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On an Extremal Property of the Chebyshev Polynomials: Maximizing Pairs of Coefficients of Bounded Polynomials

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Let $T_n(x) = \sum_{k=0}^{n} t_{n,k}x^k$ denote the n-th Chebyshev polynomial of the first kind which is even or odd, according to the parity of n, and let $P_n(x) = \sum_{k=0}^{n} a_k x^k$ denote any real polynomial of degree $\leq n$ which satisfies $|P_n(x)| \leq 1$ for $|x| \leq 1$ or merely $|P_n(\cos\frac{(n-i)\pi}{n})| \leq 1$ for $0 \leq i \leq n$. It is known that the maximal value of the sum of the moduli of two consecutive coefficients of $P_n(x)$ is then attained if $P_n(x) = \pm T_n(x)$:

$$|a_{k-1}| + |a_k| \leq |t_{n,k-1}| + |t_{n,k}| = |t_{n,k}|, \text{ provided } k \equiv n \bmod 2. \quad \text{(i)}$$

Inequality (i) is referred to as Szegő's coefficient inequality [22, p. 673].

This implies V. A. Markov's coefficient inequality [17, p. 56], [25, p. 167] for single coefficients, which in turn implies Chebyshev's inequality for the leading coefficient of $P_n(x)$, see [15, p. 385], [23, p. 68 and p. 108]. In an attempt to extend (i) to pairs of coefficients $|a_j|+|a_k|$ with $j < k-1$ we obtain, for $n \geq 6$:

Theorem. *Additionally to Szegő's coefficient inequality (i) there holds*

$$|a_{k-1-2m}| + |a_k| \leq |t_{n,k}|, \quad \text{(ii)}$$

provided $k \equiv n \bmod 2$ *and* $k \leq \frac{2n}{3}$, *where* $m \geq 1$ *is an integer.*

Addendum 1. The range of inequality (ii) can be improved by replacing $k \leq \frac{2n}{3}$ (and $k \equiv n \bmod 2$) with $k \leq k^*$, where $k^* = k^*(n) = n - 2q^* < n$ and $q^* = \lceil \frac{n^2-2n}{6n+4} \rceil$.

Addendum 2. The range of inequality (ii) can be further improved by replacing $k \leq k^*$ with $k \leq k^{**}$, where $k^{**} = k^{**}(n) = n-2q^{**} < n$, in which the positive integer q^{**} stems from the solution of a certain cubic equation.

The magnitude of k^{**} results from the numerically verified inequality $|k^{**}+1-\lceil \frac{n}{\sqrt{2}} \rceil| \leq 1$. By a counterexample we demonstrate that (ii) does

not hold if one allows $k \le n$, where $n \ge 6$. The marginal cases $n \le 5$ are treated separately. Our result can be extended to $P_n(x)$ bounded symmetrically on $[-1, 1]$ in the sense of [5], [24]. The related issue of maximizing pairs of coefficients of bounded polynomials in the complex domain is treated in [3], [15, pp. 125–130], [22, pp. 637–641]. In the course of the proof we provide explicit values for the elements of the inverse of the Vandermonde matrix associated with the extremal points of $T_n(x)$.

Keywords and Phrases: Coefficient, counterexample, elementary symmetric function, estimate, extension, extremal (points, polynomial, problem, property), generalization, height, inequality (Chebyshev, Markov, Szegő), inverse, majorant, pair of coefficients, polynomial (Chebyshev, Rogosinski, bounded), unit interval, Vandermonde matrix.

"We all know that the extreme is beautiful."
Blagovest Sendov in his Reminiscence of Borislav Bojanov,
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1. Introduction

Let Φ_n denote the linear space of real algebraic (univariate) polynomials of degree $\le n$ with elements P_n given in power form by $P_n(x) = \sum_{k=0}^{n} a_k x^k$ ($n \ge 1$, a_k real), and let B_n denote the unit ball in Φ_n with respect to the interval $I = [-1, 1]$ and the uniform norm $\|P_n\|_\infty = \sup_{x \in I} |P_n(x)|$:

$$B_n = \{P_n \in \Phi_n : \|P_n\|_\infty \le 1\}. \tag{1.1}$$

The n-th Chebyshev polynomial of the first kind with respect to I, T_n, belongs to B_n since $T_n(x) = \cos(n \arccos x)$ if $x \in I$, and plays an outstanding role in providing solutions to extremal coefficient problems for polynomials bounded (pointwise) in the uniform norm. It is well-known that $T_n(x) = \sum_{k=0}^{n} t_{n,k} x^k$ satisfies the three-term recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad n \ge 2, \tag{1.2}$$

with $T_0(x) = 1$ and $T_1(x) = x$, and is hence an even resp. odd polynomial, according to the parity of n. So $t_{n,k} = 0$, if $n - k$ is odd, whereas, if $n - k$ is even, the coefficients $t_{n,k}$ are nonzero integers given in descending order by:

$$t_{n,n-2m} = \frac{(-1)^m n 2^{n-2m-1}}{n-m} \binom{n-m}{m}, \qquad 0 \le m \le \left\lfloor \frac{n}{2} \right\rfloor. \tag{1.3}$$

The extremal points of T_n are the alternation points $x^*_{n,i}$ where $T_n(x^*_{n,i}) = (-1)^{n-i}$. These points are

$$x^*_{n,i} = \cos \frac{(n-i)\pi}{n}, \qquad 0 \le i \le n, \tag{1.4}$$

with $x^*_{n,i} + x^*_{n,n-i} = 0$ (symmetry with respect to the origin), and ordering

$$-1 = x^*_{n,0} < x^*_{n,1} < \cdots < x^*_{n,n-1} < x^*_{n,n} = 1 \tag{1.5}$$

(see [14] or [23] for more information on T_n).

