CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 227-264
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

On Markov-Duffin-Schaeffer Inequalities with a Majorant

Geno Nikolov* and Alexei Shadrin

1. Introduction

V. Markov [4] proved that if $p \in \mathcal{P}_n$, the space of algebraic polynomials of degree n, and

$$|p(x)| \le 1, \qquad x \in [-1,1],$$

then on the same interval

$$\|p^{(k)}\| \le T_n^{(k)}(1) \tag{1.1}$$

for $k = 1, \dots, n$, with equality only if $p = \gamma T_n$, where $|\gamma| = 1$. Here T_n is the Chebyshev polynomial of degree n, $T_n(x) := \cos n \arccos x$ for $x \in [-1,1]$, and $\|\cdot\| := \|\cdot\|_{C[-1,1]}$ is the usual uniform norm.

Duffin and Schaeffer [2] strengthened Markov's inequality showing that it remains valid under the weaker assumption

$$|p(x)| \le 1, \qquad x \in \{\cos \tfrac{\pi i}{n}\}_{i=0}^{n},$$

where $x = \cos \frac{\pi i}{n}$ are exactly the points where $|T_n(x)| = 1$. They also showed that, if restrictions $|p(x)| \le 1$ are imposed at any other set of $(n+1)$ points in $[-1,1]$, then Markov bound (1.1) is no longer true.

Here we consider the problem of estimating the norm $\|p^{(k)}\|$ under restriction

$$|p(x)| \le \mu(x), \qquad x \in I,$$

where μ is an arbitrary non-negative majorant, and I is either the whole interval $[-1,1]$, or a discrete set δ of $n+1$ points in $[-1,1]$.

So, the problems are the following.

*This author was supported by the National Science Fund of the Bulgarian Ministry of Education, Youth and Science under Grant DDVU–02/30.

Problem 1 (Markov inequality with a majorant). For $n, k \in \mathbb{N}$, and a majorant $\mu \geq 0$, find

$$M_{k,\mu} := \sup_{|p(x)| \leq \mu(x)} \|p^{(k)}\|. \tag{1.2}$$

Problem 2 (Duffin-Schaeffer inequality with a majorant). Given $n, k \in \mathbb{N}$, and a majorant $\mu \geq 0$, find

$$D_{k,\mu} := \inf_{\delta \in [-1,1]} \sup_{|p(x)|_{|\delta} \leq \mu(x)_{|\delta}} \|p^{(k)}\|. \tag{1.3}$$

(The idea of such setting is borrowed from Duffin-Karlovitz [3].)

In these notations, the results of Markov and Duffin-Schaeffer read:

$$\mu \equiv 1 \quad \Rightarrow \quad M_{k,\mu} = D_{k,\mu} = \|T_n^{(k)}\|,$$

and the Chebyshev polynomial T_n (which oscillates most between ± 1) is extremal for both problems. In particular, the set δ_* of its $n+1$ equioscillation points gives the infimum in (1.3).

So, the question of interest is: for which other majorants μ the polynomial $\omega = \omega_\mu$ that oscillates most between $\pm\mu$, the so called snake-polynomial, gives solution to both problems (1.2)–(1.3), i.e., when do we have the equalities

$$M_{k,\mu} \stackrel{?}{=} \|\omega_\mu^{(k)}\| \stackrel{?}{=} D_{k,\mu}\,.$$

Notice that, for any majorant μ, we have

$$\|\omega_\mu^{(k)}\| \leq M_{k,\mu} \leq D_{k,\mu}\,,$$

so the problem will be settled once we show that $D_{k,\mu} \leq \|\omega_\mu^{(k)}\|$.

In this paper we establish Duffin-Schaeffer (and, thus, Markov) inequalities for a wide range of majorants μ.

2. Results

2.1. Earlier Results

There are not so many results on Markov-Duffin-Schaeffer inequalities with majorants, therefore we decided to mention all of them (to the best of our knowledge). We display them in two tables and make short comments on them. A detailed account on different proofs of classical Markov inequality and its generalizations could be found in the recent survey [14].

	Majorant $\mu(x)$	Derivative k	Degree n	Value M_k	Authors
0°	1	all k	all n	$T_n^{(k)}(1)$	V. Markov (1892)
1°	$\sqrt{1+(a^2-1)x^2}$	all k	all n	$\omega_n^{(k)}(1)$	Vidensky (1958)
2°	$\sqrt{ax^2+bx+1}$	$k=1$	all n	$\max\|\omega_n'(\pm 1)\|$	Vidensky (1958)
3°	$\sqrt{\prod_{i=1}^m (1+c_i^2x^2)}$	$k=1$	$n\ge m$	$\omega_n'(1)$	Vidensky (1959)
4°	$\sqrt{(1+c_1^2x^2)(1+c_2^2x^2)}$	all k	$n\ge c_{1,2}^2+2$	$\omega_n^{(k)}(1)$	Vidensky (1971)
5°	$\sqrt{(1-x)^{m_1}(1+x)^{m_2}}$	$k\ge\frac{m_1+m_2}{2}$	$n\ge\frac{m_1+m_2}{2}$	$\\|\omega_{n-1}^{(k)}\\|$ or $\\|\omega_n^{(k)}\\|$	Pierre and Rahman (1981)
6°	$1-x^2$	$k=1$	$n\ge 2$	$\\|\omega_n'\\|$	Pierre, Rahman and Schmeisser (1989)
7°	$\sqrt{1-x^2}$ or $1-x^2$	all k	$n\ge 2$	$\omega_n^{(k)}(1)$	from 1° and 5°-6°

Table 1. Markov inequalities with a majorant

	Majorant μ	Derivative k	Degree n	Value D_k	Authors
8°	1	all k	all n	$T_n^{(k)}(1)$	Duffin and Schaeffer (1941) Shadrin (1992)
9°	$\sqrt{1-x^2}$	$k=1$ $k\ge 2$	all n all n	$D_1>\\|\omega_n'\\|$ $D_k=\omega_n^{(k)}(1)$	Rahman and Schmeisser (1988)
10°	$1-x^2$	$k\ge 3$	all n	$\omega_n^{(k)}(1)$	Rahman and Watt (1992)
11°	any μ	$k=n,\,n-1$ $k=n-2$	all n	$\|\omega_n^{(k)}(\pm 1)\|$	Shadrin (1992) Nikolov (2002)

Table 2. Duffin-Schaeffer inequalities with a majorant

Let us make some comments.

1°-7°. All majorants are of the form

$$\mu(x) = \sqrt{R_{2m}(x)}, \qquad R_{2m} \in \mathcal{P}_{2m},$$

where R_{2m} is a non-negative polynomial of degree $\le 2m$. For the polynomials $p \in \mathcal{P}_n$ with a majorant of this kind, Vidensky [15] found (a kind of) explicit majorant $V_k(x)$ for $|p^{(k)}(x)|$ (i.e., a bound for the pointwise Markov inequality). Also, in this case, there is an explicit (again, to a certain extent) expression for the corresponding snake-polynomial ω_n of degree n if $n \ge m$ (see § 10.1).

1°-4°. Those are particular cases where Vidensky managed to proceed from an intermediate pointwise estimate $|p^{(k)}(x)| \le V_k(x)$ to a bound for the uniform norm $\|p^{(k)}\|$.

4°. As we have mentioned, for $\mu = \sqrt{R_{2m}}$, a natural restriction on degree n of ω_n is

$$n \ge m,$$

thus restriction on n in 4° looks artificial (and we remove it in our results).

5°. For this case, Pierre and Rahman applied the original variational approach of V. Markov. The exact value of $\|\omega_n^{(k)}\|$ is generally not known unless it is equal to $\omega_n^{(k)}(\pm 1)$ or $\omega_n^{(k)}(0)$, this is perhaps why, in 5°, two candidates for the extremal function appear. We will show that, for symmetric majorants, we have $M_{k,\mu} = \omega_n^{(k)}(1)$.

6°. This case required special consideration as it was not covered by 5°.

7°. We put this case in a separate line to compare it with the corresponding results in Duffin-Schaeffer-type inequalities.

8°-10°. There are two different proofs of the classical Duffin-Schaeffer inequality (with the unit majorant). The cases 9°-10° were obtained using the original Duffin-Schaeffer method [2], but the second method [12] is applicable for those majorants as well. However, further extensions of both methods, even to the majorants $(1-x^2)^{m/2}$, are hardly possible, and that was our motivation for searching a new method.

9°. The case $k = 1$ for the majorant $\mu(x) = \sqrt{1-x^2}$, when $\omega_n^{(1)}(1) = M_1 < D_1$, is very interesting as it shows that equality $M_k = D_k$ should not be always expected.

10°. Comparing 7° with 10° we see that, for the majorant $\mu(x) = 1-x^2$, the Markov-type inequality $M_{k,\mu} = \omega_n^{(k)}(1)$ holds for all $k \ge 1$, while the Duffin-Schaeffer-type result $D_{k,\mu} = \omega_n^{(k)}(1)$ is established only for $k \ge 3$. However, the previous case suggests that, for the majorants $(1-x^2)^{m/2}$, the equality $M_{k,\mu} = D_{k,\mu}$ might not be true for small k.

11°. Since Duffin-Schaeffer inequality holds for $n-2 \le k \le n$ for any majorant μ, and, on the other hand, $k \ge 1$, we may (and shall) assume further that the degree n of the snake-polynomial ω_n satisfies $n \ge 4$.

2.2. New Results

Definition 2.1. Denote by Ω the class of polynomials ω such that

$$0)\qquad \omega(x) = c\prod_{i=1}^{n}(x-t_i), \quad t_i \in [-1,1];$$

$$1a)\qquad \|\omega\|_{C[0,1]} = \omega(1), \qquad\qquad 1b)\qquad \|\omega\|_{C[-1,0]} = |\omega(-1)|;$$

$$2)\qquad \omega(x) = \sum_{i=0}^{n} a_i T_i(x), \quad a_i \ge 0.$$

In particular, this class contains all odd and even polynomials with positive Chebyshev expansion 2), i.e., polynomials of the form

$$\omega(x) = \sum_i a_{2i+\nu}T_{2i+\nu}(x), \quad a_{2i+\nu} \ge 0, \quad \nu \in \{0,1\}.$$

Equality 1a) follows, of course, from 2), but assumptions 1)–2) are independent in the sense that they are used at different stages of the proof, and may well be relaxed. For example, we strongly believe that our main Theorem 2.1 is valid under assumption 2) only.

Our main result (with respect to the Markov-Duffin-Schaeffer inequalities with a majorant) is the following.

Theorem 2.1. *Given a majorant μ, let ω_μ be the corresponding snake-polynomial of degree n. Then we have*

$$\omega_\mu^{(k-1)} \in \Omega \quad\Rightarrow\quad M_{k,\mu} = D_{k,\mu} = \omega_\mu^{(k)}(1)\,.$$

Example 2.1. The following table gives some examples of majorants to which this theorem can be applied.

Duffin-Schaeffer inequalities with a majorant

	Majorant μ	*Derivative* k	*Degree* n	*Value* D_k
12°	$\sqrt{\prod_{i=1}^{m}(1+c_i^2x^2)}$	*all* k	$n \ge m$	$\omega_n^{(k)}(1)$
13°	$(1-x^2)^{m/2}$	$k \ge m+1$	$n \ge m$	$\omega_n^{(k)}(1)$
14°	$\sqrt{R_m(x^2)}$	$k \ge m+1$	$n \ge m$	$\omega_n^{(k)}(1)$
15°	*any* $\mu(x) = \mu(-x)$	$k \ge n/2$	*all* n	$\omega_n^{(k)}(1)$
16°	$\sqrt{(1+c_1^2x^2)(1+(a_2^2-1)x^2)}$	$k \ge 2$	$n \ge 2$	$\omega_n^{(k)}(1)$
17°	$\sqrt{ax^2+bx+1}$	$k = 2$ $k \ge 3$	$n \ge \frac{1}{\mu(\pm 1)}$ *all* n	$\lvert\omega_n^{(k)}(\pm 1)\rvert$

12°. This case extends the Markov-type results 3°-4° of Vidensky to arbitrary k, and also strengthens them in the spirit of Duffin-Schaeffer.

13°-14°. The case 13° is of course a particular case of 14°. It covers previous Duffin-Schaeffer-type results 9°-10°, and strengthens the corresponding Markov-type inequality 5° for symmetric majorants. Note that our Duffin-Schaeffer-type results are valid starting from $k = m + 1$, while those of Markov type start with $k = m$, but this could be a necessary restriction. The case 14° shows that the restriction $k \geq m + 1$ provides, in fact, Markov-Duffin-Schaeffer-type results for all symmetric polynomial majorants.

15°. This is an expected extension of 11°.

16°. This majorant is of the form which is in a sense intermediate between the cases 12° and 13°.

