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Explicit Weighted Min-Max Polynomials on the Disc

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We consider the problem of finding a best weighted uniform approximation on the unit disc to the bivariate monomials $x^n y^m$, $n, m \in \mathbb{N}$, by polynomials in two variables of lower degree with real coefficients. We give explicit solutions to this problem for two types of weight functions, continuous and positive on the unit disc.

1. Introduction

Let $\mathcal{D} := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ denote the unit disc and let Π_N^2, $N \in \mathbb{N}$, denote the set of polynomials in two variables with real coefficients of total degree at most N, i.e.,

$$\Pi_N^2 := \{P : P(x,y) = \sum_{0 \leq k+l \leq N} a_{k,l} x^k y^l,\ a_{k,l} \in \mathbb{R}\}.$$

Let w be a continuous function on $\mathcal{D}$ such that $w(x,y) > 0$ for all $(x,y) \in \mathcal{D}$. As usual, we define the weighted uniform norm on the set of continuous functions on $\mathcal{D}$ by $\|f\|_w := \max_{(x,y)\in\mathcal{D}} |f(x,y)w(x,y)|$. For the bivariate monomial $x^n y^m$, $n, m \in \mathbb{N}_0$, $n + m \geq 1$, we look for a polynomial $p^* \in \Pi_{n+m-1}^2$ such that $x^n y^m - p^*(x,y)$ has the least weighted uniform norm on $\mathcal{D}$, that is,

$$\|x^n y^m - p^*\|_w := \inf_{p \in \Pi_{n+m-1}^2} \|x^n y^m - p\|_w. \tag{1}$$

We call $x^n y^m - p^*(x,y)$ *a min-max polynomial on $\mathcal{D}$ with respect to the weight function w* (or simply *a min-max polynomial on $\mathcal{D}$* if $w(x,y) = 1$ for all $(x,y) \in \mathcal{D}$), and *minimum deviation* the value

$$E_{n+m-1}(x^n y^m; w) := \|x^n y^m - p^*\|_w.$$

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Concerning the existence, uniqueness and characterization of min-max polynomials, see [6, 11, 12], where a more general setting than the one of this problem is considered. In what follows we will use the characterization of min-max polynomials in terms of *extremal signatures*, see e.g. [11, Theorem 2], and also [5, 12] for some examples of extremal signatures in several dimensions.

In the non-weighted case, i.e., $w(x,y)=1$ for all $(x,y)\in\mathcal{D}$, the first solution to this problem was given by Gearhart in [4]. He has shown that

$$G_{n,m}(x,y):=\frac{1}{2^{n+m}}\big(U_n(x)U_m(y)+U_{n-2}(x)U_{m-2}(y)\big)=x^ny^m+e(x,y), \quad (2)$$

where $e(x,y)\in\Pi^2_{n+m-1}$, is a min-max polynomial on the disc and the minimum deviation is

$$E_{n+m-1}(x^ny^m)=\frac{1}{2^{n+m-1}}. \quad (3)$$

As usual, U_n denotes the Chebyshev polynomial of the second kind defined by

$$U_n(x):=\frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \qquad n\in\mathbb{N}_0, \quad x\in[-1,1],$$

and $U_{-1}(x)=0$, $U_{-2}(x)=-1$. A second family of min-max polynomials was introduced by Reimer by means of a generating function, see [10, Theorem 2]. Other min-max polynomials for special degrees of monomials were given by Bojanov, Haußmann and Nikolov [2], Braß [3], Newman and Xu [9], where a connection between approximation problems on the disc and on the triangle $\Delta:=\{(x,y)\in\mathbb{R}^2 : 0\le x,y\le 1,\ x+y\le 1\}$ is used, see [2, Proposition 2]. Another class of min-max polynomials was considered recently in [7], see also Remark 1 below. The extremal signature corresponding to the min-max polynomials to the bivariate monomial is given by $2(n+m)$ points $(\cos\varphi_i,\sin\varphi_i)$ on the boundary of the disc with alternating sign. More precisely, the φ_i's are the zeros of $\sin(n+m)\varphi$ if m is even, respectively of $\cos(n+m)\varphi$ if m is odd, see [4].

In this paper we give explicitly min-max polynomials for the problem (1) for two types of weight functions:

1) $w(x,y)=1/(\rho_k(x)\rho_l(y))$, where $\rho_k(x)$ and $\rho_l(y)$ are monic polynomials in one variable with real coefficients, positive on the interval $[-1,1]$;

2) $w(x,y)=1/\sqrt{\prod_{i=1}^k(-2a_ix-2b_iy+a_i^2+b_i^2+1)}$, $k=1,2,3$, where the parameters a_i,b_i, $i=1,2,3$, satisfy some conditions, see Section 2 below.

Finally, we mention the paper [8] which deals with a complex analogue of the non-weighted problem (1). As a consequence of some results there, see Proposition 6 and the proof of Theorems 1 and 3, and the above mentioned connection between approximation on the unit disc and on the triangle, min-max polynomials on $\mathcal{D}$ for the monomials with respect to the weight function x^ky^l, $k,l\ge 0$, are obtained.

