(h)) = \frac{\partial t_i(h)}{\partial h_j} = (-1)^{n-i} \delta_{ij}.
$$

Note that if $i \geq 1$, then $v'(t_i(h)) = 0$, while $\frac{\partial t_i(h)}{\partial h_j} = 0$. This implies $G_j(t_i(h)) = (-1)^{n-i} \delta_{ij}$, $i = 0, \ldots, n$. Comparing with the definition of $g_j$, we conclude that $G_j(x) = (-1)^{n-i} g_j(x)$. In order to finish the proof, we differentiate the last equality with respect to $x$ interchanging the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial h_j}$.

**Proof of Theorem 1.** Suppose first that $k = 1$. Let $\xi(h)$ be an extremal point of $v'(h; x)$, i.e. $\xi(h) = 0$ or $v'(h; \xi(h)) = 0$. We shall show that $|v'(h; \xi(h))|$ is a strictly increasing function of $h_j$, $j = 1, \ldots, n$ in the domain $H$. To this end, we shall prove that

$$
\text{sign} \left( \frac{\partial}{\partial h_j} v'(h; \xi(h)) \right) = \text{sign} \left( v'(h; \xi(h)) \right), \quad j = 1, \ldots, n.
$$

There are two cases to be considered.

**Case 1.** $\xi(h) > 0$, i.e. $v''(\xi(h)) = 0$. Then we have

$$
\frac{\partial}{\partial h_j} v'(h; \xi(h)) = \left. \frac{\partial}{\partial h_j} v'(h; x) \right|_{x=\xi(h)} + v''(\xi(h)) \frac{\partial \xi(h)}{\partial h_j} = (-1)^{n-j} g_j'(\xi(h)).
$$

(We have used (10) for the last equality.)

It is seen that the zeros of $g_j$ and $v'$ interlace, hence by Lemma 1, the zeros $\eta_1 < \cdots < \eta_n$ of $g_j'$ and the zeros $\xi_1(h) < \cdots < \xi_n(h)$ of $v''$ interlace strictly, namely

$$
\eta_1 < \xi_1(h) < \cdots < \eta_n < \xi_n(h).
$$

We set for brevity $t_i := t_i(h)$ and $\xi_i := \xi_i(h)$ for all admissible values of $i$. Let us suppose that $\xi = \xi_i$ for some $i \in \{1, \ldots, n\}$. Since $\xi_i \in (t_i, t_{i+1})$
and \( v'(x) < 0 \) for \( x > t_n \), we have \( \text{sign} \, v'(\xi) = (-1)^{n-i+1} \). On the other hand, \( \text{sign} \, \{g_j'(x) : x \in (\eta_n, \infty)\} = (-1)^{n-j+1} \) and \( \xi_i \in (\eta_i, \eta_{i+1}) \), which implies \( \text{sign} \, g_j'(\xi) = (-1)^{i+j+1} \). Consequently, making use of (12) we obtain \( \text{sign} \, \frac{\partial}{\partial h_j} v'(\xi) = (-1)^{n-j} (-1)^{i+j+1} = \text{sign} \, v'(\xi) \), which completes the proof of (11) in Case 1.

**Case 2.** \( \xi(h) = 0 \). Similarly to (12), we get

\[
\frac{\partial}{\partial h_j} v'(h; 0) = (-1)^{n-j} g_j'(0).
\]

Now we have \( g_j'(0) = (-1)^{i+1} \), hence \( \text{sign} \, \frac{\partial}{\partial h_j} v'(h; 0) = (-1)^{n-1} \). Since \( v' \) changes its sign at the points \( t_1, \ldots, t_n \) and \( v'(x) < 0 \) for \( x > t_n \), we have \( \text{sign} \, v'(0) = (-1)^{n+1} \) and (11) is proved.

Next we shall investigate the dependence of \( \text{sign} \, v'(h; \xi(h)) \) on the parameter \( h_0 \), i.e. we shall determine the sign of \( \frac{\partial}{\partial h_0} v'(h; \xi(h)) \). Suppose first that \( \xi = \xi_i \) for some \( i \in \{1, \ldots, n\} \). As in (12)

\[
\frac{\partial}{\partial h_0} |v'(h; \xi_i)| = \text{sign} \, v'(h; \xi_i) \cdot (-1)^n g_0'(\xi_i).
\]

Furthermore, the zeros of \( g_0 \) and \( v' \) coincide, hence \( g_0(x) = cv'(x) \) and \( g_0' (\xi_i) = cv'(\xi_i) = 0 \), i.e. \( \text{sign} \, \frac{\partial}{\partial h_0} |v'(h; \xi_i)| = 0 \). It remains to consider the case \( \xi(h) = 0 \).