17°. This is an example of a non-symmetric majorant.

3. Preliminaries

We are dealing with the Markov-Duffin-Schaeffer problem with a majorant μ, where we want to find the values

$$M_{k,\mu} = \sup_{|p(x)| \leq \mu(x)} \|p^{(k)}\| \,, \quad D_{k,\mu} := \inf_{\delta \in [-1,1]} \sup_{|p(x)|_{|\delta} \leq \mu(x)_{|\delta}} \|p^{(k)}\| \,.$$

For any $\mu \geq 0$ there is a unique polynomial $\omega_\mu \in \mathcal{P}_n$, the so-called snake-polynomial, that oscillates $n+1$ times between $\pm\mu$, i.e., such that

$$|\omega_\mu(x)| \leq \mu(x) \,, \quad x \in [-1, 1] \,,$$

and on some set of $n+1$ points $\delta_* = (\tau_i^*)_{i=0}^n$ we have

$$\omega_\mu(\tau_i^*) = (-1)^i \mu(\tau_i^*) \,.$$

The question of interest is: for which majorants μ it is this polynomial ω_μ that gives extremum to both values above (as in the case $\mu \equiv 1$), in particular, whether it is the set δ_* that gives the infimum value $D_{k,\mu}$.

Notice that, for any majorant μ, we have

$$\|\omega_\mu^{(k)}\| \leq M_{k,\mu} \leq D_{k,\mu} \leq D_{k,\mu}^*,$$

where

$$D_{k,\mu}^* := \sup_{|p(x)|_{|\delta_*} \leq \mu(x)_{|\delta_*}} \|p^{(k)}\| \,.$$

so it would be enough to prove that $D_{k,\mu}^* = \|\omega_\mu^{(k)}\|$. However, even with the simplest majorants, the location of the nodes τ_i^* is not known explicitly, so we have to find some arguments that avoid the use of them.

It is clear that the snake-polynomial ω_μ has n zeros inside the interval $[-1,1]$, i.e., $\omega_\mu(x) = c\prod_{i=1}^n (x-t_i)$, and that these zeros interlace with the "touch-points" (τ_i^*), i.e.,

$$-1 \le \tau_0^* < t_1 < \tau_1^* < \cdots < t_n < \tau_n^* \le 1\,.$$

Denote by Δ_ω the class of knot-sequences $\delta = (\tau_i)$ with the same interlacing properties,

$$-1 \le \tau_0 < t_1 < \tau_1 < \cdots < t_n < \tau_n \le 1\,.$$

Then

$$\begin{aligned} D_{k,\mu}^* = \sup_{|p(x)|_{|\delta_*} \le \mu(x)_{|\delta_*}} \|p^{(k)}\| \;&=\; \sup_{|p(x)|_{|\delta_*} \le |\omega(x)|_{|\delta_*}} \|p^{(k)}\| \\ &\le \sup_{\delta\in\Delta_\omega} \sup_{|p(x)|_{|\delta} \le |\omega(x)|_{|\delta}} \|p^{(k)}\| =: S_{k,\omega}\,, \end{aligned}$$

and respectively

$$\|\omega_\mu^{(k)}\| \le M_{k,\mu} \le D_{k,\mu} \le D_{k,\mu}^* \le S_{k,\omega}\,. \tag{3.1}$$

So, now, we may try to evaluate the value $S_{k,\omega}$ in terms of $\|\omega_\mu^{(k)}\|$. It turns out that the pointwise problem, of finding the values

$$s_{k,\omega}(x) := \sup_{\delta\in\Delta_\omega} \sup_{|p(x)|_{|\delta} < |\omega(x)|_{|\delta}} |p^{(k)}(x)|\,, \qquad x \in [-1,1],$$

has a remarkable solution.

Proposition 3.1 ([13], [5]). *Let* $\omega(x) = \prod_{i=1}^n (x-t_i)$, $t_i \in [-1,1]$. *Then*

$$s_{k,\omega}(x) = \max\left\{ |\omega^{(k)}(x)|,\ \max_{1\le i\le n} |\phi_i^{(k)}(x)| \right\}, \tag{3.2}$$

where

$$\phi_i(x) := \omega(x)\frac{1-xt_i}{x-t_i}, \qquad i = 1,\ldots,n. \tag{3.3}$$

Proof. The original proof by Shadrin [13] followed the idea of Duffin-Karlovits [3]: it was shown that variation of any single knot $\tau_i \in (t_i, t_{i+1})$ does not result in a local extremum of the value $p^{(k)}(x)$, hence the value $s_{k,\omega}(x)$ is achieved when τ_i is either t_i ot t_{i+1}. A simpler proof based on the properties of Lagrange interpolating polynomials was given later by Nikolov [5]. □

The polynomials ϕ_i are quite interesting. They have the same zeros as ω except one t_i, and, because of the factor $\frac{1-xt_i}{x-t_i}$, they satisfy the inequalities

$$|\phi_i(z)| \ge |\omega(z)|, \quad z \in \mathcal{D}_1, \qquad |\phi_i(z)| < |\omega(z)|, \quad z \notin \overline{\mathcal{D}_1},$$

where $\mathcal{D}_1$ is the unit open disc in the complex plane. These polynomials may be viewed as the most extreme case of the Zolotarev-like polynomials.

From the pointwise equality (3.2), since $S_{k,\omega} = \max_x s_{k,\omega}(x)$, it follows that $S_{k,\omega}$ is just the maximum of the max-norms of the polynomials on the right-hand side. Moreover, we can make a minor simplification using the fact that

$$s_{k,\omega}(\pm 1) = |\omega^{(k)}(\pm 1)|\,,$$

which means that the values of $\phi_i^{(k)}$ at the endpoints are inessential, and therefore it is only the local maxima of $\phi_i^{(k)}$s that matters. To this end, we introduce the "local norm"

$$\|f\|_* := \max\{|f(x)| : f'(x) = 0,\ x \in (-1,1)\}\,,$$

and the following statement is immediate.

Corollary 3.1. *Let* $\omega(x) = \prod_{i=1}^{n}(x - t_i)$*,* $t_i \in [-1,1]$*. Then*

$$S_{k,\omega} = \max\left\{\, \|\omega^{(k)}\|,\ \max_{1\le i\le n} |\phi_i^{(k)}\|_* \right\}. \tag{3.4}$$

From this corollary and the chain of inequalities in (3.1) we obtain the statement that gives a new way (other than those in [2] and [12]) of deriving Markov-Duffin-Schaeffer inequalities with a majorant.

Proposition 3.2. *Given a majorant* $\mu \ge 0$*, let* $\omega_\mu \in \mathcal{P}_n$ *be the corresponding snake-polynomial. If*

$$\max_i \|\phi_i^{(k)}\|_* \le \|\omega_\mu^{(k)}\|, \tag{3.5}$$

then

$$M_{k,\mu} = D_{k,\mu} = \|\omega_\mu^{(k)}\| \quad \big(= S_{k,\omega}\big). \tag{3.6}$$

An advantage of studying the inequality (3.5) is that this is purely a polynomial problem on the class of polynomials ω having all their zeros in $[-1,1]$, with rather simple and explicitly given polynomials ϕ_i involved.

A disadvantage is that the high derivatives of ϕ_i-s are still difficult to analyze, but we may reduce the problem to studying the behaviour of $\widehat{\phi}_i^{(m)}$ for small m, say, $m = 0,1$, where $\widehat{\phi}_i$ are the polynomials defined in the same way as in (3.3) but with respect to $\widehat{\omega} = \omega_\mu^{(k-m)}$.

Corollary 3.2. *Given a majorant* $\mu \ge 0$*, let* $\omega_\mu \in \mathcal{P}_n$ *be the snake-polynomial associated with* μ*, and let* $\widehat{\omega} := \omega_\mu^{(k-m)}$*. If*

$$\max_i \|\widehat{\phi}_i^{(m)}\|_* \le \|\widehat{\omega}^{(m)}\| \quad \big(= \|\omega^{(k)}\|\big), \tag{3.7}$$

then

$$M_{k,\mu} = D_{k,\mu} = \|\omega_\mu^{(k)}\| \quad \big(= S_{m,\widehat{\omega}}\big). \tag{3.8}$$

Proof. The proof is based on the fact that if a polynomial p satisfies

$$|p(\tau_i)| \le |\omega(\tau_i)|\,,$$

where $\delta = (\tau_i)$ is any set of $n+1$ points which interlace with the zeros of ω, then its derivative of any order $k-m$ satisfies similar inequalities:

$$|p^{(k-m)}(\eta_j)| \le |\omega^{(k-m)}(\eta_j)|\,,$$

where $\widehat{\delta} = (\eta_j)$ is some set of $(n+1)-(k-m)$ points which interlace with the zeros of $\omega^{(k-m)}$. Therefore, with $\widehat{\omega} := \omega^{(k-m)}$, we have

$$\begin{aligned} s_{k,\omega}(x) &= \sup_{\delta\in\Delta_\omega} \sup_{|p(x)|_{|\delta} < |\omega(x)|_{|\delta}} |p^{(k)}(x)| \\ &\le \sup_{\widehat{\delta}\in\Delta_{\widehat{\omega}}} \sup_{|q(x)|_{|\widehat{\delta}} < |\widehat{\omega}(x)|_{|\widehat{\delta}}} |q^{(k)}(x)| = s_{m,\widehat{\omega}}(x)\,, \end{aligned}$$

hence

$$\|\omega_\mu^{(k)}\| \le M_{k,\mu} \le D_{k,\mu} \le D_{k,\mu}^* \le S_{k,\omega} \le S_{m,\widehat{\omega}}\,.$$

From assumption (3.7), due to (3.4), we obtain $S_{m,\widehat{\omega}} = \|\widehat{\omega}^{(m)}\| = \|\omega_\mu^{(k)}\|$, and that implies (3.8). □

Nikolov [7] proved that

$$\widehat{\omega} = \omega^{(k)} = T_n^{(k)} \quad \Rightarrow \quad \text{(3.7) is valid with } m = 0 \quad \text{(hence (3.8) for } \mu \equiv 1),$$

and that gives one more proof of the classical Duffin-Schaeffer inequality.

In this paper, using some ideas from [7], we show that (3.7) is true with $m = 1$ for the polynomials $\widehat{\omega}$ from the class Ω which we defined in (2.1) in the following way.

$$\begin{array}{lll} 0) & \widehat{\omega}(x) = \prod_{i=1}^n (x-t_i) & \\ 1a) & \|\widehat{\omega}\|_{C[0,1]} = |\widehat{\omega}(1)|, & 1b) \quad \|\widehat{\omega}\|_{C[-1,0]} = |\widehat{\omega}(-1)| \\ 2) & \widehat{\omega} = \sum_{j=0}^n a_j T_j, \quad a_j \ge 0. & \end{array}$$

Namely, we prove the following statement.

Theorem 3.1. *Let $\widehat{\omega} \in \Omega$. Then*

$$\max_i \|\widehat{\phi}_i'\|_* \le \|\widehat{\omega}'\|\,.$$

From this result, Theorem 2.1 easily follows.

Proof of Theorem 2.1. By the assumption of Theorem 2.1, $\widehat{\omega} := \omega^{(k-1)}$ belongs to the class Ω. By Theorem 3.1, this inclusion implies the inequalities (3.7) which in turn, by Corollary 3.2, imply (3.8), i.e. the statement of the theorem. □

The rest of the paper consists of two parts. In the first part (§4-§8), we prove Theorem 3.1. The proof is a bit lengthy, so we describe its structure in §4. In the second part (§9-§10), we take some particular μ's and k's (given in Example 2.1), and verify that, for the snake-polynomial ω_μ, the polynomial $\widehat{\omega} = \omega^{(k-1)}$ belongs to the class Ω. Thus, for those particular majorants, by Theorem 2.1, we have

$$M_{k,\mu} = D_{k,\mu} = \omega_\mu^{(k)}(1).$$

4. Structure of the Proof of Theorem 3.1

The proof consists of three parts.