The roots of the topic of coefficient inequalities for polynomials $P_n \in B_n$ trace back to Chebyshev's pioneering paper [4] on polynomials with fixed leading coefficient which deviate least from zero on I (implying the estimate $|a_n| \le t_{n,n} = 2^{n-1}$, see e.g. [15, p. 385], [23, p. 68 and p. 108], [25, p. 10 and p. 162], [27, p. 86]), and to an explicit question concerning the magnitude of the three coefficients of bounded quadratic parabolas, raised by the chemist D. I. Mendeleev and answered by A. A. Markov, see e.g. [1, p. 31] and [20, pp. 329–330].

V. A. Markov in his celebrated paper of 1892 [12, pp. 80–81], [13, p. 248] (see also [2, p. 248], [15, p. 423], [17, p. 56], [25, p. 167]) settled the general case by providing exact estimates for all the $n+1$ coefficients of $P_n \in B_n$, in terms of the nonzero coefficients of T_n and T_{n-1}:

Theorem A. *Let $P_n \in B_n$ with $P_n(x) = \sum\limits_{k=0}^{n} a_k x^k$, then*

$$|a_k| \le |t_{n,k}| = \frac{n 2^{k-1} (\frac{n+k-2}{2})!}{k!\,(\frac{n-k}{2})!}, \qquad \text{if } n-k \text{ is even,} \tag{1.6}$$

$$|a_k| \le |t_{n-1,k}| = \frac{(n-1) 2^{k-1} (\frac{n+k-3}{2})!}{k!\,(\frac{n-k-1}{2})!}, \qquad \text{if } n-k \text{ is odd,} \tag{1.7}$$

with equality if $P_n = \pm T_n$ resp. $P_n = \pm T_{n-1}$.

Actually, (1.7) follows as a corollary from (1.6) by considering the coefficients of the polynomial $P_{n-1}(x) = (P_n(x) + (-1)^{n+1} P_n(-x))/2$ and applying the triangle inequality.

Some 50 years later G. Szegő found a striking strengthening of V. A. Markov's coefficient inequality (Theorem A) by considering two consecutive coefficients of $P_n(x)$, a_k (with $n-k$ even) and its predecessor coefficient, a_{k-1}:

Theorem B. *Let $P_n \in B_n$ with $P_n(x) = \sum\limits_{k=0}^{n} a_k x^k$, then*

$$|a_{k-1}| + |a_k| \le |t_{n,k}|, \qquad \text{if } n-k \text{ is even,} \tag{1.8}$$

with equality if $P_n = \pm T_n$ and setting $a_{-1} = 0$.

This result was published, without proof, by Erdös in [7, p. 1176], based on Szegő's oral communication, see also [22, p. 679]. To the best of our knowledge, Munch [16, p. 26] was the first to provide a proof for (1.8), without however quoting [7]. His proof works under the assumption $P_n \in C_n$ which is weaker than $P_n \in B_n$ (see [23, p. 139]), where

$$C_n = \{P_n \in \Phi_n : |P_n(x^*_{n,i})| \le 1 \text{ for } 0 \le i \le n\}, \tag{1.9}$$

with $x^*_{n,i}$ given by (1.4).

Obviously, $\pm T_n$ belong to both classes C_n and B_n. We thus have, see also [22, p. 673]:

Theorem C. *Let $P_n \in B_n$ or $P_n \in C_n$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$, then*

$$|a_{k-1}| + |a_k| \leq |t_{n,k}|, \qquad \textit{if } n-k \textit{ is even}, \tag{1.10}$$

with equality if $P_n = \pm T_n$ and setting $a_{-1} = 0$.

This inequality has been rediscovered by several authors, see [20], [21] for details. It immediately implies, for $n - k$ even,

$$\begin{aligned} &|a_k| \leq |t_{n,k}|, &&\text{if } P_n \in C_n, \text{ and in particular,} \\ &|a_n| \leq 2^{n-1}, &&\text{(Chebyshev's coefficient inequality),} \end{aligned} \tag{1.11}$$

and this refinement of the first part of V. A. Markov's coefficient inequality, (1.6), had been noticed earlier by Shohat [26, p. 687], see also [22, p. 672]. The second part of Theorem A, (1.7), does not hold true if $P_n \in B_n$ is relaxed to $P_n \in C_n$, as simple examples show. As was pointed out by Rogosinski [24, p. 10], the following polynomial replaces T_{n-1} as an extremizer for the coefficients a_k of arbitrary $P_n \in C_n$, if $n - k$ is odd:

$$\Pi_{n-1} \in C_n \text{ with } \Pi_{n-1}(x) = \sum_{k=0}^{n-1} c_{n-1,k} x^k, \text{ which is defined by}$$

$$\begin{aligned} &\Pi_{n-1}(x^*_{n,i}) = (-1)^{i+1}, && \text{if } 0 \leq i \leq \frac{n}{2} - 1 \\ &\Pi_{n-1}(x^*_{n,i}) = 0, && \text{if } i = \frac{n}{2} \\ &\Pi_{n-1}(x^*_{n,i}) = (-1)^{i}, && \text{if } \frac{n}{2} + 1 \leq i \leq n, \end{aligned} \tag{1.12}$$

provided n is even, resp.

$$\begin{aligned} &\Pi_{n-1}(x^*_{n,i}) = (-1)^{i}, && \text{if } 0 \leq i \leq \frac{n-1}{2} \\ &\Pi_{n-1}(x^*_{n,i}) = (-1)^{i+1}, && \text{if } \frac{n+1}{2} \leq i \leq n, \end{aligned} \tag{1.13}$$

provided n is odd. We thus have, see also [5, p. 2744]:

Theorem D. *Let $P_n \in C_n$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$, then*

$$|a_k| \leq |c_{n-1,k}|, \qquad \textit{if } n-k \textit{ is odd}, \tag{1.14}$$

with equality if $P_n = \pm \Pi_{n-1}$.