2. Main Results

In order to state our main result in the case of the first type of weight functions, let $k \in \mathbb{N}_0$ and $\rho_k(x) = \prod_{j=1}^{k}(x - a_j) = x^k + \cdots$ be a polynomial of degree k with real coefficients, positive on $[-1, 1]$. We denote

$$c_k = \prod_{j=1}^{k} |2z_j|, \qquad z_j = a_j - \sqrt{a_j^2 - 1}, \quad j = 1, \ldots, k, \tag{4}$$

where it is chosen the branch of the square root with $|z_j| < 1$, $j = 1, \ldots, k$. Furthermore, let $U_n(x; 1/\rho_k) = (2^n/c_k)x^n + \cdots$, $n \geq k$, denote the weighted Chebyshev polynomial of the second kind, defined by

$$\frac{U_n(\cos\varphi; 1/\rho_k)}{\rho_k(\cos\varphi)} = \frac{\sin((n + 1 - k)\varphi + \phi_1(\varphi))}{\sin\varphi}, \qquad \varphi \in [0, \pi], \tag{5}$$

where $\phi_1(\varphi)$ is such that

$$e^{i\phi_1(\varphi)} = \prod_{j=1}^{k} \frac{e^{i\varphi} - z_j}{1 - e^{i\varphi}\bar{z}_j}, \tag{6}$$

see e.g. [1, p. 249–254]. In a completely similar manner we introduce the polynomials $\rho_l(y)$, $l \in \mathbb{N}_0$, and the corresponding weighted Chebyshev polynomials of the second kind $U_m(y; 1/\rho_l)$, $m \geq l$.

Theorem 1. *Let $n, m \in \mathbb{N}$, $n \geq k + 2$, $m \geq l + 2$ and*

$$w(x, y) = \frac{1}{\rho_k(x)\rho_l(y)}.$$

Then the polynomial

$$\begin{aligned} P_{n,m}(x, y; w) &= \frac{c_k c_l}{2^{n+m}} \left[U_n(x; 1/\rho_k)U_m(y; 1/\rho_l) + U_{n-2}(x; 1/\rho_k)U_{m-2}(y; 1/\rho_l)\right] \\ &= x^n y^m + e(x, y), \end{aligned}$$

where $e(x, y) \in \Pi^2_{n+m-1}$, is a min-max polynomial on $\mathcal{D}$ with respect to the weight function w. The minimum deviation is $c_k c_l/2^{n+m-1}$.

In the following we state the results regarding the second class of weights.

Theorem 2. *Let $n, m \in \mathbb{N}$, $n, m \geq 3$, $a_i, b_i \in \mathbb{R}$, $i = 1, 2, 3$, be such that $a_1^2 + b_1^2 < 3 - 2\sqrt{2}$, $a_2^2 + b_2^2 < 3 - 2\sqrt{2}$, $a_3^2 + b_3^2 < 1$ and*

$$w_3(x, y) = \frac{1}{\sqrt{\prod_{i=1}^{3}(-2a_i x - 2b_i y + a_i^2 + b_i^2 + 1)}}.$$

Then the polynomial

$$\begin{aligned} P_{n,m}(x,y;w_3) &= \frac{1}{8}\,(x-a_1)(x-a_2)(x-a_3)G_{n-3,m}(x,y) \\ &\quad + \frac{1}{8}\big[(x-a_1)(x-a_2)(y-b_3) + (x-a_1)(y-b_2)(x-a_3) \\ &\qquad\qquad + (y-b_1)(x-a_2)(x-a_3)\big]G_{n-2,m-1}(x,y) \\ &\quad + \frac{1}{8}\big[(x-a_1)(y-b_2)(y-b_3) + (y-b_1)(x-a_2)(y-b_3) \\ &\qquad\qquad + (y-b_1)(y-b_2)(x-a_3)\big]G_{n-1,m-2}(x,y) \\ &\quad + \frac{1}{8}\,(y-b_1)(y-b_2)(y-b_3)G_{n,m-3}(x,y) \\ &= x^n y^m + e(x,y), \end{aligned}$$

where $e(x,y) \in \Pi^2_{n+m-1}$, *is a min-max polynomial on* $\mathcal{D}$ *with respect to the weight function* w_3. *The minimum deviation is* $1/2^{n+m-1}$.

The polynomial $P_{n,m}(x,y;w_3)$ defined in Theorem 2 is obtained as a result of a step-by-step construction. In the first two steps of this construction, the following polynomials are obtained, see Section 3 below. For $n,m \geq 1$ we determine

$$\begin{aligned} P_{n,m}(x,y;w_1) &= \frac{1}{2}\,(x-a_1)G_{n-1,m}(x,y) + \frac{1}{2}\,(y-b_1)G_{n,m-1}(x,y) \\ &= x^n y^m + e(x,y), \end{aligned} \tag{7}$$

where $e(x,y) \in \Pi^2_{n+m-1}$ and $a_1, b_1 \in \mathbb{R}$ satisfy $a_1^2 + b_1^2 < 1$. The polynomial $P_{n,m}(x,y;w_1)$ is a min-max polynomial on $\mathcal{D}$ with respect to the weight function

$$w_1(x,y) = \frac{1}{\sqrt{-2a_1x - 2b_1y + a_1^2 + b_1^2 + 1}}. \tag{8}$$

For $n,m \geq 2$ we find

$$\begin{aligned} P_{n,m}(x,y;w_2) &= \frac{1}{4}\,(x-a_1)(x-a_2)G_{n-2,m}(x,y) \\ &\quad + \frac{1}{4}\,[(x-a_1)(y-b_2) + (x-a_2)(y-b_1)]G_{n-1,m-1}(x,y) \\ &\quad + \frac{1}{4}\,(y-b_1)(y-b_2)G_{n,m-2}(x,y) \\ &= x^n y^m + e(x,y), \end{aligned} \tag{9}$$

where $e(x,y) \in \Pi^2_{n+m-1}$ and $a_1, b_1, a_2, b_2 \in \mathbb{R}$ are such that $a_1^2 + b_1^2 < 1$ and $a_2^2 + b_2^2 < 1$. The polynomial $P_{n,m}(x,y;w_2)$ is a min-max polynomial on $\mathcal{D}$ with respect to the weight function

$$w_2(x,y) = \frac{1}{\sqrt{\prod_{i=1}^{2}(-2a_ix - 2b_iy + a_i^2 + b_i^2 + 1)}}. \tag{10}$$

Remark 1. Setting the parameters a_i, b_i, $i = 1, 2, 3$, to zero in the above definitions of $P_{n,m}(x, y; w_1)$, $P_{n,m}(x, y; w_2)$, and $P_{n,m}(x, y; w_3)$ we obtain min-max polynomials in the non-weighted case. For $P_{n,m}(x, y; w_1)$ with $a_1 = b_1 = 0$, see also [7].