Step 1. In § 5, we introduce two functions $\psi_1(x,t)$ and $\psi_2(x,t)$ with the properties

$$\phi_i''(x) = 0 \quad \Rightarrow \quad \phi_i'(x) = \psi_\nu(x, t_i).$$

That means that both $\psi_\nu(\cdot, t_i)$ interpolate ϕ_i' at the points of their local extrema, therefore

$$\max_i \|\phi_i'\|_{*[0,1]} \le \max_{x\in[0,1]} \max_{t_i} \min_{\nu=1,2} |\psi_\nu(x,t_i)| \,. \tag{4.1}$$

Step 2. In § 6-§8, we show that, if

$$\|\omega\|_{C[0,1]} \le \omega(1), \tag{4.2}$$

then, with some specific functions f_j of the form (4.3) below, we have

1) $|\psi_1(x,t_i)| \le \max\{|f_1(x)|, |f_2(x)|, |f_3(x)|\}, \quad 0 \le x \le 1, \quad -1 \le \dfrac{x-t_i}{1-xt_i} \le \dfrac{1}{2}$;
2) $|\psi_2(x,t_i)| \le \max\{|f_1(x)|, |f_2(x)|\}, \quad t_1 \le x \le 1$;
3) $|\psi_2(x,t_i)| \le \max\{|f_1(x)|, |f_2(x)|, |f_4(x)|\} \quad 0 \le x \le t_1, \quad \dfrac{1}{2} \le \dfrac{x-t_i}{1-xt_i} \le 1.$

Combined with (4.1), these inequalities imply that

$$\max_i \|\phi_i'\|_{*[0,1]} \le \max_{1\le j\le 4} \|f_j\|_{C[0,1]} \,,$$

and, by symmetry, on the interval $[-1,0]$, we have

$$\max_i \|\phi_i'\|_{*[-1,0]} \le \max_{1\le j\le 4} \|\widetilde{f}_j\|_{C[-1,0]} \,,$$

where $\widetilde{f}_j(x) := f_j(-x)$.

Step 3. The functions $|f_j|$ are of the form

$$|f_j(x)| = |f_j(\omega,x)| = |a_j(x)\omega''(x) + b_j(x)\omega'(x)| + c_j \|\omega'\| \,, \tag{4.3}$$

i.e., they are semi-linear in ω. In § 8, we show that, for $\omega = T_i$, they admit the estimate

$$\|f_j(T_i)\|_{C[0,1]} \le T_i'(1) \qquad \big(\text{ thus } \quad \|\widetilde{f}_j(T_i)\|_{C[-1,0]} \le |T_i'(-1)| = T_i'(1)\ \big).$$

This implies that the same estimate is valid for polynomials ω with positive Chebyshev expansion, i.e., if

$$\omega = \sum a_i T_i, \qquad a_i \ge 0\,, \tag{4.4}$$

then

$$\|f_j(\omega)\|_{C[0,1]} \le \omega'(1)\,.$$

Indeed, since $f(\omega, x)$ is semi-linear in ω, and $a_i \ge 0$, we have

$$\begin{aligned} \|f(\omega)\| = \|f\big(\sum a_i T_i\big)\| &\le \|\sum a_i f(T_i)\| \\ &\le \sum a_i \|f(T_i)\| \le \sum a_i T_i'(1) = \omega'(1). \end{aligned}$$

Hence, for polynomials ω which satisfy (4.2) and (4.4), i.e., for ω from the class Ω, we have

$$\max_i \|\phi_i'\|_* \le \omega'(1),$$

and that concludes the proof of Theorem 3.1.

Remark 4.1. In Step 2, we used the condition $\|\omega\|_{C[0,1]} = \omega(1)$ only in the case 3, when dealing with the function f_4, but we believe that well-behaving majorants for $\|\phi_i\|_*$ of the form (4.3) exist for any ω.

5. Majorants for $\|\phi_i'\|_*$

Set

$$\begin{aligned} \omega(x) &= \prod_{i=1}^{n} (x - t_i)\,, \quad t_i \in [-1,1], \\ \omega_i(x) &= \frac{\omega(x)}{x - t_i}\,, \\ \phi_i(x) &= \frac{1 - x t_i}{x - t_i}\,\omega(x)\,. \end{aligned}$$

We would like to estimate the "local norm" $\|\phi_i'\|_*$, i.e., the largest absolute value of the local extrema of ϕ_i' inside $[0,1]$.

For any (twice differentiable) function f, the function

$$g(x) := f'(x) + c(x) f''(x)$$

interpolates f' at the points of its local extrema, hence

$$\|f'\|_{*[0,1]} \le \|g\|_{[0,1]}\,.$$

Respectively, we are going to construct two majorants for the local extrema of ϕ_i' in the form

$$\psi_\nu(x,t_i) = \phi_i'(x) + c_\nu(x,t_i)\phi_i''(x)\,, \qquad \nu = 1,\,2\,.$$

To this end, set

$$\phi(x,t) := \frac{1-xt}{x-t}\,\omega(x),$$

so that

$$\phi_i^{(k)}(x) = \phi^{(k)}(x,t_i)\,.$$

Since $\frac{1-xt}{x-t} = \frac{1-t^2}{x-t} - t$, we have

$$\phi'(x,t) = \frac{1-xt}{x-t}\,\omega'(x) - \frac{1-t^2}{(x-t)^2}\,\omega(x)\,, \tag{5.1}$$

$$\phi''(x,t) = \frac{1-xt}{x-t}\,\omega''(x) - 2\,\frac{1-t^2}{(x-t)^2}\,\omega'(x) + 2\,\frac{1-t^2}{(x-t)^3}\,\omega(x)\,. \tag{5.2}$$

Lemma 5.1. *Let* $\phi_i''(x) = 0$. *Then*

$$\phi_i'(x) = \psi_1(x,t_i)\,,$$

where

$$\begin{aligned}\psi_1(x,t) &:= \phi'(x,t) + \frac{1}{2}(x-t)\phi''(x,t)\\ &= \frac{1}{2}(1-xt)\,\omega''(x) - t\,\omega'(x)\,.\end{aligned} \tag{5.3}$$

Proof. The proof of (5.3) is straightforward from the definition and (5.1)-(5.2) as

$$\phi'(x,t) = -t\omega'(x) + \frac{1-t^2}{x-t}\,\omega'(x) - \frac{1-t^2}{(x-t)^2}\,\omega(x)\,.$$ □

Lemma 5.2. *Let* $\phi_i''(x) = 0$. *Then*

$$\phi_i'(x) = \psi_2(x,t_i)\,,$$

where

$$\begin{aligned}\psi_2(x,t) &:= \phi'(x,t) + \frac{1}{2}\,\frac{x-t}{1-xt}\,(1-x^2)\,\phi''(x,t)\\ &= \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{x-t}{1-xt}\,\omega'(x) - \frac{x(1-t^2)}{(x-t)(1-xt)}\,\omega(x).\end{aligned} \tag{5.4}$$

Proof. From the definition and expressions (5.1)-(5.2), for the factor at $\omega'(x)$ we obtain

$$\frac{1}{x-t}\Big((1-xt)-\frac{(1-t^2)(1-x^2)}{1-xt}\Big)=\frac{1}{x-t}\,\frac{(x-t)^2}{1-xt}=\frac{x-t}{1-xt}\,,$$

and for the factor at $\omega(x)$

$$-\frac{1-t^2}{(x-t)^2}\Big(1-\frac{1-x^2}{1-xt}\Big)=-\frac{x(1-t^2)}{(x-t)(1-xt)}\,. \qquad \square$$

Proposition 5.1. *For any polynomial ω with all its zeros in $[-1,1]$, we have*

$$\max_i \|\phi_i'\|_{*[0,1]} \le \max_{x\in[0,1]}\ \max_{t_i\in[-1,1]}\ \min\{\psi_1(x,t_i),\psi_2(x,t_i)\}\,,$$

where $\psi_\nu(x,t)$ are given in (5.3)–(5.4).

6. Majorants for $\psi_1(x,t)$ and $\psi_2(x,t)$

6.1. The case $0\le x\le 1,\ \ -1\le \dfrac{x-t_i}{1-xt_i}\le \dfrac{1}{2}$.

Lemma 6.1. *Let*

$$0\le x\le 1, \qquad -1\le \frac{x-t_i}{1-xt_i}\le\frac{1}{2}\,.$$

Then

$$|\psi_1(x,t_i)|\le \max\{|f_1(x)|,\,|f_2(x)|,\,|f_3(x)|\}\,,$$

where

$$f_{1,2}(x):=\frac{1}{2}(1-x^2)\,\omega''(x)\pm\omega'(x)\,,$$

$$f_3(x):=\frac{1-x^2}{2-x}\,\omega''(x)-\frac{2x-1}{2-x}\,\omega'(x)\,.$$

Proof. The function

$$\psi_1(x,t)=\frac{1}{2}(1-xt)\,\omega''(x)-t\,\omega'(x)\,,$$

is linear in t, thus, for any given x and any $t\in[a,b]$, we have the estimate

$$|\psi_1(x,t)|\le\max\{|\psi_1(x,a)|,\,|\psi_1(x,b)|\}\,. \qquad (6.1)$$

The condition $-1\le \dfrac{x-t_i}{1-xt_i}\le\dfrac{1}{2}$ is equivalent to

$$\frac{2x-1}{2-x}\le t_i\le 1,$$

thus, we can use (6.1) with $a = \frac{2x-1}{2-x}$ and $b = 1$. Then $1 - xa = 1 - \frac{x(2x-1)}{2-x} = \frac{2(1-x^2)}{2-x}$, so that

$$\psi_1(x,t)\Big|_{t=a} = \frac{1-x^2}{2-x}\omega''(x) - \frac{2x-1}{2-x}\omega'(x) =: f_3(x)\,,$$

while

$$\psi_1(x,t)\Big|_{t=1} = \frac{1}{2}(1-x)\omega''(x) - \omega'(x) =: g(x)$$

Since $|1-x| \le |1-x^2|$ on the interval $[0,1]$, we clearly have

$$|g(x)| \le \left|\frac{1}{2}(1-x^2)\omega''(x)\right| + |\omega'(x)| = \max\{|f_1(x)|,\,|f_2(x)|\}\,. \qquad \square$$

6.2. The case $t_1 \le x \le 1$.

Lemma 6.2. *Let $-1 \le t_i \le t_1 \le x \le 1$. Then*

$$|\psi_2(x,t_i)| \le \max\{|f_1(x)|,\,|f_2(x)|\}$$

with $f_{1,2}(x)$ as in Lemma 6.1.

Proof. By definition (5.4), we have

$$\psi_2(x,t_i) = \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{x-t_i}{1-xt_i}\,\omega'(x) - \frac{x(1-t_i^2)}{(x-t_i)(1-xt_i)}\,\omega(x)\,.$$

Because $\omega'(x) = \sum \omega_i(x)$, and because $\frac{x(1-t^2)}{1-xt} = \frac{x-t}{1-xt} + t$, we obtain

$$\begin{aligned}\psi_2(x,t_i) &= \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{x-t_i}{1-xt_i}\sum_{j=1}^{n}\omega_j(x) - \frac{x(1-t_i^2)}{1-xt_i}\,\omega_i(x)\\ &= \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{x-t_i}{1-xt_i}\sum_{j\ne i}\omega_j(x) - t_i\,\omega_i(x)\,.\end{aligned}$$

Now, since $|\frac{x-t_i}{1-xt_i}| \le 1$ and $|t_i| \le 1$, the absolute value of the sum of the last two terms does not exceed $\sum_{i=1}^{n}|\omega_i(x)|$. But for $t_1 \le x \le 1$ all the summands $\omega_i(x)$ are positive, hence

$$\sum_{i=1}^{n}|\omega_i(x)| = \sum_{i=1}^{n}\omega_i(x) = \omega'(x)\,.$$

Therefore,

$$|\psi_2(x,t_i)| \le \left|\frac{1}{2}(1-x^2)\,\omega''(x)\right| + |\omega'(x)| = \max\{|f_1(x)|,\,|f_2(x)|\}\,. \qquad \square$$

6.3. The case $0 \le x \le t_1,\ \frac{1}{2} \le \frac{x - t_i}{1 - xt_i} \le 1$.

Consider again the function ψ_2 defined in (5.4):

$$\psi_2(x,t) = \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{x-t}{1-xt}\,\omega'(x) - \frac{x(1-t^2)}{(x-t)(1-xt)}\,\omega(x)\,.$$

Lemma 6.3. *Let*

$$0 \le x \le t_1, \qquad \frac{1}{2} \le \frac{x-t}{1-xt} \le 1, \qquad \|\omega\|_{C[0,1]} = \omega(1)\,.$$

Then

$$|\psi_2(x,t_i)| \le \max\{|f_1(x)|,\,|f_2(x)|,\,|f_4(x)|\}\,, \tag{6.2}$$

with $f_{1,2}(x)$ as in Lemma 6.1, and

$$f_4(x) := \Big|\frac{1}{2}(1-x^2)\,\omega''(x) + \frac{1}{2}\omega'(x)\Big| + \frac{1}{4}\,\|\omega'\|\,. \tag{6.3}$$

Proof. For a fixed $x \in [0,t_1]$, set

$$\gamma := \frac{x-t}{1-xt}\,.$$

Since $(1-xt)^2 = (x-t)^2 + (1-x^2)(1-t^2)$, we have

$$\frac{x(1-t^2)}{(x-t)(1-xt)} = \frac{\frac{(1-t^2)(1-x^2)}{(1-xt)^2}}{\frac{x-t}{1-xt}}\,\frac{x}{1-x^2} = \frac{1-\gamma^2}{\gamma}\,\frac{x}{1-x^2}\,,$$

and therefore, for a fixed x,

$$\psi_2(x,t) := \psi(\gamma) := \frac{1}{2}(1-x^2)\,\omega''(x) + \gamma\,\omega'(x) - \frac{1-\gamma^2}{\gamma}\,\frac{x}{1-x^2}\,\omega(x)\,. \tag{6.4}$$