We note in passing that inequality (1.7) does hold true if one assumes $P_n \in C_n \cap C_{n-1}$, see [20, p. 324], [22, p. 673], [23, p. 112].

In the present paper we try to extend Szegő's fundamental coefficient inequality (Theorem C, which implies Theorem B, and hence Theorem A) to pairs of coefficients that are not necessarily adjacent to each other. We thus ask:

Given $P_n \in B_n$ or $P_n \in C_n$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$, is $|t_{n,k}|$ large enough to sharply majorize $|a_j| + |a_k|$, if $j < k$ and $n - k$ is even?

The special choice $j = k - 1$ would take us back to Theorem C. To avoid trivial cases we assume that the index j with $j < k$ has a parity different from k, i.e. we assume that $n - j$ is odd. For if both $n - j$ and $n - k$ were even, then $|a_j| + |a_k|$ would be majorized by $|t_{n,j}| + |t_{n,k}|$, in view of (1.11), with equality if $P_n = \pm T_n$. We notice that if both $n - j$ and $n - k$ were assumed to be odd, then $|a_j| + |a_k|$ would be majorized by $|t_{n-1,j}| + |t_{n-1,k}|$, in view of (1.7), with equality if $P_n = \pm T_{n-1}$, if $P_n \in B_n$, resp. by $|c_{n-1,j}| + |c_{n-1,k}|$, in view of (1.14), with equality if $P_n = \pm \Pi_{n-1}$, if $P_n \in C_n$.

It will turn out (see the Theorem below) that $\pm T_n$ indeed maximizes many more pairs of coefficients of $P_n \in B_n$ or $P_n \in C_n$ than is indicated by Theorem C, and this finding reveals a new extremal property of the classical polynomial $\pm T_n$. But a single coefficient of $\pm T_n$ cannot maximize all pairs $|a_j| + |a_k|$ if $j < k$ (with $n - k$ even and $n - j$ odd) as we will show by a twofold counterexample:

For $n \geq 7$, the modulus of the coefficient $c_{n-1,n-3}$ of the above introduced Rogosinski polynomial $\Pi_{n-1} \in C_n$ is larger than $t_{n,n} = 2^{n-1}$, so that $|a_{n-3}| + |a_n| \leq |t_{n,n}|$ does not hold for all $P_n \in C_n$; for $n = 6$, we provide an explicit polynomial $P_6 \in C_6$ whose coefficients satisfy $|a_3| + |a_6| > t_{6,6} = 32$. This is why in our Theorem we will have to restrict the index k (where $n - k$ is even) by some upper bound which is smaller than n, and thus we cannot allow $k \leq n$, as is the case in Theorem C. Actually, we will provide three upper bounds in ascending order.

Our question may be considered as a special case of V. A. Markov's general problem on extremal values of coefficient functionals (see [9, p. 24], [12, p. 79], [13, p. 246]) which we state here in an extended version that includes polynomials from C_n:

Extended V. A. Markov's Coefficient Problem of 1892:

$$\text{Given the real scalars } \beta_k, 0 \leq k \leq n, \text{find the maximum of } \Big|\sum_{k=0}^{n} \beta_k a_k\Big| \quad (1.15)$$

subject to the condition that $P_n \in B_n$ or $P_n \in C_n$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$.

V. A. Markov's two-staged coefficient inequality (Theorem A) is the solution for the cases $\pm\beta_k = 1$ and $\beta_i = 0$ elsewhere, with $P_n \in B_n$.

Szegő's coefficient inequality (Theorem C) is the solution to (1.15) for the cases $\beta_{k-1} = \pm\beta_k = 1$ and $\beta_i = 0$ elsewhere, where $n - k$ is even, since (1.10) implies $|a_{k-1} \pm a_k| \leq |t_{n,k}|$ for all $P_n \in B_n$ or $P_n \in C_n$, with equality if $P_n = \pm T_n$.

A solution to our problem (with restricted index k) would give the solution to (1.15) for the cases $\beta_j = \pm\beta_k = 1$ and $\beta_i = 0$ else, where $j < k$ with $n-k$ even and $n-j$ odd, since $|a_j| + |a_k| \leq |t_{n,k}|$ implies $|a_j \pm a_k| \leq |t_{n,k}|$ for all $P_n \in B_n$ or $P_n \in C_n$, with equality if $P_n = \pm T_n$. We mention that the problem of maximizing partial coefficient sums of $P_n \in B_n$ or $P_n \in C_n$, i.e. $\beta_0 = \beta_1 = \cdots = \beta_k = 1$, if $n-k$ is even, and $\beta_i = 0$, if $i > k$, has been settled in [19] to the effect that, again, $\pm T_n$ is extremal.

2. Results and Examples

We obtain a proper extension of Szegő's coefficient inequality (Theorem C). The result is stated in three versions concerning the upper bound for the index k of the majorizing coefficients $t_{n,k}$ of T_n. These versions are ascending in sharpness but at the same time the computational complexity of the upper bound increases. In Section 4 we briefly address the question of optimality of the upper bounds for the index k. The Theorem is stated for $n \geq 6$ since the marginal cases $n \leq 5$ are treated separately.