The min-max polynomials defined by Theorem 1 and relations (7) and (9) all satisfy quadratic equations on the disc $\mathcal{D}$, see respectively, Propositions 1, 5 and 6 below. In what follows we present two methods of generating new weighted min-max polynomials on the disc.

Theorem 3. *Suppose that $n, m \in \mathbb{N}$, $n \geq k + 3$ and $m \geq l + 2$. Let $w(x, y)$ and $P_{n,m}(x, y; w)$ be defined as in Theorem 1 and $c_{n,m} = c_k c_l / 2^{n+m-1}$. Furthermore, let $Q_{\nu,\mu}(x, y) = x^\nu y^\mu + \cdots$ be any min-max polynomial on $\mathcal{D}$ with $\nu, \mu \in \mathbb{N}_0$ and $\nu + \mu \geq 0$. Then the polynomial*

$$\Big(\frac{c_{n,m}}{w(x,y)}\Big)^{\nu+\mu} Q_{\nu,\mu}\big(c_{n,m}^{-1} P_{n,m}(x, y; w) w(x, y), c_{n,m}^{-1} P_{n-1,m+1}(x, y; w) w(x, y)\big)$$
$$= x^{n(\nu+\mu)-\mu} y^{m(\nu+\mu)+\mu} + e(x, y),$$

where $e(x, y) \in \Pi^2_{(\nu+\mu)(n+m)-1}$, is a min-max polynomial on $\mathcal{D}$ with respect to the weight function $w^{\nu+\mu}$. The minimum deviation is $c_{n,m}^{\nu+\mu}/2^{\nu+\mu-1}$.

Corollary 1. *Let $\nu, \mu, m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\nu + \mu \geq 0$, and let $Q_{\nu,\mu}(x, y) = x^\nu y^\mu + \cdots$ be any min-max polynomial on $\mathcal{D}$. Then*

$$\frac{1}{2^{(n+m-1)(\nu+\mu)}} Q_{\nu,\mu}\big(2^{n+m-1} G_{n,m}(x, y), 2^{n+m-1} G_{n-1,m+1}(x, y)\big)$$
$$= x^{n(\nu+\mu)-\mu} y^{m(\nu+\mu)+\mu} + e(x, y),$$

where $e(x, y) \in \Pi^2_{(n+m)(\nu+\mu)-1}$, is a min-max polynomial on $\mathcal{D}$. The minimum deviation is $1/2^{(n+m)(\nu+\mu)-1}$.

For the special cases $m = 0$ and $\mu = 0$, see [2, Theorem 1], respectively [4, Corollary 2.1].

Theorem 4. *Let $n_1 \in \mathbb{N}$, $m_1 \in \mathbb{N}_0$, and*

$$P_{n_1,m_1}(x, y; \tilde{w}_1) = x^{n_1} y^{m_1} + \cdots, \quad P_{n_1-1,m_1+1}(x, y; \tilde{w}_1) = x^{n_1-1} y^{m_1+1} + \cdots$$

be min-max polynomials with respect to the weight function $\tilde{w}_1(x, y)$, positive on $\mathcal{D}$, satisfying

$$[P_{n_1,m_1}(x, y; \tilde{w}_1)\tilde{w}_1(x, y)]^2 + [P_{n_1-1,m_1+1}(x, y; \tilde{w}_1)\tilde{w}_1(x, y)]^2$$
$$= c^2_{n_1,m_1} - (1 - x^2 - y^2)[\tilde{w}_1(x, y)]^2 q(x, y; \tilde{w}_1) \quad (11)$$

for all $(x,y) \in \mathcal{D}$, *where* $q(x,y;\tilde{w}_1) \in \Pi^2_{2(n_1+m_1-1)}$, $q(x,y;\tilde{w}_1) \geq 0$ *on* $\mathcal{D}$ *and* $c_{n_1,m_1} > 0$ *is a real constant depending on* n_1 *and* m_1 *with*

$$\begin{aligned} P_{n_1,m_1}(\cos\varphi,\sin\varphi;\tilde{w}_1)\tilde{w}_1(\cos\varphi,\sin\varphi) &= c_{n_1,m_1}\cos((n_1+m_1)\varphi+\varphi_1) \\ P_{n_1-1,m_1+1}(\cos\varphi,\sin\varphi;\tilde{w}_1)\tilde{w}_1(\cos\varphi,\sin\varphi) &= c_{n_1,m_1}\sin((n_1+m_1)\varphi+\varphi_1) \end{aligned} \tag{12}$$

for all $\varphi \in [0,2\pi)$, *where* $\varphi_1 \in [0,2\pi)$. *Completely analogously, we consider the polynomials*

$$P_{n_2+1,m_2-1}(x,y;\tilde{w}_2) = x^{n_2+1}y^{m_2-1} + \cdots, \quad P_{n_2,m_2}(x,y;\tilde{w}_2) = x^{n_2}y^{m_2} + \cdots,$$

$n_2 \in \mathbb{N}_0, m_2 \in \mathbb{N}$. *Then the polynomial*

$$\begin{aligned} \frac{1}{2}\big[P_{n_1,m_1}(x,y;\tilde{w}_1)P_{n_2,m_2}(x,y;\tilde{w}_2) + P_{n_1-1,m_1+1}(x,y;\tilde{w}_1)P_{n_2+1,m_2-1}(x,y;\tilde{w}_2)\big] \\ = x^{n_1+n_2}y^{m_1+m_2} + e(x,y), \end{aligned}$$

where $e(x,y) \in \Pi^2_{n_1+m_1+n_2+m_2-1}$, *is a min-max polynomial on* $\mathcal{D}$ *with respect to the weight function* $\tilde{w}_1\tilde{w}_2$. *The minimum deviation is* $c_{n_1,m_1}c_{n_2,m_2}/2$.