For $\gamma \in [\frac{1}{2},1]$, the maximum of $\psi(\gamma)$ is attained either at the endpoints, or at the points where $\psi'(\gamma) = 0$. In the latter case,

$$\psi'(\gamma) = \omega'(x) + \frac{1+\gamma^2}{\gamma^2}\,\frac{x}{1-x^2}\,\omega(x) = 0,$$

hence, $\frac{x}{1-x^2}\,\omega(x) = -\frac{\gamma^2}{1+\gamma^2}\,\omega'(x)$, and putting this expression into (6.4) we obtain

$$\begin{aligned}\psi(\gamma) &= \frac{1}{2}(1-x^2)\,\omega''(x) + \Big(\gamma + \frac{\gamma(1-\gamma^2)}{1+\gamma^2}\Big)\,\omega'(x)\\ &= \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{2\gamma}{1+\gamma^2}\,\omega'(x)\,.\end{aligned}$$

So, at the points where $\psi'(\gamma) = 0$, we have

$$|\psi(\gamma)| \le \Big|\frac{1}{2}(1-x^2)\,\omega''(x)\Big| + |\omega'(x)| = \max\{|f_1(x)|,\, |f_2(x)|\}\,.$$

As to the values of $\psi(\gamma)$ in (6.4) at the endpoints of $[\frac{1}{2}, 1]$, they are

$$\begin{aligned} \psi(\gamma)\big|_{\gamma=1} &= \frac{1}{2}(1-x^2)\,\omega''(x) + \omega'(x) = f_1(x), \\ \psi(\gamma)\big|_{\gamma=\frac{1}{2}} &= \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{1}{2}\omega'(x) - \frac{3}{2}\,\frac{x}{1-x^2}\,\omega(x) =: g(x), \end{aligned} \tag{6.5}$$

and it remains to show that $|g(x)| \le |f_4(x)|$. The functions g and f_4 in (6.3) differ only in the last term which is $-\frac{3}{2}\frac{x}{1-x^2}\,\omega(x)$ for $g(x)$ and $\frac{1}{4}\,\|\omega'\|$ for $f_4(x)$. By the forthcoming Lemma 7.4,

$$\|\omega\|_{C[0,1]} = \omega(1) \quad \Rightarrow \quad \Big|\frac{\omega(x)}{1-x}\Big| \le \frac{1}{3}\,\|\omega'\|\,, \qquad 0 \le x \le t_1,$$

and because $\frac{x}{1+x} \le \frac{1}{2}$ on $[0, 1]$, we have

$$\Big|\frac{3}{2}\,\frac{x}{1-x^2}\,\omega(x)\Big| = \frac{3}{2}\,\frac{x}{1+x}\,\Big|\frac{\omega(x)}{1-x}\Big| \le \frac{1}{4}\|\omega'\|\,. \qquad \square$$

7. Estimates for $\left|\frac{\omega(x)}{1-x}\right|$, $0 \le x \le t_1$.

Lemma 7.1. *For any $\gamma \in [0, 1]$, and for any $c > 0$, we have*

$$\min\{c\gamma, 1-\gamma\} \le \frac{c}{c+1}\,.$$

Proof. When γ runs through $[0, 1]$, the value $c\gamma$ is increasing from zero, while the value $1 - \gamma$ is decreasing to zero. So, there is a γ_* for which both values coincide, and for this γ_* (equal to $1/(c + 1)$) we have

$$\min\{c\gamma,\, 1-\gamma\} \le c\gamma_* = \frac{c}{c+1}\,. \qquad \square$$

Lemma 7.2. *Let $\|\omega\|_{C[0,1]} = \omega(1)$, and let $0 < x < t_m < 1$. Then*

$$\Big|\frac{\omega(x)}{1-x}\Big| \le \frac{1}{m+1}\,\|\omega'\|.$$

Proof. Since $\omega(t_m) = 0$, we have $|\omega(x)| = |\int_{t_m}^{x} \omega'| \le (t_m - x)\|\omega'\|$, hence

$$\Big|\frac{\omega(x)}{1-x}\Big| \le \frac{t_m - x}{1-x}\,\|\omega'\|\,.$$

On the other hand, since $|\omega(x)| \le \omega(1)$ and $0 < x < t_m < 1$, we have

$$\Big|\frac{\omega(x)}{1-x}\Big| \le \frac{1-t_m}{1-x}\,\frac{\omega(1)}{1-t_m} \le \frac{1-t_m}{1-x}\,\frac{1}{m}\sum_{i=1}^{m}\frac{\omega(1)}{1-t_i} \le \frac{1-t_m}{1-x}\,\frac{1}{m}\sum_{i=1}^{n}\frac{\omega(1)}{1-t_i}$$
$$= \frac{1-t_m}{1-x}\,\frac{1}{m}\,\omega'(1)\,.$$

So,

$$\Big|\frac{\omega(x)}{1-x}\Big| \le \min\Big\{\frac{1}{m}\,\frac{1-t_m}{1-x}\,,\,\frac{t_m-x}{1-x}\Big\}\,\|\omega'\| \le \frac{1}{m+1}\|\omega'\|\,,$$

the latter inequality by Lemma 7.1, with $\gamma = \frac{1-t_m}{1-x}$ and $c = \frac{1}{m}$. □

If x is located between t_2 and t_1, then Lemma 7.2 gives the inequality $|\frac{\omega(x)}{1-x}| \le \frac{1}{2}\|\omega'\|$ which is not strong enough for our purposes. The next lemma improves it.

Lemma 7.3. *Let* $\|\omega\|_{C[0,1]} = \omega(1)$*, and let*

$$0 \le x \le 1, \qquad t_2 < x < t_1 < 1.$$

Then

$$\Big|\frac{\omega(x)}{1-x}\Big| \le \gamma\,\omega'(1), \qquad \gamma = \frac{2-\sqrt{2}}{2} < \frac{1}{3}\,. \tag{7.1}$$

Proof. Let s_1 be the rightmost zero of $\omega'(x)$, i.e.

$$t_2 < s_1 < t_1, \qquad \omega'(s_1) = 0,$$

where t_i are zeros of ω in the reverse order. Clearly, the ratio $\frac{\omega(x)}{1-x}$ attains its maximal absolute value for some x in $[s_1, t_1]$, and we will distinguish two cases for location of s_1:

$$1)\quad 0 \le s_1 \le x < t_1 < 1\,, \qquad 2)\quad s_1 < 0 \le x < t_1 < 1\,.$$

Case 1 ($0 \le s_1 \le x < t_1 < 1$).

1) By assumption, $|\omega(x)| \le \omega(1)$ on the interval $[s_1, t_1]$ so that, assuming that $\omega(1) = 1$ we let $\omega(s_1) = -a$ with some $a \in (0, 1]$.

Let p be the quadratic polynomial that interpolates ω at the points $(s_1, s_1, 1)$, i.e.

$$p(s_1) = \omega(s_1) = -a, \qquad p'(s_1) = \omega'(s_1) = 0, \qquad p(1) = \omega(1) = 1.$$

Then, for $x \in [s_1, 1]$, the Lagrange interpolation formula provides

$$\omega(x) - p(x) = \frac{1}{2!}(x-s_1)^2(x-1)\omega'''(\xi), \qquad \xi \in [s_1, 1]$$

and since $\omega'''(\xi) > 0$ for $\xi > s_1$, it follows that

$$\omega(x) \le p(x), \quad x \in [s_1, 1], \qquad p'(1) < \omega'(1)\,.$$

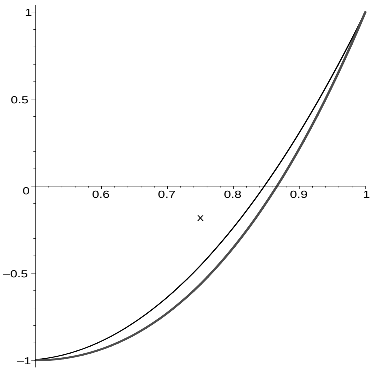

Figure 1: The graphs of ω and p

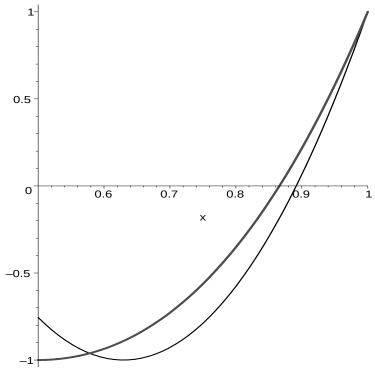

Figure 2: The graphs of ω and q

2) Let $q \in \mathcal{P}_2$ be a quadratic polynomial defined by the conditions

$$q(1) = \omega(1) = 1, \qquad q'(1) = \omega'(1), \qquad q(s_1^*) := \inf_x q(x) = -a\,, \tag{7.2}$$

and let t_1^* be its zero in the interval $[s_1^*, 1]$

$$q(t_1^*) = 0, \qquad t_1^* \in [s_1^*, 1].$$

This q is a dilation of p, namely

$$q(x) = p(\lambda x - (\lambda - 1)), \qquad \lambda := \omega'(1)/p'(1) > 1,$$

and, because the squeezing coefficient λ is greater than one, we conclude that

$$s_1 < s_1^* \qquad (\text{where } \omega(s_1) = p(s_1) = q(s_1^*) = -a).$$

When x runs through $[s_1, s_1^*]$, the value $\omega(x)$ is increasing from $\omega(s_1) = -a$, while the value $q(x)$ is decreasing to $q(s_1^*) = -a$. So, there is a point $x^* \in (s_1, s_1^*)$ for which both values coincide:

$$q(x^*) = \omega(x^*), \qquad s_1 < x^* < s_1^* < 1.$$

We see that q interpolates ω at $(x_1^*, 1, 1)$, hence, by the Lagrange interpolation formula, for $x \in [x^*, 1]$,

$$\omega(x) - q(x) = \frac{1}{2!}(x - x^*)(x-1)^2\omega'''(\xi), \qquad \xi \in [x^*, 1],$$

and we have $q(x) < \omega(x)$, for $x^* < x < 1$, in particular,

$$q(x) < \omega(x), \qquad s_1^* < x < 1\,.$$

3) It follows that, for any $x \in [s_1, t_1]$, there is a point $y_x \in [s_1^*, t_1^*]$ such that

$$\omega(x) = q(y_x) < 0, \qquad x < y_x < t_1^*\,,$$

whence

$$\Big|\frac{\omega(x)}{1-x}\Big| = \Big|\frac{q(y_x)}{1-x}\Big| < \Big|\frac{q(y_x)}{1-y_x}\Big|\,.$$

So, if

$$\max_{y\in[s_1^*,t_1^*]} \Big|\frac{q(y)}{1-y}\Big| \le \gamma q'(1), \tag{7.3}$$

then, because $q'(1) = \omega'(1)$,

$$\max_{x\in[s_1,t_1]} \Big|\frac{\omega(x)}{1-x}\Big| \le \gamma\omega'(1)\,.$$

4) Finally, let us find the least constant γ in (7.3) for the quadratic polynomial q given by conditions (7.2), i.e.,

$$q(1) = 1, \qquad q(t_1^*) = 0, \qquad q'(s_1^*) = 0, \qquad q(s_1^*) = -a\,.$$

It is easy to see that γ is maximized if $-a = -1$, and that its value does not depend on the position of s_1^*, so we may take $s_* = 0$, and consider inequality (7.3) just for the polynomial

$$q(y) = 2y^2 - 1, \qquad y \in [0, \tfrac{1}{\sqrt{2}}]\,.$$

For such a q, we have

$$\Big[\frac{q(x)}{1-x}\Big]' = 0 \;\Rightarrow\; q'(x)(1-x)+q(x) = 0 \;\Rightarrow\; 2x^2-4x+1 = 0 \;\Rightarrow\; x_0 = \frac{2-\sqrt{2}}{2}\,,$$

whence

$$\gamma = \frac{1}{q'(1)}\Big|\frac{q(x_0)}{1-x_0}\Big| = \Big|\frac{q'(x_0)}{q'(1)}\Big| = \frac{4x_0}{4} = \frac{2-\sqrt{2}}{2} < \frac{1}{3}\,.$$

Case 2 ($s_1 < 0 \le x < t_1 < 1$).