Theorem. *Let $P_n \in B_n$ or $P_n \in C_n, n \geq 6$, with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be arbitrary and let T_n with $T_n(x) = \sum_{k=0}^{n} t_{n,k} x^k$ denote the n-th Chebyshev polynomial of the first kind. The nonzero coefficients of T_n exhibit the following majorizing property for pairs of coefficients of P_n:*

$$|a_j| + |a_k| \leq |t_{n,k}| \qquad (a_{-1} = 0;\ \textit{equality if}\ P_n = \pm T_n) \tag{2.1}$$

for all $k \leq \frac{2n}{3}$ with $n-k$ even, and for all j with $j < k$ and $n-j$ odd.

The upper bound for k is thus $k^\circ = k^\circ(n) =$ the largest integer $\leq \frac{2n}{3}$ with $n - k^\circ$ even. For $j = k-1$ inequality (2.1) covers less cases than Theorem C due to $k \leq k^\circ$ compared with $k \leq n$. For $j \leq k-3$ inequality (2.1) is a proper extension of Theorem C.

Addendum 1. The upper bound k° for the index k can be improved to

$$k^* = k^*(n) = n - 2q^* < n, \qquad \text{with } q^* = \left\lceil \frac{n^2 - 2n}{6n + 4} \right\rceil, \tag{2.2}$$

so that (2.1) holds for all indices j and k with $j < k \leq k^*$ and $n-k$ even and $n-j$ odd. For a given n holds either $k^\circ = k^*$ or $k^\circ < k^*$.

Addendum 2. The upper bound k^* for the index k can be further improved to

$$k^{**} = k^{**}(n) = n - 2q^{**} < n, \text{ with } q^{**} = \left\lceil 2R\cos\left(\frac{\pi}{3} - \frac{1}{3}\arccos\frac{Q}{R^3}\right) + S \right\rceil, \tag{2.3}$$

where

$$Q = \frac{(1-4n^2)(2n^4-10n^2+35)}{1728}, \tag{2.4}$$

$$R = -\frac{1}{12}\sqrt{4n^4+4n^2+13}, \tag{2.5}$$

$$S = \frac{n^2}{6}+\frac{n}{2}-\frac{5}{12}, \tag{2.6}$$

so that (2.1) holds for all indices j and k with $j < k \le k^{**}$ and $n-k$ even and $n-j$ odd. For a given n holds either $k^* = k^{**}$ or $k^* < k^{**}$, and $n = 43$ is the first polynomial degree for which the constellation $k^\circ < k^* < k^{**}$ occurs: $k^\circ(43) = 27 < k^*(43) = 29 < k^{**}(43) = 31$.

Example 1. We choose $n = 20$ and obtain (see the Table below) $\frac{2n}{3} = 13.3\ldots$ (and hence $k^\circ = 12$), $q^* = q^{**} = 3$, and $k^* = k^{**} = 14$. The inequality (2.1) thus holds for all $k \le 12$, and in fact, according to Addendum 1 and 2, for all $k \le 14$, with $20-k$ even, and for all $j < k$ with $20-j$ odd. This renders the following set of twenty-eight sharp inequalities for pairs of coefficients of $P_{20} \in B_{20}$ or $P_{20} \in C_{20}$ with $P_{20}(x) = \sum_{k=0}^{20} a_k x^k$:

$$\begin{aligned}
k = 14: &\quad |a_j| + |a_{14}| \le |t_{20,14}| = 6553600 \quad \text{for } j = 1,3,5,7,9,11,13,\\
k = 12: &\quad |a_j| + |a_{12}| \le |t_{20,12}| = 4659200 \quad \text{for } j = 1,3,5,7,9,11,\\
k = 10: &\quad |a_j| + |a_{10}| \le |t_{20,10}| = 2050048 \quad \text{for } j = 1,3,5,7,9,\\
k = 8: &\quad |a_j| + |a_8| \le |t_{20,8}| = 549120 \quad \text{for } j = 1,3,5,7,\\
k = 6: &\quad |a_j| + |a_6| \le |t_{20,6}| = 84480 \quad \text{for } j = 1,3,5,\\
k = 4: &\quad |a_j| + |a_4| \le |t_{20,4}| = 6600 \quad \text{for } j = 1,3,\\
k = 2: &\quad |a_1| + |a_2| \le |t_{20,2}| = 200.
\end{aligned}$$

The seven inequalities above with $j = k-1$ are covered by Theorem C, whereas the twenty-one inequalities above for non-adjacent coefficients (that is, $j \le k-3$) go beyond Szegő's coefficient inequality (Theorem C).

Suppose the task is given to estimate from above the pair of coefficients $|a_{11}| + |a_{14}|$ when P_{20} varies in the unit ball B_{20}. V. A. Markov's coefficient inequality (Theorem A) would give the estimate

$$|t_{19,11}| + |t_{20,14}| = 8324096.$$

Szegő's coefficient inequality (Theorem C) would give the estimate

$$|t_{20,12}| + |t_{20,14}| = 11212800;$$

our Theorem gives the optimal upper bound

$$|t_{20,14}| = 6553600.$$

Example 2. It follows from the Table below that the first three even polynomial degrees n where the upper bound k^{**} (from Addendum 2) outperforms k^* (from Addendum 1) are $n = 16$, $n = 22$, and $n = 28$, in which cases we get the improved upper bounds $k \le k^{**} = 12$ (instead of $k \le k^* = 10$), resp. $k \le k^{**} = 16$ (instead of $k \le k^* = 14$) resp. $k \le k^{**} = 20$ (instead of $k \le k^* = 18$).

Correspondingly, the first five odd polynomial degrees n where k^{**} is larger than k^* are $n \in \{9, 15, 21, 27, 29\}$.