Corollary 2. *Let* $n_1, m_1, n_2, m_2 \in \mathbb{N}$, $n_1, m_2 \geq 3$, $n_2, m_1 \geq 2$ *and* $a_i, b_i \in \mathbb{R}$ *be such that* $a_i^2+b_i^2 < 3-2\sqrt{2}$, $i = 1,2,3,4$. *Let the polynomials* $P_{n_1,m_1}(x,y;\tilde{w}_1)$ *and* $P_{n_2,m_2}(x,y;\tilde{w}_2)$ *be of the form* (9) *with weight functions* $\tilde{w}_1(x,y)$ *and* $\tilde{w}_2(x,y)$ *as in* (10) *using parameters* a_i, b_i, $i = 1,2$, *and* a_i, b_i, $i = 3,4$, *respectively. Then the polynomial defined by Theorem 4 is a min-max polynomial on* $\mathcal{D}$ *with respect to the weight function* $1/\sqrt{\prod_{i=1}^4(-2a_ix - 2b_iy + a_i^2 + b_i^2 + 1)}$. *Moreover, the minimum deviation is* $1/2^{n_1+n_2+m_1+m_2-1}$.

Finally, we mention that any min-max polynomial to a monomial x^ny^m obtained above is characterized by the same type of extremal signature whose support consists of $2(n+m)$ points $(\cos\varphi_i, \sin\varphi_i)$ lying on the boundary of the unit disc with alternating sign. In the non-weighted case φ_i's are the zeros of $\sin((n+m)\varphi+\varphi_0)$ if m is even, and the zeros of $\cos((n+m)\varphi+\varphi_0)$ if m is odd, where $\varphi_0 \in [0,2\pi)$.

3. Proofs

Proof of Theorem 1. First we prove the inequality

$$|P_{n,m}(x,y;w)w(x,y)| \leq \frac{c_kc_l}{2^{n+m-1}} \tag{13}$$

for all $(x,y) \in \mathcal{D}$.

Let us take an arbitrary point in the interior of $\mathcal{D}$, e.g. $(x, y) = (\cos\varphi, \cos\theta)$, $\varphi, \theta \in (0, \pi)$, with $\cos^2\varphi + \cos^2\theta < 1$. Using (5) we easily get

$$\begin{aligned} &P_{n,m}(\cos\varphi, \cos\theta; w) w(\cos\varphi, \cos\theta) \\ &= \frac{c_k c_l}{2^{n+m-1}} \Big\{ \frac{\cos\varphi}{\sin\theta} \frac{\cos\theta}{\sin\varphi} \sin[(n-k)\varphi + \phi_1(\varphi)] \sin[(m-l)\theta + \phi_2(\theta)] \\ &\qquad + \cos[(n-k)\varphi + \phi_1(\varphi)] \cos[(m-l)\theta + \phi_2(\theta)] \Big\}. \end{aligned} \tag{14}$$

Since $\varphi, \theta \in (0, \pi)$ and $\cos^2\varphi + \cos^2\theta < 1$ we have $|\frac{\cos\varphi}{\sin\theta}| < 1$ and $|\frac{\cos\theta}{\sin\varphi}| < 1$. Hence, by (14) we obtain

$$\begin{aligned} &|P_{n,m}(\cos\varphi, \cos\theta; w) w(\cos\varphi, \cos\theta)| \\ &\qquad \le \frac{c_k c_l}{2^{n+m-1}} \big\{ |\sin[(n-k)\varphi + \phi_1(\varphi)]| \, |\sin[(m-l)\theta + \phi_2(\theta)]| \\ &\qquad\qquad + |\cos[(n-k)\varphi + \phi_1(\varphi)]| \, |\cos[(m-l)\theta + \phi_2(\theta)]| \big\}. \end{aligned}$$

Applying Schwarz's inequality, it follows that inequality (13) holds for all (x, y) in the interior of $\mathcal{D}$.

It remains to prove (13) for the points on $\partial\mathcal{D}$, i.e. the boundary of the unit disc. For any point $(x, y) = (\cos\varphi, \sin\varphi)$, $\varphi \in [0, 2\pi)$, combining (5), (6) and the fact that $\sin\varphi = \cos(\varphi - \frac{\pi}{2})$ yields

$$\begin{aligned} &P_{n,m}(\cos\varphi, \sin\varphi; w) w(\cos\varphi, \sin\varphi) \\ &= \begin{cases} L_{n,m,k,l} \cos\big[(n-k+m-l)\varphi + \phi_1(\varphi) + \phi_2(\varphi - \frac{\pi}{2})\big], & m-l \ \text{even}, \\ L_{n,m,k,l} \sin\big[(n-k+m-l)\varphi + \phi_1(\varphi) + \phi_2(\varphi - \frac{\pi}{2})\big], & m-l \ \text{odd}, \end{cases} \end{aligned} \tag{15}$$

where $L_{n,m,k,l} = (-1)^{\lfloor (m-l)/2 \rfloor} c_k c_l / 2^{n+m-1}$. Therefore inequality (13) holds also for all $(x, y) \in \partial\mathcal{D}$ and thus the proof of (13) is completed.