Recall that t_i are zeros of the polynomial $\omega \in \mathcal{P}_n$ in the reverse order, and s_1 is the rightmost zero of its first derivative ω', so that in the case under consideration we have

$$-1 \le t_n \le \cdots \le t_2 \le s_1 \le 0 \le t_1 \le 1\,. \tag{7.4}$$

1) Let us find out what the rightmost position of $t_1 \in [0,1]$ could be if we require that $s_1 \le 0$. It is known that, for a polynomial $\omega(x) = \prod_{i=1}^n (x - t_i)$,

the zeros s_i of ω' are monotonically increasing functions of t_i. Therefore, with $t_1 \in [0,1]$ fixed, the leftmost position s_1^* of s_1 is attained for the polynomial ω_* with the leftmost positions of all other zeros from t_2 to t_n, which are -1, i.e.,

$$\omega_*(x) = (x - t_1)(x+1)^{n-1}.$$

Then

$$\omega_*'(x) = (x+1)^{n-2}(x + 1 + (n-1)(x - t_1))$$

and its first zero s_1^* from the right satisfies

$$t_1 = \frac{ns_1^* + 1}{n-1}.$$

So, for any polynomial $\omega \in \mathcal{P}_n$ which satisfies (7.4), we have

$$s_1^* < s_1 \le 0 \quad \Rightarrow \quad t_1 \le \frac{1}{n-1} \le \frac{1}{2}, \qquad n \ge 3.$$

2) Now, consider the ratio

$$\gamma = \sup_{\omega} \sup_{x \in [0,t_1]} \frac{\omega(x)}{1-x} \frac{1}{\omega'(1)},$$

where zeros of ω satisfy

$$1 \le t_n \le \cdots \le t_2 \le 0 \le t_1 \le \frac{1}{2}. \tag{7.5}$$

Since $\omega'(1) \ge (1-t_2)\cdots(1-t_n)$, we have

$$\gamma \le \sup_{t_i \in (7.5)} \sup_{x \in [0,t_1]} \frac{t_1 - x}{1-x} \frac{x - t_2}{1-t_2} \cdots \frac{x - t_n}{1-t_n}.$$

The first factor satisfies $\frac{t_1-x}{1-x} \le \frac{1}{2}$, with equality when $x = 0$ and $t_1 = \frac{1}{2}$. The remaining factors satisfy $\frac{x-t_i}{1-t_i} \le \frac{3}{4}$, with equality when $x = t_1 = \frac{1}{2}$ and $t_i = -1$. So, in Case 2,

$$\gamma \le \frac{1}{2}\Big(\frac{3}{4}\Big)^{n-1} \le \frac{1}{2}\Big(\frac{3}{4}\Big)^{2} = \frac{9}{32} < \frac{1}{3}, \qquad n \ge 3. \qquad \square$$

On combining Lemma 7.2 (for $x < t_2$) and Lemma 7.3 (for $t_2 < x < t_1$) we obtain the following statement which we used in proving Lemma 6.3.

Lemma 7.4. *Let* $\|\omega\|_{C[0,1]} = \omega(1)$, *and let* $0 \le x \le t_1 < 1$. *Then*

$$\Big|\frac{\omega(x)}{1-x}\Big| \le \frac{1}{3}\,\|\omega'\|.$$

8. Proof of Theorem 3.1

1) We summarize the results of §5-§6 in the following statement.

Theorem 8.1. *Let ω satisfy the following condition:*

$$1a)\qquad \max_{x\in[0,1]} |\omega(x)| = \omega(1)\,.$$

Then

$$\max_i \|\phi_i'\|_{*[0,1]} \le \max_{1\le j\le 4} \|f_j(\omega;\cdot)\|_{C[0,1]},$$

where

$$\begin{aligned} f_1(\omega;x) &:= \frac{1}{2}(1-x^2)\,\omega''(x) + \omega'(x)\,, \\ f_2(\omega;x) &:= \frac{1}{2}(1-x^2)\,\omega''(x) - \omega'(x)\,, \\ f_3(\omega;x) &:= \frac{1-x^2}{2-x}\,\omega''(x) - \frac{2x-1}{2-x}\,\omega'(x)\,, \\ f_4(\omega;x) &:= \Big|\frac{1}{2}(1-x^2)\,\omega''(x) + \frac{1}{2}\,\omega'(x)\Big| + \frac{1}{4}\,\|\omega'\|\,. \end{aligned} \tag{8.1}$$

By symmetry, on the other half of the interval $[-1,1]$ we obtain the following statement.

Theorem 8.2. *Let ω satisfy the following condition:*

$$1b)\qquad \max_{x\in[-1,0]} |\omega(x)| = |\omega(-1)|\,.$$

Then

$$\max_i \|\phi_i'\|_{*[-1,0]} \le \max_{1\le j\le 4} \|\widetilde{f}_j(\omega;\cdot)\|_{C[-1,0]},$$

where

$$\begin{aligned} \widetilde{f}_1(\omega;x) &:= \frac{1}{2}(1-x^2)\,\omega''(x) - \omega'(x)\,, \\ \widetilde{f}_2(\omega;x) &:= \frac{1}{2}(1-x^2)\,\omega''(x) + \omega'(x)\,, \\ \widetilde{f}_3(\omega;x) &:= \frac{1-x^2}{2+x}\,\omega''(x) - \frac{2x+1}{2+x}\,\omega'(x)\,, \\ \widetilde{f}_4(\omega;x) &:= \Big|\frac{1}{2}(1-x^2)\,\omega''(x) - \frac{1}{2}\,\omega'(x)\Big| + \frac{1}{4}\,\|\omega'\|\,. \end{aligned} \tag{8.2}$$

2) In order to complete the proof of Theorem 3.1 (see Step 3 in §4), we need to prove the following statement.

Theorem 8.3. *If* $\omega = T_n$*, then*

$$\|f_j(T_n;\cdot)\|_{[0,1]} \le T_n'(1) = n^2, \qquad j = 1,\, 2,\, 3,\, 4\,.$$

(That implies the same estimate for $\|\widetilde{f}_j(T_n;\cdot)\|_{C[-1,0]}$*).*

We will provide the proof in several lemmas, first for f_3 and f_1, and then for f_2 and f_4. (For the sake of brevity, we write below $f_j(x)$ instead of $f_j(T_n;x)$).

Lemma 8.1. *We have*

$$|f_3(x)| := \Big|\frac{1-x^2}{2-x}\,T_n''(x) - \frac{2x-1}{2-x}\,T_n'(x)\Big| \le T_n'(1), \qquad x \in [0,1]. \tag{8.3}$$

Proof. Since

$$(x^2-1)T_n''(x) + xT_n'(x) = n^2 T_n(x), \tag{8.4}$$

the left-hand side becomes

$$\begin{aligned}\frac{1-x^2}{2-x}\,T_n''(x) - \frac{2x-1}{2-x}\,T_n'(x) &= \frac{(1-x^2)T_n''(x) - xT_n'(x)}{2-x} + \frac{1-x}{2-x}\,T_n'(x)\\ &= \frac{(1-x)T_n'(x) - n^2T_n(x)}{2-x}\,,\end{aligned}$$

and our inequality is equivalent to

$$\big|(1-x)T_n'(x) - n^2T_n(x)\big| \le (2-x)\|T_n'\| = (2-x)n^2.$$

The latter is obvious because $|T_n'(x)| \le n^2$ and $|T_n(x)| \le 1$. □

Lemma 8.2. *We have*

$$|f_2(x)| := \Big|\frac{1}{2}(1-x^2)\,T_n''(x) - T_n'(x)\Big| \le T_n'(1), \qquad x \in [0,1]. \tag{8.5}$$

Proof. By (8.4), our inequality is equivalent to

$$\Big|\frac{1}{2}(x^2-1)T_n''(x) + \frac{x}{2}T_n'(x) + \Big(1-\frac{x}{2}\Big)\,T_n'(x)\Big| = \Big|\frac{n^2}{2}T_n(x) + \Big(1-\frac{x}{2}\Big)\,T_n'(x)\Big| \le n^2,$$

and we are done once we prove that $\big|(1-\frac{x}{2})\,T_n'(x)\big| \le \frac{n^2}{2}$, that is

$$|T_n'(x)| \le \frac{n^2}{2-x} =: g(x).$$

The function g is convex and monotonically increasing on $[0,1]$, moreover $g(1) = g'(1) = n^2$, hence

$$g(x) \ge g(0) = \frac{n^2}{2}\,, \qquad g(x) \ge n^2 x.$$

1) If $x \in [x_0, 1]$, where $x_0 := \cos\frac{\pi}{n}$ is the rightmost zero of T_n', then T_n' is convex and $T_n'(x)$ varies monotonically from 0 to n^2, hence

$$0 \le T_n'(x) \le n^2 \frac{x - x_0}{1 - x_0} \le n^2 x \le g(x)\,.$$

2) If $x \in [0, x_0]$, then using Bernstein's inequality and the inequality $\sin t \ge \frac{2}{\pi} t$ for $t \in [0, \frac{\pi}{2}]$, we obtain

$$|T_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \le \frac{n}{\sqrt{1-x_0^2}} = \frac{n}{\sin\frac{\pi}{n}} \le \frac{n}{2/n} = \frac{n^2}{2} \le g(x)\,, \qquad n \ge 2.$$

3) Finally, for $n = 0, 1$, both sides of (8.5) are identical. □

Lemma 8.3. *Let* $\gamma \in [0, 2]$. *Then*

$$|g_\gamma(x)| := \left|(1-x^2)\,T_n''(x) + \gamma T_n'(x)\right| \le \frac{2}{\sqrt{3-\gamma}}\, T_n'(1), \qquad x \in [0,1]. \tag{8.6}$$

The proof of Lemma 8.3 is based on the next two lemmas.

Lemma 8.4. *Let*

$$g_\gamma(x) := (1-x^2)\,T_n''(x) + \gamma T_n'(x)\,, \qquad \gamma \in [0,2]\,.$$

Then, at the points $x \in [0,1]$ *where* $g_\gamma'(x) = 0$ *and* $x + \gamma > 0$, *we have*

$$|g_\gamma(x)| \le n^2 \sqrt{G_\gamma(y_x)},$$

where

$$G_\gamma(y_x) := \frac{(\gamma + y_x)^2}{1 + (\gamma - 1) y_x + y_x^2}\,, \qquad y_x := \frac{(n^2-1)(1-x^2)}{x+\gamma} \ge 0\,. \tag{8.7}$$

Proof. From the differential equation

$$(1-x^2) T_n''(x) = x T_n'(x) - n^2 T_n(x), \tag{8.8}$$

it follows that

$$g_\gamma(x) = (x+\gamma) T_n'(x) - n^2 T_n(x), \tag{8.9}$$

and, respectively,

$$g_\gamma'(x) = (x+\gamma)\,T_n''(x) - (n^2-1) T_n'(x).$$

If $g_\gamma'(x) = 0$, then

$$T_n''(x) = \frac{n^2-1}{x+\gamma}\, T_n'(x)\,.$$

Multiplying both sides with $(1-x^2)$ and using (8.8), we obtain

$$x\,T_n'(x) - n^2\,T_n(x) = \frac{(n^2-1)(1-x^2)}{x+\gamma}\,T_n'(x) =: y_x T_n'(x)$$

or, after rearrangement,

$$n^2\,T_n(x) = (x-y_x)\,T_n'(x). \tag{8.10}$$

Putting this expression into (8.9), we find that, at the points of local extrema,

$$g_\gamma(x) = (\gamma+y_x)\,T_n'(x). \tag{8.11}$$

Next, we square (8.10), and substitute the left-hand side by

$$n^4 T_n(x)^2 = n^4 - n^2\,(1-x^2)T_n'(x)^2\,,$$

and that gives

$$n^4 = [n^2\,(1-x^2) + (x-y_x)^2]\,T_n'(x)^2.$$

This formula expresses the value $T_n'(x)$ in terms of x, and we put this expression instead of $T_n'(x)$ into the right-hand side of (8.11) to obtain

$$g_\gamma(x)^2 = n^4\frac{(\gamma+y_x)^2}{n^2\,(1-x^2)+(x-y_x)^2}\,.$$

Finally, from definition (8.7) of y_x it follows that $n^2(1-x^2) = (x+\gamma)y_x + (1-x^2)$, so for the denominator $D_\gamma(x)$ in the expression above we have the estimate

$$D_\gamma(x) = (x+\gamma)y_x + (1-x^2) + (x-y_x)^2 = 1 + (\gamma-x)y_x + y_x^2 \ge 1 + (\gamma-1)y_x + y_x^2\,.$$