Comparing Addendum 2 with Addendum 1 we thus get, additionally to Addendum 1, the following set of new inequalities, i.e. $j \le k - 3$ (with $n - j$ odd and $n - k$ even), for pairs of non-adjacent coefficients of $P_n \in B_n$ or $P_n \in C_n$, where $6 \le n \le 30$:

$$\begin{aligned}
n = \ 9: &\quad |a_j| + |a_7| \le |t_{9,7}| = 576 \quad \text{for } j = 0, 2, 4,\\
n = 15: &\quad |a_j| + |a_{11}| \le |t_{15,11}| = 92160 \quad \text{for } j = 0, 2, 4, 6, 8,\\
n = 16: &\quad |a_j| + |a_{12}| \le |t_{16,12}| = 212992 \quad \text{for } j = 1, 3, 5, 7, 9,\\
n = 21: &\quad |a_j| + |a_{15}| \le |t_{21,15}| = 15597568\\
&\quad \text{for } j = 0, 2, 4, 6, 8, 10, 12,\\
n = 22: &\quad |a_j| + |a_{16}| \le |t_{22,16}| = 36765696\\
&\quad \text{for } j = 1, 3, 5, 7, 9, 11, 13,\\
n = 27: &\quad |a_j| + |a_{19}| \le |t_{27,19}| = 2724986880\\
&\quad \text{for } j = 0, 2, 4, 6, 8, 10, 12, 14, 16,\\
n = 28: &\quad |a_j| + |a_{20}| \le |t_{28,20}| = 6499598336\\
&\quad \text{for } j = 1, 3, 5, 7, 9, 11, 13, 15, 17,\\
n = 29: &\quad |a_j| + |a_{21}| \le |t_{29,21}| = 15386804224\\
&\quad \text{for } j = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18.
\end{aligned} \tag{2.7}$$

It also follows from the Table below that within the range $6 \le n \le 30$ the upper bound k^* is larger than k° for $n \in \{7, 8, 13, 14, 19, 20, 25, 26\}$, compare with Lemma 11 below.

2.1. The Marginal Cases $1 \le n \le 5$

To avoid tedious distinctions of low-degree cases, we present below the inequalities for pairs $|a_j| + |a_k|$ of coefficients of $P_n \in B_n$ or $P_n \in C_n$ for the first five polynomial degrees n. We confine ourselves to the instances $j \le k - 3$ (with $n - j$ odd and $n - k$ even) which are not covered by Szegő's coefficient inequality, and hence we need only to consider the marginal cases $3 \le n \le 5$.

A proof of the following five inequalities is indicated in Section 3.2 below:

$$\begin{aligned} n = 3: &\quad |a_0| + |a_3| \le |t_{3,3}| = 4, \\ n = 4: &\quad |a_1| + |a_4| \le |t_{4,4}| = 8, \\ n = 5: &\quad |a_0| + |a_3| \le |t_{5,3}| = 20, \\ &\quad |a_0| + |a_5| \le |t_{5,5}| = 16, \\ &\quad |a_2| + |a_5| \le |t_{5,5}| = 16. \end{aligned} \tag{2.8}$$

2.2. A Counterexample if $k = n$

Our Theorem does not hold any longer if we drop the upper bound k° resp. k^* resp. k^{**} on the index k and would thus allow $k \le n$, because then there exist polynomials $P_n \in C_n$ $(n \ge 6)$ with coefficients a_j $(n-j$ odd) and a_k $(n-k$ even and $j < k)$ with $|a_j| + |a_k| > |t_{n,k}|$. In particular, this is always the case for the choice $j = n-3$ and $k = n$, since we will show in Section 3.3 below that the coefficient $c_{n-1,n-3}$ of the Rogosinski polynomial $\Pi_{n-1} \in C_n$ satisfies:

$$|c_{n-1,n-3}| > t_{n,n} = 2^{n-1} \qquad \text{for all } n \ge 7. \tag{2.9}$$

Concerning the special case $n = 6$ consider the polynomial $P_6 \in C_6$ given explicitly by

$$\begin{aligned} P_6(x) &= \sum_{k=0}^{6} a_k x^k \\ &= (-1) + \frac{3+4\sqrt{3}}{3}x + \frac{43}{3}x^2 \\ &\quad + \frac{(-16-20\sqrt{3})}{3}x^3 + \frac{(-88)}{3}x^4 + \frac{16+16\sqrt{3}}{3}x^5 + 16x^6. \end{aligned}$$

Its coefficients satisfy the inequality

$$|a_3| + |a_6| = \frac{16+20\sqrt{3}}{3} + 16 = 32.88\ldots > t_{6,6} = 32.$$

2.3. The Table

The following Table displays, for $6 \le n \le 30$, the above defined values k°, k^*, k^{**}, q^*, q^{**}, and the numerically calculated value k_{opt}, see Section 4 below.