Furthermore, it follows from (15) that $|P_{n,m}(x, y; w) w(x, y)|$ attains the upper bound $c_k c_l / 2^{n+m-1}$ in (13) on the boundary of the unit disc at the points $(\cos\varphi_i, \sin\varphi_i)$ with alternating sign, where $\varphi_i \in [0, 2\pi)$ are the zeros of $\sin[(n-k+m-l)\varphi + \phi_1(\varphi) + \phi_2(\varphi - \pi/2)]$ if $m-l$ is even, and zeros of $\cos[(n-k+m-l)\varphi + \phi_1(\varphi) + \phi_2(\varphi - \pi/2)]$ if $m-l$ is odd. Next we show that these are precisely $2(n+m)$ points. By the Principle of the Argument, recall also (6) and (4), the argument of $e^{i\phi_1(\varphi)} e^{i\phi_2(\varphi - \pi/2)}$ has an increase of $2(k+l)\pi$ when φ increases from 0 to 2π. More precisely, $\phi_1(\varphi) + \phi_2(\varphi - \pi/2)$ increases from φ_0 to $\varphi_0 + 2(k+l)\pi$ when $\varphi \in [0, 2\pi)$, where $\varphi_0 = \phi_1(0) + \phi_2(-\pi/2)$ and therefore $(n-k+m-l)\varphi + \phi_1(\varphi) + \phi_2(\varphi - \pi/2)$ increases from φ_0 to $\varphi_0 + 2(n+m)\pi$ for $\varphi \in [0, 2\pi)$. Thus we get exactly $2(n+m)$ extreme points of $P_{n,m}(x, y; w) w(x, y)$ on $\partial\mathcal{D}$. The fact that these points form the support of an extremal signature with respect to Π^2_{n+m-1} follows by Shapiro's Theorem [12, Theorem 2]. In view of the characterization theorem in terms of extremal signatures [11, Theorem 2], the assertion is proved. $\square$

In the following proposition we show that the polynomials defined by Theorem 1 satisfy a quadratic equation on the disc.

Proposition 1. *Let $n, m \in \mathbb{N}$, $n \geq k+3$, $m \geq l+2$, and let $w(x,y)$ and $P_{n,m}(x,y;w)$ be defined as in Theorem 1. Then*

$$\begin{aligned}\big[P_{n,m}(x,y;w)w(x,y)\big]^2 &+ \big[P_{n-1,m+1}(x,y;w)w(x,y)\big]^2 \\ &= \Big(\frac{c_k c_l}{2^{n+m-1}}\Big)^2 - (1-x^2-y^2)w^2(x,y)q(x,y;w),\end{aligned}$$

where

$$\begin{aligned}q(x,y;w) =&\Big(\frac{c_k c_l}{2^{n+m-1}}\Big)^2 \\ &\times \big[U_{n-1}^2(x;1/\rho_k)U_m^2(y;1/\rho_l) + U_{n-2}^2(x;1/\rho_k)U_{m-1}^2(y;1/\rho_l)\big].\end{aligned}$$

Proof. For $k = l = 0$, the statement is precisely that of Proposition 2 below. Its proof, which can be found in [7], is based on the following identities for the Chebyshev polynomials of the second kind:

$$\begin{aligned}U_n(x)U_{n-2}(x) &= U_{n-1}^2(x) - 1, \\ U_n^2(x) + U_{n-2}^2(x) - 2 &= (4x^2-2)U_{n-1}^2(x),\end{aligned}$$

$n \in \mathbb{N}$. It is easy to show that their analogues for the weighted Chebyshev polynomials of the second kind also hold true:

$$\begin{aligned}\frac{U_n(x;1/\rho_k)}{\rho_k(x)}\frac{U_{n-2}(x;1/\rho_k)}{\rho_k(x)} &= \Big(\frac{U_{n-1}(x;1/\rho_k)}{\rho_k(x)}\Big)^2 - 1, \\ \Big(\frac{U_n(x;1/\rho_k)}{\rho_k(x)}\Big)^2 + \Big(\frac{U_{n-2}(x;1/\rho_k)}{\rho_k(x)}\Big)^2 - 2 &= (4x^2-2)\Big(\frac{U_{n-1}(x;1/\rho_k)}{\rho_k(x)}\Big)^2,\end{aligned}$$

$n \geq k+2$. Having these identities, the proof of the proposition is completely analogous to the one of Proposition 2. □

In order to prove Theorem 2 we need some auxiliary results. Let us introduce the polynomials

$$Q_{n,m}(x,y) := U_n(x)U_{m-2}(y) + U_{n-2}(x)U_m(y), \qquad n, m \in \mathbb{N}_0,$$

and

$$\begin{aligned}S_0(x,y) := 0, \qquad S_1(x,y) := 0,& \\ S_i(x,y) := \frac{2[(-1-(-1)^i)/2]^{i/2-1} - Q_{i,i}(x,y)}{2(1-x^2-y^2)}, \qquad i \in \mathbb{N},\ i \geq 2.&\end{aligned}$$

With these notations we have the following relations satisfied by the Gearhart polynomials defined in (2).