In this case we have

\[
\frac{\partial}{\partial h_0} |v'(h; 0)| = \text{sign} \, v'(h; 0) \cdot (-1)^n g_0'(0).
\]

Using the fact that \( g_0(0) = 1 \), we get

\[
\text{sign} \, \frac{\partial}{\partial h_0} |v'(h; 0)| = (-1)^{n+1} (-1)^n = 1.
\]

The conclusion is that \( |v'(h; \xi(h))| \) is a nondecreasing function of \( h_0 \). This finishes the proof of (5) for the first derivative.

The validity of (5) for \( k \geq 2 \) follows by induction.

Finally, let us suppose that \( h_j(v_1) < h_j(v_2) \) for some \( j \in \{1, \ldots, n\} \). It follows from (11) that all the quantities \( h_i(v') \), \( i = 0, \ldots, n \), are strictly increasing functions of \( h_j \), which implies \( h_i(v'_1) < h_i(v'_2) \) for every \( i = 0, \ldots, n \). By induction, we conclude that (5) are strict for every natural number \( k \). Similarly, if \( h_0(v_1) < h_0(v_2) \), then using (13) we obtain \( h_0(v^{(k)}_1) < h_0(v^{(k)}_2) \) for every \( k \in \mathbb{N} \). The theorem is proved.

**Lemma 3.** Let \( f \) and \( g \) be polynomials from \( \mathcal{V}_n \) with zeros \( x_1 < \cdots < x_n \) and \( y_1 < \cdots < y_n \), respectively. Suppose that

\[
x_1 \leq y_1 \leq \cdots \leq x_n \leq y_n.
\]

Then \( R(x) := f'(x)g(x) - f(x)g'(x) \) does not change its sign on \( \mathbb{R} \).
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We consider strictly, i.e. we shall prove that \( v \) does not vanish on \( \mathbb{R} \) where \( \sigma \) to Case 1.

We have \( x_1 < y_1 < \cdots < x_n < y_n \). \( \text{(15)} \)

We shall prove that \( R(x) \neq 0 \) for every \( x \in \mathbb{R} \). Let us fix a point \( \eta \in \mathbb{R} \).

Let us suppose first that \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) interlace strictly, i.e.

\[
x_1 < y_1 < \cdots < x_n < y_n.
\]

We consider \( v(x) := f(x)g(\eta) - f(\eta)g(x) \in V_n(\bar{\alpha}) \). Clearly \( v(\eta) = 0 \). Since \( v'(\eta) = R(\eta) \), it is sufficient to prove that \( v'(\eta) \neq 0 \).

Case 1. \( \eta = x_k \) for some \( k \in \{1, \ldots, n\} \). Then \( v(x) = f(x)g(\eta) \) and it has only simple zeros \( x_1, \ldots, x_n \), hence \( v'(\eta) = v'(x_k) \neq 0 \).

Case 2. \( \eta = y_k \) for some \( k \in \{1, \ldots, n\} \). This case is completely analogous to Case 1.

Case 3. \( \eta \neq \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \). Note first that \( v(x) \neq 0 \). Otherwise, \( f(x) = (f(\eta)/g(\eta))g(x) \) which contradicts (15). Let us assume the contrary, i.e. \( v'(\eta) = 0 \). Thus \( \eta \) is at least double zero of \( v \). We shall show that \( v \) has \( n+1 \) zeros, counting the multiplicities. Since \( V_n(\bar{\alpha}) \) is a Chebyshev space, this will imply \( v \equiv 0 \), a contradiction.

We have \( v(x_k) = -f(\eta)g(x_k) \). By (15), the numbers \( \{v(x_k)\}_{k=1}^n \) have alternating signs, hence there exist at least \( n-1 \) points \( z_1 < \cdots < z_{n-1} \) in \( (x_1, x_n) \), where \( v \) changes its sign. If \( \eta \notin \{z_1, \ldots, z_{n-1}\} \) then \( v \) has \( n+1 \) zeros, as desired. Suppose now that \( \eta = z_l \). Since \( z_l \) has to be at least triple zero of \( v \), we conclude again that \( v \) has \( n+1 \) zeros.