Within the Table the following three constellations occur regarding the three upper bounds: $k^\circ = k^* = k^{**}, k^\circ < k^* = k^{**}$ and $k^\circ = k^* < k^{**}$.

n	k°	$k^* = n - 2q^*$	$k^{**} = n - 2q^{**}$	q^*	q^{**}	k_{opt}
6	4	4	4	1	1	4
7	3	5	5	1	1	5
8	4	6	6	1	1	6
9	5	5	7	2	1	7
10	6	6	6	2	2	8
11	7	7	7	2	2	9
12	8	8	8	2	2	10
13	7	9	9	2	2	9
14	8	10	10	2	2	10
15	9	9	11	3	2	11
16	10	10	12	3	2	12
17	11	11	11	3	3	13
18	12	12	12	3	3	14
19	11	13	13	3	3	15
20	12	14	14	3	3	14
21	13	13	15	4	3	15
22	14	14	16	4	3	16
23	15	15	15	4	4	17
24	16	16	16	4	4	18
25	15	17	17	4	4	19
26	16	18	18	4	4	20
27	17	17	19	5	4	19
28	18	18	20	5	4	20
29	19	19	21	5	4	21
30	20	20	20	5	5	22

3. Proofs

3.1. Proof of the Theorem

We will prove the Theorem in the version of Addendum 2, i.e. with the upper bound $k^{**} < n$ valid for the index k in (2.1). We will then show that k^{**} majorizes the value k^* which can hence be chosen as a weaker but more convenient upper bound for k (Addendum 1). Finally we show that k^* majorizes k° which can hence be used as an even weaker upper bound for k. It is, on the other hand, easiest to evaluate for a given n.

To check which n is the lowest polynomial degree to satisfy the constellation $k^\circ(n) < k^*(n) < k^{**}(n)$ we may restrict the search to $n = 6m + 1$ or $n = 6m + 2$, with $m \geq 1$, see Lemma 11 below, and a straightforward evaluation of $k^{**}(6m + 1)$ and $k^{**}(6m + 2)$ according to (2.3) to (2.6) yields as solution $m = 7$ and $n = 43$, giving $k^\circ(43) = 27 < k^*(43) = 29 < k^{**}(43) = 31$.

To set the stage, let us consider an arbitrary $P_n \in C_n$. It can be represented on the grid (1.5) in Lagrange's interpolation form as $P_n(x) = \sum_{i=0}^{n} P_n(x^*_{n,i}) L_{n,i}(x)$, where $L_{n,i} \in \Phi_n$ is given by

$$L_{n,i}(x) = \frac{G_{n,i}(x)}{G_{n,i}(x^*_{n,i})}, \quad \text{with } G_{n,i}(x) = \prod_{s=0, s\neq i}^{n} (x - x^*_{n,s}) = \sum_{k=0}^{n} r_{n,i,k} x^k. \tag{3.1}$$

The coefficients $r_{n,i,k} = r_{i,k}$ (for short) of $G_{n,i}(x)$ are, up to the factor $(-1)^{n-k}$, elementary symmetric functions associated with the set of roots $\{x^*_{n,0}, x^*_{n,1}, \ldots, x^*_{n,n-1}, x^*_{n,n}\} \setminus \{x^*_{n,i}\}$, with special instances

$$r_{i,n} = 1, \quad r_{i,n-1} = -\sum_{s=0, s\neq i}^{n} x^*_{n,s}, \quad \text{and } r_{i,0} = (-1)^n \prod_{s=0, s\neq i}^{n} x^*_{n,s}. \tag{3.2}$$

Combining coefficients of like powers we obtain from (3.1) for the k-th coefficient of P_n with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ the representation

$$a_k = \sum_{i=0}^{n} \frac{P_n(x^*_{n,i}) r_{i,k}}{G_{n,i}(x^*_{n,i})}. \tag{3.3}$$

The crucial numbers

$$V_{n,i,k} = V_{i,k} \text{ (for short) } = \frac{r_{i,k}}{G_{n,i}(x^*_{n,i})} \tag{3.4}$$

are, for each i, the coefficients of $L_{n,i}(x)$ with respect to the grid (1.5), and can be identified with the elements of the inverse of the Vandermonde matrix associated with the extremal points of T_n. This Vandermonde matrix v_n and its inverse V_n are given by

$$V_n = (V_{i,k})_{0\leq i,k\leq n} = (v_n)^{-1}, \tag{3.5}$$

where

$$v_n = (v_{i,k})_{0\leq i,k\leq n}, \quad \text{with } v_{i,k} = (x^*_{n,k})^i \text{ and } x^*_{n,k} = \cos\frac{(n-k)\pi}{n}. \tag{3.6}$$

In the course of the proof we will utilize properties of the matrix V_n and therefore we need to delve into the structure of V_n. There is a vast literature on the inversion of the (arbitrary) Vandermonde matrix, see e.g. [8] and the references given therein, but for the particular case of the Vandermonde matrix and its inverse associated with the extremal points of T_n there seem to be only few references, such as [6] and [11]. For our purpose we find it appropriate to start from the scratch.

The zero-symmetry of the $x^*_{n,i}$'s induces a special form for the polynomials $G_{n,i}(x)$:

for n even:

$$G_{n,i}(x) = x^2 \prod_{s=\frac{n}{2}+1, s\neq n-i}^{n} (x^2 - (x^*_{n,s})^2) - x x^*_{n,n-i} \times \prod_{s=\frac{n}{2}+1, s\neq n-i}^{n} (x^2 - (x^*_{n,s})^2), \qquad \text{if } 0 \leq i \leq \frac{n}{2} - 1, \tag{3.7}$$

$$G_{n,i}(x) = \prod_{s=\frac{n}{2}+1}^{n} (x^2 - (x^*_{n,s})^2), \qquad \text{if } i = \frac{n}{2}, \tag{3.8}$$

$$G_{n,i}(x) = x^2 \prod_{s=\frac{n}{2}+1, s\neq i}^{n} (x^2 - (x^*_{n,s})^2) + x x^*_{n,i} \prod_{s=\frac{n}{2}+1, s\neq i}^{n} (x^2 - (x^*_{n,s})^2), \qquad \text{if } \frac{n}{2} + 1 \leq i \leq n. \tag{3.9}$$

for n odd:

$$G_{n,i}(x) = x \prod_{s=\frac{n+1}{2}, s\neq n-i}^{n} (x^2 - (x^*_{n,s})^2) - x^*_{n,n-i} \prod_{s=\frac{n+1}{2}, s\neq n-i}^{n} (x^2 - (x^*_{n,s})^2), \qquad \text{if } 0 \leq i \leq \frac{n-1}{2}, \tag{3.10}$$

$$G_{n,i}(x) = x \prod_{s=\frac{n+1}{2}, s\neq i}^{n} (x^2 - (x^*_{n,s})^2) + x^*_{n,i} \prod_{s=\frac{n+1}{2}, s\neq i}^{n} (x^2 - (x^*_{n,s})^2), \qquad \text{if } \frac{n+1}{2} \leq i \leq n. \tag{3.11}$$