Proposition 2. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then*

$$
\begin{aligned}
&[G_{n,m}(x,y)]^2 + [G_{n-1,m+1}(x,y)]^2 \\
&\qquad = \frac{1}{2^{2(n+m-1)}} - \frac{1-x^2-y^2}{2^{2(n+m-1)}}\big[U_{n-1}^2(x)U_m^2(y) + U_{n-2}^2(x)U_{m-1}^2(y)\big].
\end{aligned}
$$

Proposition 3. *Let $n \in \mathbb{N}$ and $m, i \in \mathbb{N}_0$ be such that $i \le n-1$. Then*

$$
\begin{aligned}
&G_{n,m}(x,y)G_{n-i,m+i}(x,y) + G_{n-1,m+1}(x,y)G_{n-(i+1),m+(i+1)}(x,y) \\
&= \frac{[(-1-(-1)^i)/2]^{i/2}}{2^{2(n+m-1)}} - \frac{1-x^2-y^2}{2^{2(n+m-1)}}\big[U_{n-1}(x)U_{n-i-1}(x)U_m(y)U_{m+i}(y) \\
&\qquad + U_{n-2}(x)U_{n-i-2}(x)U_{m-1}(y)U_{m+i-1}(y) - S_i(x,y)\big].
\end{aligned}
$$

Complete proofs of Propositions 2 and 3 can be found in [7].

Proposition 4. *Let $n \in \mathbb{N}$, $m, k \in \mathbb{N}_0$ be such that $k \le n-1$ and let $a_i \in \mathbb{R}$, $i = 0, \ldots, k$. Then the following identity holds:*

$$
\begin{aligned}
&\Big(\sum_{i=0}^{k} a_i G_{n-i,m+i}(x,y)\Big)^2 + \Big(\sum_{i=0}^{k} a_i G_{n-(i+1),m+(i+1)}(x,y)\Big)^2 \\
&= \frac{1}{2^{2(n+m-1)}}\Big[\Big(\sum_{i=0}^{[k/2]}(-1)^i a_{2i}\Big)^2 + \Big(\sum_{i=0}^{[(k-1)/2]}(-1)^i a_{2i+1}\Big)^2\Big] \\
&\quad - \frac{1-x^2-y^2}{2^{2(n+m-1)}}\Big[\Big(\sum_{i=0}^{k} a_i U_{n-1-i}(x)U_{m+i}(y)\Big)^2 \\
&\quad + \Big(\sum_{i=0}^{k} a_i U_{n-2-i}(x)U_{m-1+i}(y)\Big)^2 - 2\sum_{i=1}^{k}\Big(\sum_{j=0}^{k-i} a_j a_{j+i}\Big)S_i(x,y)\Big].
\end{aligned}
\tag{16}
$$

Proof. The assertion follows immediately from Propositions 2 and 3. □

Theorem 5. *Let $n, m \in \mathbb{N}$, $a_1, b_1 \in \mathbb{R}$ be such that $a_1^2 + b_1^2 < 1$ and $w_1(x,y)$ be given by (8). Then the polynomial $P_{n,m}(x,y;w_1)$ defined in (7) is a min-max polynomial on $\mathcal{D}$ with respect to the weight function w_1 and the minimum deviation is $1/2^{n+m-1}$.*

Proof. By the definition (7) of $P_{n,m}(x,y;w_1)$, Schwarz's inequality and Proposition 2, it follows that

$$
\begin{aligned}
|P_{n,m}(x,y;w_1)| &\le \frac{1}{2}\sqrt{(x-a_1)^2+(y-b_1)^2}\,\sqrt{G_{n,m-1}^2(x,y)+G_{n-1,m}^2(x,y)} \\
&\le \frac{\sqrt{-2a_1x-2b_1y+a_1^2+b_1^2+1}}{2^{n+m-1}}
\end{aligned}
$$

holds for all $(x,y) \in \mathcal{D}$. Therefore, using (8), we obtain

$$|P_{n,m}(x,y;w_1)w_1(x,y)| \leq \frac{1}{2^{n+m-1}}, \qquad (x,y) \in \mathcal{D}.$$

Let us set

$$p_1(x,y) = \frac{x-a_1}{\sqrt{-2a_1x-2b_1y+a_1^2+b_1^2+1}},$$
$$q_1(x,y) = \frac{y-b_1}{\sqrt{-2a_1x-2b_1y+a_1^2+b_1^2+1}}.$$

From the identity

$$(x-a_1)^2+(y-b_1)^2 = (-2a_1x-2b_1y+a_1^2+b_1^2+1)-(1-x^2-y^2)$$

it follows that on the boundary of the unit disc we have

$$p_1^2(x,y)+q_1^2(x,y)=1, \qquad (x,y) \in \partial\mathcal{D},$$

i.e.

$$p_1^2(\cos\varphi,\sin\varphi)+q_1^2(\cos\varphi,\sin\varphi)=1 \tag{17}$$

for all $\varphi \in [0,2\pi)$.

If $z_1 = a_1+ib_1$, then relation (17) can be written as

$$\Big(\frac{\operatorname{Re}(z-z_1)}{|z-z_1|}\Big)^2+\Big(\frac{\operatorname{Im}(z-z_1)}{|z-z_1|}\Big)^2=1, \qquad z=e^{i\varphi}.$$

Since $|z_1|<1$, it follows by a simple application of the Principle of the Argument that

$$p_1(\cos\varphi,\sin\varphi)=\cos(\varphi+\varphi_0) \quad \text{and} \quad q_1(\cos\varphi,\sin\varphi)=\sin(\varphi+\varphi_0), \tag{18}$$

where $\varphi_0 \in [0,2\pi)$. Hence, using the fact that on the boundary of the unit disc the Gearhart polynomials are equal to

$$G_{n,m}(\cos\varphi,\sin\varphi)=\begin{cases}\dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\cos(n+m)\varphi, & \text{if } m \text{ is even},\\[2mm] \dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\sin(n+m)\varphi, & \text{if } m \text{ is odd},\end{cases} \tag{19}$$

see [4], and (18), we conclude that

$$P_{n,m}(\cos\varphi,\sin\varphi;w_1)w_1(\cos\varphi,\sin\varphi)$$
$$=\begin{cases}\dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\cos((n+m)\varphi+\varphi_0), & \text{if } m \text{ is even},\\[2mm] \dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\sin((n+m)\varphi+\varphi_0), & \text{if } m \text{ is odd}.\end{cases}$$