That proves (8.7). □

Lemma 8.5. *Let*

$$G_\gamma(y) := \frac{(\gamma+y)^2}{1+(\gamma-1)y+y^2}\,. \tag{8.12}$$

Then, for any $y \in [0,\infty]$ *and for any* $\gamma \in [0,2]$,

$$G_\gamma(y) \le \frac{4}{3-\gamma}\,.$$

Proof. For a fixed $\gamma \in [0,2]$, we need to determine the maximum of the value $G_\gamma(y)$ over $y \ge 0$. We have

$$G_\gamma(0) = \gamma^2, \qquad G_\gamma(\infty) = 1\,,$$

while differentiation with respect to y gives

$$\begin{aligned} G'_\gamma(y) = 0 \ &\Rightarrow\ (\gamma + y)\left[2(1 + (\gamma - 1)y + y^2) - (\gamma + y)((\gamma - 1) + 2y)\right] = 0 \\ &\Leftrightarrow\ (\gamma + y)\left[2 - (1 + \gamma)y - \gamma(\gamma - 1)\right] = 0\,. \end{aligned}$$

From the two roots

$$y_1 = -\gamma, \qquad y_2 = \frac{2 + \gamma - \gamma^2}{1 + \gamma} = 2 - \gamma\,,$$

only the second one should be considered, and we have

$$G_\gamma(y_2) = \frac{4}{1 + (\gamma - 1)(2 - \gamma) + (2 - \gamma)^2} = \frac{4}{3 - \gamma}\,.$$

So, for $\gamma \in [0, 2]$, we have

$$G_\gamma(y) \le \max\left\{\gamma^2, 1, \frac{4}{3 - \gamma}\right\}.$$

Now, clearly $\frac{4}{3-\gamma} \ge 1$, and we also have $\frac{4}{3-\gamma} \ge \gamma^2$, because

$$4 - (3 - \gamma)\gamma^2 = 4 - 3\gamma^2 + \gamma^3 = (1 + \gamma)(4 - 4\gamma + \gamma^2) = (1 + \gamma)(2 - \gamma)^2 \ge 0\,,$$

hence

$$G_\gamma(y) \le \frac{4}{3 - \gamma}\,. \qquad \square$$

Proof of Lemma 8.3. For $x + \gamma > 0$ inequality (8.6) follows from the preceding two lemmas. The case $x = \gamma = 0$ is verified directly. $\square$

Corollary 8.1. *We have*

$$|f_1(x)| := \left|\frac{1}{2}(1 - x^2)T''_n(x) + T'_n(x)\right| \le T'_n(1), \tag{8.13}$$

$$|f_4(x)| := \left|\frac{1}{2}(1 - x^2)T''_n(x) + \frac{1}{2}T'_n(x)\right| + \frac{1}{4}\|T'_n\|_{[0,1]} \le T'_n(1).$$

Proof. For the first inequality we apply the estimate (8.6) with $\gamma = 2$,

$$|f_1(x)| = \frac{1}{2}\left|g_\gamma(x)\right|_{\gamma=2} \le T'_n(1),$$

For the second one, the same estimate with $\gamma = 1$ gives

$$\left|\frac{1}{2}(1 - x^2)T''_n(x) + \frac{1}{2}\,T'_n(x)\right| = \frac{1}{2}\left|g_\gamma(x)\right|_{\gamma=1} \le \frac{1}{\sqrt{2}}\,T'_n(1),$$

and, because $\frac{1}{\sqrt{2}} < \frac{3}{4}$, we obtain $|f_4(x)| \le \left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right) T'_n(1) \le T'_n(1)$. $\square$

9. Derivatives of $(x^2-1)^m T_n(x)$ and $(x^2-1)^m T_n'(x)$

In this section we find the orders k of the derivatives of $f(x) := (x^2 - 1)^m T_n(x)$ and $g(x) := (x^2-1)^m T_n'(x)$ that have positive Chebyshev expansions. These results will be used in the next section for establishing the same property for the derivatives of certain snake-polynomials.

Lemma 9.1. *Let*

$$f(x) := (x^2-1)^m T_n(x).$$

Then

$$f^{(k)}(x) = \sum a_i T_i(x), \quad a_i \ge 0 \ \ \forall n \qquad \Leftrightarrow \qquad k \ge 2m\,.$$

Proof. We will use the fact that both $T_n'(x)$ and $xT_n(x)$ have positive Chebyshev expansions.

1a) For $m = 1$, we use also the differential equation for T_n to obtain

$$\begin{aligned} [(x^2-1)\,T_n(x)]'' &= (x^2-1)\,T_n''(x) + 4x\,T_n'(x) + 2\,T_n(x) \\ &= (n^2+2)\,T_n(x) + 3x\,T_n'(x) \\ &= \sum a_j T_j(x)\,, \qquad a_j \ge 0\,. \end{aligned}$$

1b) And for $m \ge 2$,

$$\begin{aligned} [(x^2-1)^m\,T_n(x)]'' &= (x^2-1)^m\,T_n''(x) + 4x\,m(x^2-1)^{m-1}\,T_n'(x) \\ &\quad + \left[2m(x^2-1)^{m-1} + 4x^2\,m(m-1)(x^2-1)^{m-2}\right]\,T_n(x) \\ &= (x^2-1)^{m-1}\left\{(x^2-1)\,T_n''(x) + 4mx\,T_n'(x) + [2m + 4m(m-1)]\,T_n\right\} \\ &\quad + \ (x^2-1)^{m-2}\,4m(m-1)\,T_n(x) \\ &= (x^2-1)^{m-1}\sum a_j T_j + (x^2-1)^{m-2}\sum b_j T_j\,, \qquad a_j, b_j \ge 0\,, \end{aligned}$$

so that

$$\begin{aligned} \left[(x^2-1)^m\,T_n(x)\right]^{(2m)} &= \left\{\ \left[(x^2-1)^m T_n(x)\right]''\ \right\}^{(2(m-1))} \\ &= \left\{(x^2-1)^{m-1}\sum a_j T_j + (x^2-1)^{m-2}\sum b_j T_j\right\}^{(2(m-1))}, \end{aligned}$$

and we apply the induction hypothesis to the last two terms.

2) Now, let us prove that condition $k \ge 2m$ is also necessary for $f^{(k)}$ to have a positive Chebyshev expansion, if n is big enough. We shall use the fact that if $f^{(k)}$ possesses positive Chebyshev expansion, then necessarily $f^{(k)}(0) \le f^{(k)}(1)$. We have

$$f^{(k)}(x) = \sum_{s=0}^{k} \binom{k}{s} \left[(x^2-1)^m\right]^{(s)} \cdot T_n^{(k-s)}(x),$$

and since (for $n = s \mod 2$)

$$T_n^{(s)}(0) = \mathcal{O}(n^s), \qquad T_n^{(s)}(1) = \mathcal{O}(n^{2s})\,,$$

we have

$$f^{(k)}(0) = (x^2-1)^m\big|_{x=0}\cdot T_n^{(k)}(0) + \cdots = \mathcal{O}(n^k)\,,$$

while

$$f^{(k)}(1) \;=\; \binom{k}{m}[(x^2-1)^m]^{(m)}\big|_{x=1}\cdot T_n^{(k-m)}(1) + \cdots \;=\; \mathcal{O}(n^{2k-2m})\,.$$

Hence

$$|f^{(k)}(0)| \le |f^{(k)}(1)|, \quad n \ge n_0 \quad \Leftrightarrow \quad k \le 2k-2m \quad \Leftrightarrow \quad 2m \le k\,. \qquad \square$$

Lemma 9.2. *Let*

$$g(x) := (x^2-1)^m T_n'(x).$$

Then

$$g^{(k)}(x) = \sum a_i T_i(x), \quad a_i \ge 0 \quad \forall\, n \qquad \Leftrightarrow \qquad k \ge 2m-1\,.$$

1a) Similarly to the previous case, for $m=1$ we obtain

$$\begin{aligned}[(x^2-1)\,T_n'(x)]' &= (x^2-1)\,T_n''(x) + 2xT_n'(x) = n^2\,T_n(x) + x\,T_n'(x)\\ &= \sum a_j T_j(x), \qquad a_j \ge 0.\end{aligned}$$

1b) And for $m \ge 2$,

$$[(x^2-1)^m\,T_n'(x)]'' = (x^2-1)^{m-1}\sum a_j T_j'(x) + (x^2-1)^{m-2}\sum b_j T_j'(x)$$

with a_j, $b_j \ge 0$, so that

$$\begin{aligned}\big[(x^2-1)^m\,T_n'(x)\big]^{(2m-1)} &= \Big\{(x^2-1)^{m-1}\sum a_j T_j'(x)\Big\}^{(2(m-1)-1)}\\ &\quad + \Big\{(x^2-1)^{m-2}\sum b_j T_j'(x)\Big\}^{(2(m-1)-1)},\end{aligned}$$

and we apply the induction assumption to the last two terms.

2) Necessity. We have, for $n = k \mod 2$

$$g^{(k)}(0) = \mathcal{O}(n^{k+1}), \qquad g^{(k)}(1) = \mathcal{O}(n^{2(k+1)-2m}),$$

hence

$$|g^{(k)}(0)| \le |g^{(k)}(1)|, \;\; n \ge n_0 \quad \Leftrightarrow \quad k+1 \le 2(k+1)-2m \quad \Leftrightarrow \quad 2m-1 \le k. \;\square$$

10. Duffin-Schaeffer Inequalities for Various Majorants

10.1. Requisites

The material in this subsection is borrowed from Vidensky [15].

1) Let R_{2m} be a polynomial of degree $\le 2m$, which is non-negative on $[-1, 1]$, i.e.

$$R_{2m} \in \mathcal{P}_{2m}, \quad R_{2m}(x) > 0, \qquad x \in [-1, 1].$$

Then, for any $n \ge m$, it can be represented in the form

$$R_{2m}(x) = P_n^2(x) + (1 - x^2)Q_{n-1}^2(x),$$

where P_n and Q_{n-1} satisfy the following conditions:

a) $P_n \in \mathcal{P}_n$ and $Q_{n-1} \in \mathcal{P}_{n-1}$;
b) all zeros of P_n and Q_{n-1} lie in $[-1, 1]$ and interlace;
c) the leading coefficients of P_n and Q_{n-1} are positive;
d) P_n is the snake-polynomial for $\mu = \sqrt{R_{2m}}$.

Moreover,

$$\begin{aligned} P_{m+n}(x) &= \mathrm{Re}\left[P_m(x) + i\sqrt{1-x^2}Q_{m-1}(x)\right]\left[T_n(x) + i\sqrt{1-x^2}U_{n-1}(x)\right] \\ &= P_m(x)T_n(x) + (x^2 - 1)Q_{m-1}(x)U_{n-1}(x)\,, \end{aligned}$$

2) For $n \ge m$, the polynomials P_n satisfy three-term recurrence relation

$$P_{n+1}(x) = 2x\, P_n(x) - P_{n-1}(x)$$

and they are orthogonal with respect to the weight function $\frac{1}{\mu(x)} \frac{1}{\sqrt{1-x^2}}$, i.e.,

$$\int_{-1}^{1} x^k\, P_n(x) \frac{1}{\mu(x)} \frac{dx}{\sqrt{1-x^2}} = 0, \qquad k = 0, \dots, n-1, \quad n \ge m.$$

3) For the special case

$$R_{2m}(x) = \prod_{j=1}^{m} (1 + (a_j^2 - 1)x^2),$$

the formula for P_{m+n} takes the form

$$P_{m+n}(x) = \mathrm{Re}\left\{ \prod_{j=1}^{m} \left(a_j x + i\sqrt{1-x^2}\right)\left[T_n(x) + i\sqrt{1-x^2}U_{n-1}(x)\right] \right\}, \qquad (10.1)$$

where $U_m = T'_{m+1}/(m+1)$ is the m-th Chebyshev polynomial of the second kind.

3) Also, for the majorants $\mu = \sqrt{R_{2m}}$, Vidensky [15] established the following bound in the pointwise Markov inequality:

$$m_{k,\mu}(x) := \sup_{|p(x)|\le\mu(x)} |p^{(k)}(x)| \le V_k(x)\,,$$

where

$$V_k(x) = \left|\big(P_{m+n}(x) + i\sqrt{1-x^2}Q_{m+n-1}\big)^{(k)}\right|.$$

In particular,

$$V_1(x) = \sqrt{\frac{[nP(x)+xQ(x)+(x^2-1)Q'(x)]^2+(1-x^2)[P'(x)+nQ(x)]^2}{1-x^2}}\,, \quad (10.2)$$

where $P = P_m$, $Q = Q_{m-1}$.

10.2. The Majorant $\mu(x) = \sqrt{R_m(x^2)}$

Lemma 10.1. *Let $\mu(x) = \sqrt{R_m(x^2)}$, where $R_m(x)$ is a polynomial of degree m, non-negative in $[0,1]$. Then, for any $k \ge m$, and for any $n \ge 0$, we have*

$$\omega_{m+n}^{(k)}(x) = \sum_{i=0}^{m+n} a_i T_i(x), \qquad a_i \ge 0. \quad (10.3)$$

Proof. We have

$$\omega_{m+n}(x) = P_{m+n}(x) = P_m(x)T_n(x) + (x^2-1)Q_{m-1}(x)\frac{1}{n}\,T_n'(x)\,,$$

where both polynomials P_m and Q_{m-1} are either odd or even, all their (symmetric) zeros are in $[-1,1]$, and they have positive leading coefficients. Consider separately two cases.