Step 2. Now we shall consider the general case, where there can be equalities in (14). We introduce the points

\[
t_k(\epsilon) = \begin{cases} y_k, & \text{if } x_k < y_k < x_{k+1}, \\ y_k + \epsilon, & \text{if } x_k = y_k, \\ y_k - \epsilon, & \text{if } y_k = x_{k+1}. \end{cases}
\]

We have \( x_1 < t_1(\epsilon) < \cdots < x_n < t_n(\epsilon) \), provided \( \epsilon > 0 \) is sufficiently small. Let \( g_{\epsilon} \in V_n(\bar{\alpha}) \) be the unique polynomial from \( V_n(\bar{\alpha}) \) which satisfies the conditions:

\[
g_{\epsilon}(t_k(\epsilon)) = 0, \quad k = 1, \ldots, n,
\]

\[
g_{\epsilon}(t^*) = g(t^*),
\]

where \( t^* \) is an arbitrary point, different from \( \{y_k\}_{k=1}^n \). Applying Step 1 to \( f \) and \( g_{\epsilon} \), we conclude that

\[
R_{\epsilon}(x) := f'(x)g_{\epsilon}(x) - f(x)g_{\epsilon}'(x)
\]

does not vanish on \( \mathbb{R} \). In order to determine the sign of \( R_{\epsilon}(x) \), we consider \( R_{\epsilon}(x_0) \). It is not difficult to see that sign \( f'(x_0) = \sigma \) and sign \( g_{\epsilon}(x_0) = -\delta \), where \( \sigma := \text{sign} \{f(x) : x \to \infty\} \) and \( \delta := \text{sign} \{g(x) : x \to \infty\} \). Hence sign \( R_{\epsilon}(x) = -\sigma \delta \) for every sufficiently small \( \epsilon \).

It follows from the definition of \( g_{\epsilon} \) that \( g_{\epsilon}(x) \to g(x) \) and \( g_{\epsilon}'(x) \to g'(x) \) as \( \epsilon \to 0 \), for every \( x \in \mathbb{R} \). Therefore \( R_{\epsilon}(x) \to R(x) \) as \( \epsilon \to 0 \), which implies that sign \( R(x) \) is equal to \(-\sigma \delta \) or 0. Lemma 3 is proved. \( \square \)
Lemma 4. Let $\psi \in C^1[0, \infty) \cap C^2(0, \infty)$ be a convex and increasing on $[0, \infty)$ function, such that $\psi(0) = \psi'(0) = 0$. Then the integral

$$I(h) := \int_0^\infty \psi(|v'(h; x)|) \, dx$$

is an increasing function of every argument $h_j$, $j = 0, \ldots, n$, in the domain $H$. Moreover, if $\psi$ is strictly increasing, then $I(h)$ is strictly increasing, too.

Proof. We fix an index $j \in \{0, \ldots, n\}$. Differentiating $I$ with respect to $h_j$ we get

$$\frac{\partial I}{\partial h_j} = \int_0^\infty \psi'(|v'(h; x)|) \text{ sign } v'(h; x) \frac{\partial}{\partial h_j} v'(h; x) \, dx. \tag{16}$$

According to Lemma 2,

$$\frac{\partial}{\partial h_j} v'(h; x) = (-1)^{n-j} g_j'(x), \tag{17}$$

where $g_j(x) = g_j(h; x)$ is the unique polynomial from $V_n(\bar{a})$, which satisfies the conditions $g_j(t_i) = \delta_{ij}$ for $i = 0, \ldots, n$.

Substituting (17) in (16), we obtain

$$\frac{\partial I}{\partial h_j} = \int_0^\infty x_j(x) \, dx, \tag{18}$$

where

$$x_j(x) := (-1)^{n-j} \psi'(|v'(h; x)|) \text{ sign } v'(h; x) g_j'(x).$$

The condition $\psi'(0) = 0$ implies that $\chi_j$ is a continuous function. Next we introduce the set $E_j(\delta) := \mathbb{R}^+ \setminus (t_j - \delta, t_j + \delta)$. If $I_j(\delta) := \int_{E_j(\delta)} \chi_j(x) \, dx$ then by the continuity of $\chi_j$

$$\lim_{\delta \to 0} I_j(\delta) = \int_0^\infty \chi_j(x) \, dx. \tag{19}$$

We transform $I_j(\delta)$ as follows:

$$I_j(\delta) = \int_{E_j(\delta)} (-1)^{n-j} \psi'(|v'(x)|) \text{ sign } v'(x) \left\{ v'(x) \frac{g_j(x)}{v'(x)} \right\}' \, dx$$