Therefore, $|P_{n,m}(x,y;w_1)w_1(x,y)|$ attains its maximum $1/2^{n+m-1}$ on $\partial\mathcal{D}$ at the points $(\cos\varphi_i, \sin\varphi_i)$, $i = 1, \ldots, 2(n+m)$, with alternating sign. Moreover, φ_i's are the zeros in $[0, 2\pi)$ of $\sin((n+m)\varphi + \varphi_0)$ if m is even, and the zeros in $[0, 2\pi)$ of $\cos((n+m)\varphi + \varphi_0)$ if m is odd. Since by Shapiro's Theorem, see [12, Theorem 2], these points form the support of an extremal signature with respect to Π^2_{n+m-1}, the assertion follows by Theorem 2 in [11]. $\square$

Proposition 5. *Let $n, m \in \mathbb{N}$, $n \geq 2$ and $a_1, b_1 \in \mathbb{R}$, $a_1^2 + b_1^2 < 1$. Furthermore, let $w_1(x,y)$ be the function defined in* (8) *and $P_{n,m}(x,y;w_1)$ the polynomial given by* (7)*. Then the following relation holds:*

$$[P_{n,m}(x,y;w_1)w_1(x,y)]^2 + [P_{n-1,m+1}(x,y;w_1)w_1(x,y)]^2 = \frac{1}{2^{2(n+m-1)}} - (1-x^2-y^2)w_1^2(x,y)q(x,y;w_1)$$

where

$$q(x,y;w_1) = \frac{1}{2^{2(n+m-1)}}\left[p^2_{n,m}(x,y;w_1) + p^2_{n-1,m-1}(x,y;w_1) + 1\right]$$

and

$$p_{n,m}(x,y;w_1) = (x-a_1)U_{n-2}(x)U_m(y) + (y-b_1)U_{n-1}(x)U_{m-1}(y).$$

Proof. The identity follows immediately from (16) with $k = 1$, $a_0 = \frac{1}{2}(y-b_1)$ and $a_1 = \frac{1}{2}(x-a_1)$, and then by replacing m by $m-1$. $\square$

Theorem 6. *Let $n, m \in \mathbb{N}$, $n, m \geq 2$, $a_i, b_i \in \mathbb{R}$ be such that $a_i^2 + b_i^2 < 1$, $i = 1, 2$, and $w_2(x,y)$ be given by* (10)*. Then the polynomial $P_{n,m}(x,y;w_2)$ defined in* (9) *is a min-max polynomial on $\mathcal{D}$ with respect to the weight function w_2 and the minimum deviation is $1/2^{n+m-1}$.*

Proof. The proof is analogous to that of Theorem 5 taking into account that

$$P_{n,m}(x,y;w_2) = \frac{1}{2}(x-a_2)P_{n-1,m}(x,y;w_1) + \frac{1}{2}(y-b_2)P_{n,m-1}(x,y;w_1)$$

holds for all $(x,y) \in \mathcal{D}$. Also, Proposition 5 is to be used in this case. $\square$

Proposition 6. *Let $n, m \in \mathbb{N}$, $n \geq 3$, $m \geq 2$, and $a_i, b_i \in \mathbb{R}$ be such that $a_i^2 + b_i^2 < 3 - 2\sqrt{2}$, $i = 1, 2$. Furthermore, let $w_2(x,y)$ be defined by* (10) *and let $P_{n,m}(x,y;w_2)$ be the polynomial given in* (9)*. Then the following identity holds:*

$$[P_{n,m}(x,y;w_2)w_2(x,y)]^2 + [P_{n-1,m+1}(x,y;w_2)w_2(x,y)]^2 = \frac{1}{2^{2(n+m-1)}} - (1-x^2-y^2)w_2^2(x,y)q(x,y;w_2)$$

where

$$q(x,y;w_2) = \frac{1}{2^{2(n+m-1)}}\big[p_{n,m}^2(x,y;w_2) + p_{n-1,m-1}^2(x,y;w_2) \\ + (x-a_1)^2 + (y-b_1)^2 + (x-a_2)^2 + (y-b_2)^2 \\ - 4(x-a_1)(x-a_2)(y-b_1)(y-b_2) + (1-x^2-y^2)\big]$$

is non-negative on the unit disc and

$$p_{n,m}(x,y;w_2) = (x-a_1)(x-a_2)U_{n-3}(x)U_m(y) \\ + [(x-a_1)(y-b_2) + (x-a_2)(y-b_1)]U_{n-2}(x)U_{m-1}(y) \\ + (y-b_1)(y-b_2)U_{n-1}(x)U_{m-2}(y).$$