1) The case $m = 2m_0$. Then

$$P_m(x) = P_{2m_0}(x) = c\prod_{i=1}^{m_0}(x^2-t_i^2) = c\prod_{i=1}^{m_0}(x^2-1+a_i^2)$$
$$= \sum_{i=0}^{m_0} b_i^2(x^2-1)^{m_0-i}\,,$$

and

$$(x^2-1)Q_{m-1}(x) = (x^2-1)Q_{2m_0-1}(x) = cx(x^2-1)\prod_{i=1}^{m_0-1}(x^2-s_i^2)$$
$$= cx(x^2-1)\prod_{i=1}^{m_0-1}(x^2-1+c_i^2) = x\sum_{i=0}^{m_0} d_i^2(x^2-1)^{m_0-i}\,.$$

Hence,

$$\omega_\mu(x) = P_{m+n}(x) = \Big[\sum_{i=0}^{m_0} b_i^2(x^2-1)^{m_0-i}\Big]T_n(x) + \Big[\sum_{i=0}^{m_0} d_i^2(x^2-1)^{m_0-i}\Big]xT_n'(x)\,,$$

and conclusion (10.3) follows by Lemmas 9.1-9.2, if $k \ge 2m_0 =: m$.

2) The case $m = 2m_0 - 1$. Similarly, we obtain

$$P_m(x) = P_{2m_0-1}(x) = x\sum_{i=0}^{m_0-1} b_i^2(x^2-1)^{m_0-1-i}\,,$$

and

$$(x^2-1)Q_{m-1}(x) = (x^2-1)Q_{2m_0-2}(x) = \sum_{i=0}^{m_0} d_i^2(x^2-1)^{m_0-i}\,.$$

Hence,

$$\omega_\mu(x)=P_{m+n}(x)=\Big[\sum_{i=0}^{m_0-1} b_i^2(x^2-1)^{m_0-1-i}\Big]xT_n(x)+\Big[\sum_{i=0}^{m_0} d_i^2(x^2-1)^{m_0-i}\Big]T_n'(x)\,,$$

and conclusion (10.3) follows by Lemmas 9.1-9.2, if $k \ge 2m_0 - 1 =: m$. □

Applying Theorem 2.1 we obtain the following Duffin-Schaeffer-type result.

Theorem 10.1 (Example 2.1, 13°-14°). *Let* $\mu(x) = \sqrt{R_m(x^2)}$, *where* R_m *is a polynomial of degree* m, *non-negative in* $[0,1]$. *Then we have*

$$M_{k,\mu} = D_{k,\mu} = \omega_\mu^{(k)}(1), \qquad k \ge m+1, \quad n \ge m\,.$$

10.3. Arbitrary Even Majorant $\mu(x) = \mu(-x)$

Lemma 10.2. *Let*

$$\mu(x) = \mu(-x).$$

Then, for any n, *and for any* $k \ge \frac{n-1}{2}$, *we have*

$$\omega_n^{(k)}(x) = \sum_{i=1}^{n} a_i T_i(x), \qquad a_i \ge 0. \tag{10.4}$$

Proof. We have

$$\omega_n(x) = P_n(x),$$

where P_n is either odd or even, all its (symmetric) zeros are in $[-1,1]$ and it has a positive leading coefficient. Consider again two cases.

1) The case $n = 2n_0$. Then, as in the proof of Lemma 10.1, we conclude that

$$P_n(x) = P_{2n_0}(x) = \sum_{i=0}^{n_0} b_i^2 (x^2 - 1)^{n_0 - i}\,. \tag{10.5}$$

Hence

$$P_n^{(n_0)}(x) = \sum_{i=0}^{n_0} b_i^2 L_{n_0-i}^{(i)}(x)\,,$$

where

$$L_m(x) := \frac{d^m}{dx^m}(x^2-1)^m$$

is the Legendre polynomial of degree m (with a different normalization, the classical one has an additional factor $\frac{1}{2^m\,m!}$). Since L_m is known to have a positive Chebyshev expansion, i.e.,

$$L_m(x) = \sum_{i=0}^{m} a_i T_i(x), \qquad a_i \ge 0,$$

the same is true for its derivatives (because $T_i^{(\ell)}$ have positive expansions). Hence (10.4) holds true for $k \ge n_0 = n/2$, i.e., for all $k \ge \frac{n-1}{2}$.

2) The case $n = 2n_0 - 1$. We may write

$$P_n(x) = P_{2n_0-1}(x) = x \sum_{i=0}^{n_0-1} b_i^2 (x^2-1)^{n_0-1-i} = \frac{d}{dx} Q_{2n_0}(x)\,,$$

where

$$Q_{2n_0}(x) = \sum_{i=0}^{n_0-1} c_i^2 (x^2-1)^{n_0-i}, \quad c_i^2 = \frac{b_i^2}{2(n_0-i)},$$

so that

$$P_n^{(n_0-1)}(x) = Q_{2n_0}^{(n_0)}(x).$$

The polynomial Q_{2n_0} has the same form as the polynomial P_{2n_0} in (10.5), hence its n_0-th derivative has a positive Chebyshev expansion. So, we have (10.4) for $k \ge n_0 - 1 = \frac{n-1}{2}$. □

Now, application of Theorem 2.1 gives the following

Theorem 10.2 (Example 2.1, 15°). *Let $\mu(x)$ be any even majorant,*

$$\mu(x) = \mu(-x), \qquad x \in [-1,1].$$

Then we have

$$M_{k,\mu} = D_{k,\mu} = \omega_\mu^{(k)}(1), \qquad k \ge \frac{n}{2}, \quad n \in \mathbb{N}.$$

10.4. The Majorant $\mu(x) = \sqrt{\prod_{i=1}^{m} (1 + c_i^2 x^2)}$

Lemma 10.3. *Let*

$$\mu^2(x) = \prod_{i=1}^{m}(1 + (a_i^2 - 1)x^2)\,, \qquad a_i \geq 1.$$

Then, for any $n \geq 0$, we have

$$\omega_{n+m}(x) = \sum_{j=0}^{n+m} b_j T_j(x), \qquad b_j \geq 0. \tag{10.6}$$

Proof. With $x = \cos t$, we have

$$\begin{aligned}
&\big(ax+i\sqrt{1-x^2}\big)\big(T_n(x)+i\sqrt{1-x^2}U_{n-1}(x)\big) = (a\cos t+i\sin t)(\cos nt+i\sin nt)\\
&\quad= \Big(\frac{a+1}{2}(\cos t + i\sin t) + \frac{a-1}{2}(\cos t - i\sin t)\Big)(\cos nt + i\sin nt)\\
&\quad= \frac{a+1}{2}(\cos(n+1)t + i\sin(n+1)t) + \frac{a-1}{2}(\cos(n-1)t + i\sin(n-1)t)\\
&\quad= \frac{a+1}{2}\big(T_{n+1}(x)+i\sqrt{1-x^2}U_n(x)\big)+\frac{a-1}{2}\big(T_{n-1}(x)+i\sqrt{1-x^2}U_{n-2}(x)\big)\,,
\end{aligned}$$

therefore, in finding expression for the snake polynomial (see (10.1))

$$P_{n+m}(x) = \mathrm{Re}\Big\{\prod_{j=1}^{m}\big(a_j x + i\sqrt{1-x^2}\big)\big[T_n(x) + i\sqrt{1-x^2}U_{n-1}\big]\Big\}\,,$$

we may proceed by induction. In particular, we have: for $m = 1$,

$$P_{n+1}(x) = \frac{a_1+1}{2}\,T_{n+1}(x) + \frac{a_1-1}{2}\,T_{n-1}(x)\,;$$

for $m = 2$,

$$\begin{aligned}
P_{n+2}(x) = {}&\frac{(a_1+1)}{2}\frac{(a_2+1)}{2}\,T_{n+2}(x)+\Big(\frac{(a_1+1)}{2}\frac{(a_2-1)}{2}+\frac{(a_1-1)}{2}\frac{(a_2+1)}{2}\Big)T_n(x)\\
&+\frac{(a_1-1)}{2}\frac{(a_2-1)}{2}\,T_{n-2}(x)\,,
\end{aligned} \tag{10.7}$$

and, generally,

$$P_{m+n}(x) \quad = \quad \frac{1}{2^m}\sum_{e_i}\prod_{j=1}^{m}(a_j + e_{ij})\,T_{n+|e_i|}\,,$$

where summation is extended over all vectors $e_i = (e_{i,1}, \ldots, e_{im})$, $e_{ij} = \pm 1$ and $|e_i| := \sum e_{ij}$. So, if all $a_j \geq 1$, then the Chebyshev coefficients of P_{m+n} are non-negative. □

Thus, the following statement is true.

Theorem 10.3 (Example 2.1, 12°). *Let*

$$\mu^2(x) = \prod_{i=1}^{m} (1 + c_i^2 x^2)\,.$$

Then, for all $k \geq 1$, and for all $n \geq m$,

$$M_{k,\mu} = D_{k,\mu} = \omega_n^{(k)}(1)\,, \qquad k \geq 1, \quad n \geq m\,.$$

10.5. The Majorant $\mu(x) = \sqrt{(1 + c_1^2 x^2)(1 + (a_2^2 - 1)x^2)}$

Theorem 10.4 (Example 2.1, 16°). *Let*

$$\mu^2(x) = (1 + (a_1^2 - 1)x^2)(1 + (a_2^2 - 1)x^2), \qquad a_1 \geq 1.$$

Then we have

$$M_{k,\mu} = D_{k,\mu} = \omega_n^{(k)}(1), \qquad k \geq 2, \quad n \geq 2.$$

Proof. It suffices to prove that the first derivative of the snake-polynomial $\omega_\mu = P_{n+2}$ in (10.7) has a positive Chebyshev expansion. Denote the coefficients of the Chebyshev expansion of P_{n+2} in (10.7) by A,B and C, respectively:

$$P_{n+2}(x) = AT_{n+2}(x) + BT_n(x) + CT_{n-2}(x)$$

and note that

$$A = \frac{a_1+1}{2}\frac{a_2+1}{2}\,, \quad A+B = \frac{a_1+1}{2}\,a_2 + \frac{a_1-1}{2}\frac{a_2+1}{2}\,, \quad A+B+C = a_1 a_2\,,$$

hence

$$a_1 \geq 1, a_2 \geq 0 \quad \Rightarrow \quad A > 0, \quad A+B \geq 0, \quad A+B+C \geq 0. \tag{10.8}$$

Since

$$T_m'(x) = m\left(T_{m-1}(x) + T_{m-3}(x) + \cdots\right)\,,$$

we obtain

$$P_{n+2}'(x) = A'T_{n+1}(x) + B'T_{n-1}(x) + C'\left(T_{n-3}(x) + T_{n-5}(x) + \cdots\right)\,,$$

where

$$A' = (n+2)A, \qquad B' = (n+2)A + nB, \qquad C' = (n+2)A + nB + (n-2)C,$$

and all these constants are positive because of (10.8). □

10.6. The Majorant $\mu(x) = \sqrt{ax^2 + bx + 1}$

Here we will treat the the case of a non-symmetric majorant of the form

$$\mu^2(x) = ax^2 + bx + 1 = (\alpha x + \beta)^2 + \gamma^2(1 - x^2).$$

where we will assume that

$$\mu(-1) \le \mu(1) \quad \Leftrightarrow \quad b \ge 0.$$

Equating the coefficients we obtain

$$\beta^2 + \gamma^2 = 1, \qquad \alpha^2 - \gamma^2 = a, \qquad 2\alpha\beta = b \tag{10.9}$$

whence

$$\alpha = \frac{\mu(1) + \mu(-1)}{2} \ge 0, \quad \beta = \frac{\mu(1) - \mu(-1)}{2} \in [0,1], \quad \gamma = \sqrt{1 - \beta^2} \in [0,1]\,.$$

The corresponding snake-polynomial has the form

$$\omega_{n+1}(x) = (\alpha x + \beta) T_n(x) + \frac{\gamma}{n}\,(x^2 - 1) T_n'(x) \tag{10.10}$$

$$= \frac{\alpha + \gamma}{2}\, T_{n+1}(x) + \beta T_n(x) + \frac{\alpha - \gamma}{2}\, T_{n-1}(x). \tag{10.11}$$

In order to get

$$M_{k,\mu} = D_{k,\mu} = \omega_n^{(k)}(1)$$

for a particular k, we need to verify two conditions (of $\omega^{(k-1)}$ belonging to the class Ω):

$$1b) \quad \|\omega^{(k-1)}\|_{C[-1,0]} = |\omega^{(k-1)}(-1)| \tag{10.12}$$

$$2) \quad \omega^{(k-1)} = \sum a_i T_i, \qquad a_i \ge 0.$$

(The right end-point condition (1a) follows from (2).)