$$= \int_{E_j(\delta)} (-1)^{n-j} \psi'(|v'(x)|) \text{ sign } v'(x) \left\{ v''(x) \frac{g_j(x)}{v'(x)} + v'(x) \left( \frac{g_j(x)}{v'(x)} \right)' \right\} \, dx$$

$$= \int_{E_j(\delta)} (-1)^{n-j} \frac{g_j(x)}{v'(x)} \, d\psi(|v'(x)|)$$

$$+ \int_{E_j(\delta)} (-1)^{n-j} \psi'(|v'(x)|)|v'(x)| \left( \frac{g_j(x)}{v'(x)} \right)' \, dx$$

$$=: A_j(\delta) + B_j(\delta).$$
Let us suppose that \( j \geq 1 \). We integrate by parts \( A_j(\delta) \) and obtain
\[
A_j(\delta) = C_j(\delta) - \int_{E_j(\delta)} (-1)^{n-j} \psi(|v'(x)|) \left( \frac{g_j(x)}{v'(x)} \right)' \, dx,
\]
where
\[
C_j(\delta) := (-1)^{n-j} \left[ \frac{g_j(x)}{v'(x)} \psi(|v'(x)|) \bigg|_{t_j}^{t_j+\delta} + \frac{g_j(x)}{v'(x)} \psi(|v'(x)|) \bigg|_{t_j}^{\infty} \right].
\]
Bringing together \( B_j(\delta) \) and the integral in (20), we get
\[
I_j(\delta) = C_j(\delta) + \int_{E_j(\delta)} (-1)^{n-j} \left[ \psi(|v'(x)|)|v'(x)| - \psi(|v'(x)|) \right] \left( \frac{g_j(x)}{v'(x)} \right)' \, dx.
\]
From convexity of \( \psi \) and conditions \( \psi(0) = \psi'(0) = 0 \) we infer that the term in the square brackets in the last integral is nonnegative for every \( x \). We set
\[
H(x) := (-1)^{n-j} \left( \frac{g_j(x)}{v'(x)} \right)' = (-1)^{n-j} \frac{h(x)}{(v'(x))^2},
\]
where \( h(x) := g'_j(x)v'(x) - g_j(x)v''(x) \). We shall show that \( H(x) \) is nonnegative for every \( x \in E_j(\delta) \). It is seen that the zeros of \( g_j(x) \) and \( v'(x) \) interlace. According to Lemma 3, \( h(x) \) does not change its sign on \( \mathbb{R} \). But sign \( v''(t_j) = (-1)^{n-j+1} \), hence sign \( h(t_j) = (-1)^{n-j} \). This implies \( H(x) \geq 0 \) for \( x \in E_j(\delta) \).

Note that \( g_j(0) = 0 \) and \( \lim_{t \to 0} \frac{v'(t)}{t^2} = \psi'(0) = 0 \). We also have \( v''(t_j) = 0 \) and \( \lim_{t \to \infty} v''(x) = 0 \). Therefore \( \lim_{\delta \to 0} C_j(\delta) = 0 \). Letting \( \delta \to 0 \) in (21), by using (18) and (19), we obtain
\[
\frac{\partial I}{\partial h_j} = \int_0^\infty \left[ \psi'(|v'(x)|)|v'(x)| - \psi(|v'(x)|) \right] H(x) \, dx \geq 0.
\]
It remains to prove that if \( \psi(t) \) is strictly increasing, then (22) holds true as a strict inequality. Since \( v(x) \neq 0 \) and \( H(x) \neq 0 \), it is sufficient to show that \( f(t) := \psi'(t)t - \psi(t) > 0 \) for every \( t > 0 \). Indeed, \( f(0) = 0 \) and \( f'(t) \geq 0 \) for every \( t \geq 0 \), hence \( f(t) \geq 0 \) for \( t > 0 \). Suppose that there exists a point \( t_0 > 0 \) such that \( f(t_0) = 0 \). Then \( f(t) \equiv 0 \) on \((0, t_0)\), which implies \( \psi''(t) = 0 \) for every \( t \in (0, t_0) \). But \( \psi(0) = \psi'(0) = 0 \), hence \( \psi(t) \equiv 0 \) on \((0, t_0)\), which contradicts the strict monotonicity of \( \psi \). This completes the case \( j \geq 1 \).