It follows from this representation that for each index k with $n-k$ even the coefficient $r_{i,k}$ of the even resp. odd part of $G_{n,i}(x)$ is, up to the factor $(-1)^{n-k}$, an elementary symmetric function associated with the following sets of positive roots:

$$X^2_{n,i} = \{(x^*_{n,\frac{n}{2}+1})^2, (x^*_{n,\frac{n}{2}+2})^2, \ldots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\} \setminus \{(x^*_{n,n-i})^2\},$$

if n is even and $0 \leq i \leq \frac{n}{2} - 1$;

$$X^2_{n,\frac{n}{2}} = \{(x^*_{n,\frac{n}{2}+1})^2, (x^*_{n,\frac{n}{2}+2})^2, \ldots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\},$$

if n is even and $i = \frac{n}{2}$;

$$X^2_{n,i} = \{(x^*_{n,\frac{n+1}{2}})^2, (x^*_{n,\frac{n+1}{2}+1})^2, \ldots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\} \setminus \{(x^*_{n,n-i})^2\},$$

if n is odd and $0 \le i \le \frac{n-1}{2}$.

If $X = \{x_1, x_2, \ldots, x_{w-1}, x_w\}$ is a set of positive distinct real numbers, we will denote by $C_q = C_q(X)$ the q-th (positive) elementary symmetric function of the variables from X, i.e. C_q is the sum of the products, q at a time, of the x_u's, with $C_0 = 1, C_1 = \sum_{u=1}^{w} x_u, \ldots, C_w = \prod_{u=1}^{w} x_u$, compare with [15, p. 72].

For later reference we set, for k with $n-k$ even,

$$|r_{i,k}| = |r_{i,n-2q}| = C_q(X^2_{n,i}) \tag{3.12}$$

with $0 \le i, q \le \frac{n}{2} - 1$ if n is even, and $0 \le i, q \le \frac{n-1}{2}$ if n is odd;

$$|r_{\frac{n}{2},k}| = |r_{\frac{n}{2},n-2q}| = C_q(X^2_{n,\frac{n}{2}}) \tag{3.13}$$

with $0 \le q \le \frac{n}{2}$ if n is even.

Next we turn to the denominator of $V_{i,k}$ in (3.4). It follows by inspection that the numbers $G_{n,i}(x^*_{n,i})$ alternate in sign (see also Lemma 2 below):

$$\operatorname{sign}\,(G_{n,i}(x^*_{n,i})) = (-1)^{n-i}, \qquad 0 \le i \le n. \tag{3.14}$$

These observations suffice to state a "skeleton theorem" concerning the matrix V_n: All the elements of V_n are uniquely determined if one knows the elements in the upper halves of the columns $(V_{i,k})_{0 \le i \le n}$, where only those columns need to be considered whose column index k is such that $n-k$ is even, see also [20, p. 341]. More precisely, we have:

Lemma 1. *If $n-k$ is even and $0 \le i \le \lfloor \frac{n}{2} \rfloor$, then*

$$V_{i,k} = (-1)^k V_{n-i,k} \qquad \text{and} \qquad V_{i,k-1} = x^*_{n,i} V_{i,k}. \tag{3.15}$$

Proof. The proof follows immediately from (3.4) to (3.11), and (3.14). □

With this information at hand we continue with the proof of our Theorem. We deduce from (3.3), with $j < k$ and $n-k$ even and $n-j$ odd, that

$$\begin{aligned} |a_j + a_k| &= \Big|\sum_{i=0}^{n} P_n(x^*_{n,i})V_{i,j} + \sum_{i=0}^{n} P_n(x^*_{n,i})V_{i,k}\Big| = \Big|\sum_{i=0}^{n} P_n(x^*_{n,i})(V_{i,j} + V_{i,k})\Big| \\ &\le \sum_{i=0}^{n} |P_n(x^*_{n,i})||V_{i,j} + V_{i,k}| \le \sum_{i=0}^{n} |V_{i,j} + V_{i,k}|. \end{aligned}$$

This upper bound can be split into $\sum_{i=0}^{\frac{n}{2}-1} |V_{i,j}+V_{i,k}| + \sum_{i=\frac{n}{2}+1}^{n} |V_{i,j}+V_{i,k}| + |V_{\frac{n}{2},k}|$, if n is even (note that $V_{\frac{n}{2},j}=0$), resp. into $\sum_{i=0}^{\frac{n-1}{2}} |V_{i,j}+V_{i,k}| + \sum_{i=\frac{n-1}{2}+1}^{n} |V_{i,j}+V_{i,k}|$, if n is odd. Now, Lemma 1 implies:

$$|a_j + a_k| \le \sum_{i=0}^{\frac{n}{2}-1} |V_{i,j}+V_{i,k}| + \sum_{i=0}^{\frac{n}{2}-1} |V_{i,j}-V_{i,k}| + |V_{\frac{n}{2},k}|, \qquad \text{if } n \text{ is even,}$$

$$|a_j + a_k| \le \sum_{i=0}^{\frac{n-1}{2}} |V_{i,j}+V_{i,k}| + \sum_{i=0}^{\frac{n-1}{2}} |V_{i,j}-V_{i,k}|, \qquad \text{if } n \text{ is odd.}$$