1) Case $k = 1$.

In this case, ω has a positive Chebyshev expansion if

$$\alpha \ge \gamma \quad \Leftrightarrow \quad a \ge 0\,.$$

It is also clear, that the "left end-point condition" (10.12) will be satisfied if

$$\mu(-1) \ge \mu(0) \quad \Leftrightarrow \quad a \ge b.$$

Thus we have the following statement.

Theorem 10.5. *Let*

$$\mu(x) = \sqrt{ax^2 + bx + 1}, \quad \textit{where} \quad a \ge b \ge 0.$$

Then, for all $n \ge 1$,

$$M_{1,\mu} = D_{1,\mu} = \omega_n'(1).$$

2) Case $k=2$.

In this case, since $xT_n(x)$, $T_i'(x)$ and $[(x^2-1)T_n'(x)]'$ have positive Chebyshev expansions, it follows that

$$\omega'(x) = \Big[(\alpha x+\beta)T_n(x)\Big]' + \Big[\frac{\gamma}{n}(x^2-1)T_n'(x)\Big]' = \sum a_i T_i\,, \quad a_i \ge 0,$$

i.e., ω_μ' has a positive Chebyshev expansions for any $\mu = \sqrt{ax^2+bx+1}$ with $b \ge 0$.

However, the "left end-point" property is not always fulfilled. For example, for $\mu(x) = x+1$, and odd n, we have

$$\omega(x) = (x+1)T_n(x) \quad \Rightarrow \quad |\omega'(-1)| = 1 < n = |\omega'(0)|\,.$$

Below we give a sufficient condition which provides the "left end-point" property (10.12) for the first derivative of

$$\omega(x) = (\alpha x+\beta)T_n(x) + \frac{\gamma}{n}(x^2-1)T_n'(x).$$

By Vidensky result (10.2), with $P(x) = \alpha x+\beta$ and $Q(x)=\gamma$, we have

$$\begin{aligned}|\omega'(x)|^2 \le V_1(x)^2 &= \frac{[n(\alpha x+\beta)+\gamma x]^2 + (1-x^2)(\alpha+n\gamma)^2}{1-x^2}\,,\\ &= \frac{[(n\alpha+\gamma)x+n\beta]^2}{1-x^2} + (\alpha+n\gamma)^2\end{aligned}$$

with equality attained at $n+1$ points.

Let us show that $V_1(x)$ (which is unbounded at $x=1$ for $\alpha, \beta, \gamma \ge 0$) has exactly one point of extremum (which is necessarily a minimum) inside $[-1,0]$. We have

$$V_1'(x) = 0 \quad \Leftrightarrow \quad 2[(n\alpha+\gamma)x+n\beta)](n\alpha+\gamma)(1-x^2) + [(n\alpha+\gamma)x+n\beta)]^2 2x = 0$$

which is equivalent to two conditions

$$1) \qquad (n\alpha+\gamma)x+n\beta = 0 \quad \Leftrightarrow \quad x = -\frac{n\beta}{n\alpha+\gamma} \quad (=: x_1)$$

$$2) \qquad (n\alpha+\gamma)(1-x^2) + [(n\alpha+\gamma)x+n\beta]x = 0 \quad \Leftrightarrow \quad x = -\frac{n\alpha+\gamma}{n\beta} \quad (=: x_2)\,.$$

and, since $x_1 = 1/x_2 \in [-1,0]$, there is exactly one extremum inside the interval.

Therefore, the "left end-point" condition (10.12) will be fulfilled for all n if

$$|\omega'(-1)| \ge V_1(0).$$

We have

$$\begin{aligned}V_1(0) &= \sqrt{(n\beta)^2 + (\alpha+n\gamma)^2)} = \sqrt{n^2+2\alpha\gamma n+\alpha^2} \le n+\alpha,\\ |\omega'(-1)| &= \frac{\alpha+\gamma}{2}(n+1)^2 - \beta n^2 + \frac{\alpha-\gamma}{2}(n-1)^2 = (\alpha-\beta)n^2 + 2\gamma n + \alpha\,,\end{aligned}$$

where in the first line we used relations $\beta^2 + \gamma^2 = 1$ and $\gamma \le 1$ from (10.9).

So, it is sufficient to require

$$(\alpha - \beta)n^2 + 2\gamma n + \alpha \ge n + \alpha \quad \Leftrightarrow \quad \alpha - \beta \ge \frac{1 - 2\gamma}{n}.$$

Since $\alpha - \beta \ge 0$ by definition, the latter is true if

$$\gamma \ge \frac{1}{2} \quad \Leftrightarrow \quad \mu(1) - \mu(-1) \le \sqrt{3} \qquad (\text{since } 2\beta = 2\sqrt{1-\gamma^2}),$$

with a possibility $\mu(-1) = 0$. Another option is, when $\mu(-1) > 0$,

$$2) \quad \gamma < \frac{1}{2} \quad \text{and} \quad n \ge \frac{1-2\gamma}{\alpha - \beta} = \frac{1-2\gamma}{\mu(-1)}\,.$$

Applying Theorem 2.1, we obtain the following statement:

Theorem 10.6 (Example 2.1, 17°, $k = 2$). *Let*

$$\mu(x) = \sqrt{ax^2 + bx + 1}, \qquad b \ge 0.$$

If $\mu(1) - \mu(-1) \le \sqrt{3}$, then for all $n \ge 1$, otherwise for all $n \ge \frac{1}{\mu(-1)}$ if $\mu(-1) > 0$, we have

$$M_{2,\mu} = D_{2,\mu} = \omega_n''(1)\,.$$

The case $k \ge 3$.

Let us show that, for $m \ge 2$, the left end-point condition

$$\|\omega^{(m)}\|_{C[-1,0]} = |\omega^{(m)}(-1)|, \quad m \ge 2,$$

is fulfilled for any $\alpha, \beta, \gamma \ge 0$. We have

$$\begin{aligned}\omega^{(m)}(x) &= \big[(\alpha x + \beta)T_n(x)\big]^{(m)} + \frac{\gamma}{n}\big[(x^2-1)T_n'(x)\big]^{(m)} \\ &= \alpha\big[(x+1)T_n(x)\big]^{(m)} + \big[(\beta-\alpha)T_n(x)\big]^{(m)} + \frac{\gamma}{n}\big[(x^2-1)T_n'(x)\big]^{(m)}\,.\end{aligned}$$

Since $\alpha, \gamma \ge 0$ and $\beta - \alpha \le 0$, all the terms in the last line have the same sign $(-1)^{n-m+1}$ at $x = -1$. Also, $\big[(x^2-1)T_n'(x)\big]^{(m)}$ and $T_n^{(m)}(x)$ have positive Chebyshev expansions for $m \ge 2$, hence satisfy the left-end property, therefore it suffices to prove the left-end property only for the first term.

The latter is equivalent to the right-end property for the polynomial $[(x-1)T_n(x)]^{(m)}$, i.e. we need to prove that

$$g_m(x) := |(x-1)T_n^{(m)}(x) + mT_n^{(m-1)}(x)| \le mT_n^{(m-1)}(1)\,, \qquad x \in [0,1]\,.$$

For $m = 2$ and $x \in [0,1]$ we have, by (8.13) and (8.5),

$$g_2(x) \le |(x^2-1)T_n''(x)| + 2|T_n'(x)| = 2\max\{|f_1(T_n;x)|, |f_2(T_n;x)|\} \le 2T_n'(1)\,.$$

Since $T_n^{(m-1)} = \sum a_i T_i'$, hence $T_n^{(m)} = \sum a_i T_i''$, with the same $a_i \ge 0$, the result for $m = 2$ implies, for $m > 2$,

$$\begin{aligned}\big|(x-1)T_n^{(m)}(x) + 2T_n^{(m-1)}(x)\big| &= \Big|\sum a_i\big((x-1)T_i''(x) + 2T_i'(x)\big)\Big| \\ &\le \sum a_i\big|(x-1)T_i''(x) + 2T_i'(x)\big| \\ &\le 2\sum a_i T_i'(1) = 2T_n^{(m-1)}(1)\,.\end{aligned}$$

Consequently, for $m > 2$ we have

$$\begin{aligned}g_m(x) &\le |(x-1)T_n^{(m)}(x) + 2T_n^{(m-1)}(x)| + (m-2)|T_n^{(m-1)}(x)| \\ &\le mT_n^{(m-1)}(1).\end{aligned}$$

Now Theorem 2.1 implies

Theorem 10.7 (Example 2.1, 17°, $k \ge 3$). *Let*

$$\mu(x) = \sqrt{ax^2 + bx + 1}, \qquad b \ge 0.$$

Then we have

$$M_{k,\mu} = D_{k,\mu} = \omega_n^{(k)}(1)\,, \qquad k \ge 3, \qquad n \ge 1\,.$$

References

[1] B. Bojanov and G. Nikolov, Duffin and Schaeffer type inequality for ultrasphrical polynomials, *J. Approx. Theory* **84** (1996), no. 2, 129–138.

[2] R. J. Duffin and A. S. Shaeffer, A refinement of an inequality of the brothers Markoff, *Trans. Amer. Math. Soc.* **50** (1941), 517–528.

[3] R. J. Duffin and L. A. Karlovitz, The Markoff-Duffin-Schaeffer inequalities abtsracted, *Proc. Nat. Acad. Sci. USA* **82** (1985), 955–957.

[4] W. Markov, "On functions which deviate least from zero in a given interval", Sankt-Petersburg, 1892; German transl.: *Math. Ann.* **77** (1916), 213–258.

[5] G. Nikolov, Inequalities of Duffin-Schaeffer type, *SIAM J. Math. Anal.* **33** (2001), no. 3, 686–698.

[6] G. Nikolov, Snake polynomials and Markov-type inequalities, *in* "Approximation Theory: A volume dedicated to Blagovest Sendov" (B. Bojanov, Ed.), pp. 342–352, DARBA, Sofia, 2002.

[7] G. Nikolov, Inequalities of Duffin-Schaeffer type. II, *East J. Approx.* **11** (2005), no. 2, 147–168.

[8] R. Pierre and Q. I. Rahman, On a problem of Turan about polynomials. II, *Canad. J. Math.* **33** (1981), no. 3, 701–733.

[9] R. Pierre, Q. I. Rahman, and G. Schmeisser, On polynomials with curved majorants, *J. Approx. Theory* **57** (1989), no. 2, 211–222.

[10] Q. I. Rahman and G. Schmeisser, Markov-Duffin-Schaeffer inequality for polynomials with a circular majorant, *Trans. Amer. Math. Soc.* **310** (1988), no. 2, 693–702.

[11] Q. I. Rahman and A. O. Watt, Polynomials with a parabolic majorant and the Duffin-Schaeffer inequality, *J. Approx. Theory* **69** (1992), no. 3, 338–354.

[12] A. Yu. Shadrin, Interpolation with Lagrange polynomials. A simple proof of Markov inequality and some of its generalizations, *Approx. Theory Appl.* **8** (1992), no. 3, 51–61.

[13] A. Yu. Shadrin, On Markov-Duffin-Schaeffer inequalities with weight, a manuscript.

[14] A. Yu. Shadrin, Twelve proofs of the Markov inequality, *in* "Approximation theory: a volume dedicated to Borislav Bojanov", pp. 233–298, Prof. M. Drinov Acad. Publ. House, Sofia, 2004.

[15] V. S. Videnskii, On estimates of derivatives of a polynomial, *Izvestiya Akad. Nauk SSSR, Ser. Mat.* **15** (1951), 401–420 [in Russian].

[16] V. S. Videnskii, A generalization of V. A. Markoff inequalities, *Dokl. Akad. Nauk SSSR* **120** (1958), 447–449 [in Russian].

[17] V. S. Videnskii, Generalizations of Markov's theorem on the evaluation of a polynomial derivative, *Doklady Akad. Nauk SSSR* **125** (1959), 15–18 [in Russian].

[18] V. S. Videnskii, Least upper bounds for the successive derivatives of a polynomial on an interval, *Izv. Vys. Uchebn. Zaved., Ser. Mat.* **106** (1971), no. 3, 18–22 [in Russian].

Geno Nikolov

Department of Mathematics
Universlty of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
E-mail: geno@fmi.uni-sofia.bg

Alexei Shadrin

Department of Applied Mathematics and Theoretical Physics (DAMTP)
Cambridge University
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM
E-mail: a.shadrin@damtp.cam.ac.uk