Suppose now that \( j = 0 \). It follows from (16) and (17) that
\[
\frac{\partial I}{\partial h_0} = \int_0^\infty \psi'(|v'(x)|) \text{sign} v'(x) \cdot (-1)^n g_0(x) \, dx.
\]
On the other hand, the zeros of \( g_0 \) and \( v' \) coincide, hence \( g_0(x) = cv'(x) \). The condition \( g_0(0) = 1 \) gives \( c = 1/|v'(0)| = (-1)^{n+1}/|v'(0)| \). Therefore

\[
\frac{\partial I}{\partial h_0} = -\frac{1}{|v'(0)|} \int_0^\infty \psi'(|v'(x)|) \text{sign} v'(x) \cdot v''(x) \, dx \\
= -\frac{1}{|v'(0)|} \psi(|v'(x)|)|_0^\infty = \frac{\psi(|v'(0)|)}{|v'(0)|} \geq 0.
\]

Note that \( v'(0) \neq 0 \) since \( v \) is an oscillating polynomial. Consequently, \( \frac{\partial I}{\partial h_0} > 0 \) provided \( \psi(t) \) is a strictly increasing function.

**Lemma 5.** If \( \psi \in \Psi \), then the integral \( I(h) \) is a strictly increasing function of every argument \( h_j, j = 0, \ldots, n \), in the domain \( H \).

**Proof.** Let us suppose first that \( \psi(t) = t \). Recall that \( 0 =: t_0 < t_1 < \cdots < t_n \) are the extremal points of \( v(h; x) \). Let us set \( t_{n+1} := \infty \). In this case \( I(h) \) can be computed in explicit form as follows:

\[
I = \int_0^\infty |v'(h; x)| \, dx = \sum_{k=0}^n \int_{t_k}^{t_{k+1}} |v'(h; x)| \, dx = h_0 + 2 \sum_{k=1}^n h_k.
\]

Clearly, this is a strictly increasing function of \( h_0, \ldots, h_n \).

Next we consider the general case, i.e. \( \psi \) is an arbitrary function from \( \Psi \). Since \( \psi(0) = 0 \), \( \psi(t) \) can be represented as

\[
\psi(t) = \psi'(0)t + \tilde{\psi}(t),
\]

where \( \tilde{\psi}(t) := \psi(t) - \psi'(0)t \) is an increasing and convex function, such that \( \tilde{\psi}(0) = \tilde{\psi}'(0) = 0 \). Consequently,

\[
\int_0^\infty \psi(|v'(h; x)|) \, dx = \psi'(0) \int_0^\infty |v'(h; x)| \, dx + \int_0^\infty \tilde{\psi}(|v'(h; x)|) \, dx \\
= \psi'(0) \left( h_0 + 2 \sum_{k=1}^n h_k \right) + \int_0^\infty \tilde{\psi}(|v'(h; x)|) \, dx.
\]

If \( \psi'(0) = 0 \), then \( \tilde{\psi}(t) = \psi(t) \) is strictly increasing and the statement follows from Lemma 4. Otherwise, the first summand is strictly increasing, while according to Lemma 4, the second summand is increasing. Therefore \( I(h) \) is strictly increasing. Lemma 5 is proved.

**Proof of Theorem 2.** For a fixed \( j \in \{0, \ldots, n\} \), let \( h^{(1)} = (h_0^{(1)}, \ldots, h_n^{(1)}) \) and \( h^{(2)} = (h_0^{(2)}, \ldots, h_n^{(2)}) \) be two vectors from \( H \), whose components satisfy the conditions: \( h_j^{(1)} < h_j^{(2)} \) and \( h_i^{(1)} = h_i^{(2)} \) for all \( i \neq j \). For \( k \geq 2 \) Theorem 1 gives

\[
h_i(v^{(k-1)}(h^{(1)}), \cdot) \leq h_i(v^{(k-1)}(h^{(2)}), \cdot), \quad i = 0, \ldots, n.
\]  

(23)
Moreover, at least the inequality (23) for \( i = j \) is strict. Note that the same is true for \( k = 1 \) according to the assumptions for \( h^{(1)} \) and \( h^{(2)} \).

Applying Lemma 5 for \( v^{(k-1)}(h^{(1)}; \cdot) \) and \( v^{(k-1)}(h^{(2)}; \cdot) \), we conclude that \( I_k(h^{(1)}) < I_k(h^{(2)}) \). Theorem 2 is proved. \( \square \)

References


Markov Type Inequalities for Oscillating Exponential Polynomials


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