Proof. The proposition follows immediately from identity (16) with $k=2$, $a_0 = \frac{1}{4}(y-b_1)(y-b_2)$, $a_1 = \frac{1}{4}[(x-a_1)(y-b_2) + (x-a_2)(y-b_1)]$ and $a_2 = \frac{1}{4}(x-a_1)(x-a_2)$ and then by replacing m by $m-2$. The inequalities $a_i^2 + b_i^2 < 3 - 2\sqrt{2}$, $i=1,2$, give a sufficient condition for

$$(x-a_1)^2 + (y-b_1)^2 + (x-a_2)^2 + (y-b_2)^2 - 4(x-a_1)(y-b_1)(x-a_2)(y-b_2),$$

and hence for $q(x,y;w_2)$, to be non-negative on the unit disc. □

Proof of Theorem 2. The proof is similar to that of Theorem 5. Since

$$P_{n,m}(x,y;w_3) = \frac{1}{2}(x-a_3)P_{n-1,m}(x,y;w_2) + \frac{1}{2}(y-b_3)P_{n,m-1}(x,y;w_2)$$

holds for all $(x,y) \in \mathcal{D}$, we just apply Proposition 6. □

Proof of Theorem 3. Let us denote the polynomial from the theorem by $P_{n(\nu+\mu)-\mu,m(\nu+\mu)+\mu}(x,y;w^{\nu+\mu})$. One can easily check that it is indeed of the form $x^{n(\nu+\mu)-\mu}y^{m(\nu+\mu)+\mu}$+ terms of lower degree. Proposition 1 yields that if $(x,y) \in \mathcal{D}$ then $(c_{n,m}^{-1}P_{n,m}(x,y;w)w(x,y), c_{n,m}^{-1}P_{n-1,m+1}(x,y;w)w(x,y)) \in \mathcal{D}$, too. Consequently, having in mind (3),

$$|P_{n(\nu+\mu)-\mu,m(\nu+\mu)+\mu}(x,y;w^{\nu+\mu})w(x,y)^{\nu+\mu}| \\ = |c_{n,m}^{\nu+\mu}Q_{\nu,\mu}(c_{n,m}^{-1}P_{n,m}(x,y;w)w(x,y), c_{n,m}^{-1}P_{n-1,m+1}(x,y;w)w(x,y))| \\ \le \frac{c_{n,m}^{\nu+\mu}}{2^{\nu+\mu-1}}$$

holds for all $(x,y) \in \mathcal{D}$. In addition, by (15) and (19) combined with the fact that all min-max polynomials to a monomial agrees on the boundary of the unit disc, see [4, Theorem 2.2], we can easily show that the function $P_{n(\nu+\mu)-\mu,m(\nu+\mu)+\mu}(\cos\varphi,\sin\varphi;w^{\nu+\mu})w^{\nu+\mu}(\cos\varphi,\sin\varphi)$ has an increase of $2(\nu+\mu)(n+m)\varphi$, when φ increases from 0 to 2π. Hence, the maximum modulus is attained at $2(n+m)(\nu+\mu)$ points on the boundary of the disc with alternating sign. Since by [12, Theorem 2] these points form the support of an extremal signature with respect to $\Pi^2_{(n+m)(\nu+\mu)-1}$, the statement follows from [11, Theorem 2]. □

Proof of Corollary 1. The assertion is an immediate consequence of Theorem 3 with $w \equiv 1$ and the Gearhart polynomials defined by (2), taking into consideration also Proposition 2 and relation (19). □

Proof of Theorem 4. Let us denote the polynomial from the theorem by $P_{n_1+n_2,m_1+m_2}(x,y;\tilde{w}_1\tilde{w}_2)$. Then combining Schwarz's inequality, relation (11) for $P_{n_1,m_1}(x,y;\tilde{w}_1)$ and the corresponding one for $P_{n_2,m_2}(x,y;\tilde{w}_2)$, we have

$$\begin{aligned}
&|P_{n_1+n_2,m_1+m_2}(x,y;\tilde{w}_1\tilde{w}_2)\tilde{w}_1(x,y)\tilde{w}_2(x,y)| \\
&\quad \le \frac{1}{2}\sqrt{[P_{n_1,m_1}(x,y;\tilde{w}_1)\tilde{w}_1(x,y)]^2 + [P_{n_1-1,m_1+1}(x,y;\tilde{w}_1)\tilde{w}_1(x,y)]^2} \\
&\qquad \sqrt{[P_{n_2,m_2}(x,y;\tilde{w}_2)\tilde{w}_2(x,y)]^2 + [P_{n_2+1,m_2-1}(x,y;\tilde{w}_2)\tilde{w}_2(x,y)]^2} \\
&\quad \le \frac{1}{2}\, c_{n_1,m_1} c_{n_2,m_2}
\end{aligned}$$

for all $(x,y) \in \mathcal{D}$. Moreover, due to (12) and the corresponding relation for $P_{n_2,m_2}(x,y;\tilde{w}_2)$, we easily get that on the boundary of the unit disc we have

$$\begin{aligned}
&P_{n_1+n_2,m_1+m_2}(\cos\varphi,\sin\varphi;\tilde{w}_1\tilde{w}_2)\,\tilde{w}_1(\cos\varphi,\sin\varphi)\,\tilde{w}_2(\cos\varphi,\sin\varphi) \\
&\quad = \frac{c_{n_1,m_1}c_{n_2,m_2}}{2}\big[\cos((n_1+m_1)\varphi+\varphi_1)\sin((n_2+m_2)\varphi+\varphi_2) \\
&\qquad\qquad + \sin((n_1+m_1)\varphi+\varphi_1)\cos((n_2+m_2)\varphi+\varphi_2)\big] \\
&\quad = \frac{c_{n_1,m_1}c_{n_2,m_2}}{2}\sin((n_1+m_1+n_2+m_2)\varphi+\varphi_1+\varphi_2)
\end{aligned}$$

for all $\varphi \in [0,2\pi)$. Therefore, the maximum modulus $c_{n_1,m_1}c_{n_2,m_2}/2$ is attained at $2(n_1+m_1+n_2+m_2)$ points on the boundary of the disc with alternating sign. Since by [12, Theorem 2] these points form an extremal signature with respect to $\Pi^2_{n_1+m_1+n_2+m_2-1}$, the statement of the theorem follows from [11, Theorem 2]. □

Proof of Corollary 2. The assertion is an immediate application of Theorem 4 for the min-max polynomials defined by (9). □

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