The obvious identity $|\gamma+\delta| + |\gamma-\delta| = 2\max\{|\gamma|, |\delta|\}$, valid for any real numbers γ and δ, further implies that

$$|a_j + a_k| \le 2\sum_{i=0}^{\frac{n}{2}-1} \max\{|V_{i,j}|, |V_{i,k}|\} + |V_{\frac{n}{2},k}|, \qquad \text{if } n \text{ is even,} \tag{3.16}$$

$$|a_j + a_k| \le 2\sum_{i=0}^{\frac{n-1}{2}} \max\{|V_{i,j}|, |V_{i,k}|\}, \qquad \text{if } n \text{ is odd.} \tag{3.17}$$

Suppose now that we had $|V_{i,j}| \le |V_{i,k}|$ for all i in question. We could then conclude as follows, see (3.15):

$$|a_j + a_k| \le 2\sum_{i=0}^{\frac{n}{2}-1} |V_{i,k}| + |V_{\frac{n}{2},k}| = \sum_{i=0}^{n} |V_{i,k}|, \qquad \text{if } n \text{ is even,}$$

$$|a_j + a_k| \le 2\sum_{i=0}^{\frac{n-1}{2}} |V_{i,k}| = \sum_{i=0}^{n} |V_{i,k}|, \qquad \text{if } n \text{ is odd.}$$

It follows from the oscillating properties of the polynomials T_n (and Π_{n-1}), see [20, p. 340] and the references given there, that $\sum_{i=0}^{n} |V_{i,k}| = |t_{n,k}|$, if $n-k$ is even (and $\sum_{i=0}^{n} |V_{i,k}| = |c_{n-1,k}|$, if $n-k$ is odd), and hence we would get $|a_j + a_k| \le |t_{n,k}|$. Applying the same conclusion to the polynomial $P_n^\circ \in C_n$, where $P_n^\circ(x) = P_n(-x)$, we would likewise get $|a_j - a_k| \le |t_{n,k}|$, and hence altogether $|a_j| + |a_k| \le |t_{n,k}|$, as required, since $|\gamma| + |\delta| = \max\{|\gamma+\delta|, |\gamma-\delta|\}$ is valid for any real numbers γ and δ.

According to Lemma 1 we have, for $j = k-1, V_{i,k-1} = x_{n,i}^* V_{i,k}$, and $|x_{n,i}^*| \le 1$, so that we get $|V_{i,k-1}| \le |V_{i,k}|$ for all admissible k and i and thus we arrive at a proof of (1.10). But we are particularly interested in the cases where $j \le k-3$ and $n-j$ is odd.

The basic problem is thus to

(∗) *find out for which values of* k*, with* $n-k$ *even, there holds* $|V_{i,j}| \leq |V_{i,k}|$*, for all* $j \leq k-3$ *with* $n-j$ *odd, and for all* i *with* $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$*.*

In an attempt to approach the problem (∗) we confine ourselves to find out for which values of k, with $l < k$ and both $n-l$ and $n-k$ even, there holds $|V_{i,l}| \leq |V_{i,k}|$ for all admissible i since this implies $|V_{i,l-1}| = |x^*_{n,i}||V_{i,l}| \leq |V_{i,l}| \leq |V_{i,k}|$ for all i under consideration. But we are aware that this confinement may produce less admissable values of k than is asked for in (∗). The reason is that it may happen, for some i, that $|V_{i,j}| \leq |V_{i,k}|$ holds, but $|V_{i,j+1}| \leq |V_{i,k}|$ does not hold ($j \leq k-3$ with $n-j$ odd). This case occurs for the first time for the polynomial degree $n = 10$ and $k = 8$ and $i = j = 5$, see Remark 7 below.

We will thus compare the magnitude of the elements in the i-th row of the upper half of the matrix V_n, and will thereby consider only those columns of V_n whose column index has the same parity as n. To this end, we have to delve even deeper into the structure of V_n.

We shall first need the explicit value of the denominator of $V_{i,k}$ and obtain (thus extending (3.14)), see also [6, p. 1395]:

Lemma 2.

$$G_{n,i}(x^*_{n,i}) = (-1)^{n-i} n 2^{2-n}, \qquad \textit{if } i = 0 \textit{ or } i = n, \tag{3.18}$$
$$G_{n,i}(x^*_{n,i}) = (-1)^{n-i} n 2^{1-n}, \qquad \textit{if } 1 \leq i \leq n-1. \tag{3.19}$$

Proof. Consider $S_{n+1} \in \Phi_{n+1}$ given by

$$S_{n+1}(x) = \prod_{s=0}^{n}(x - x^*_{n,s}) = \sum_{p=0}^{n+1} \sigma_{n+1,p} x^p. \tag{3.20}$$

Differentiating S_{n+1} at $x = x^*_{n,i}$ yields $S'_{n+1}(x^*_{n,i}) = G_{n,i}(x^*_{n,i}), 0 \leq i \leq n$. Since the interior extrema $x^*_{n,1}, x^*_{n,2}, \ldots, x^*_{n,n-1}$ of T_n are the zeros of $U_{n-1} = T'_n/n$ (the Chebyshev polynomial of the second kind of degree $n-1$, with leading coefficient 2^{n-1}, see [23, p. 7]), we get

$$S_{n+1}(x) = 2^{1-n}(x^2 - 1)U_{n-1}(x), \tag{3.21}$$
$$S'_{n+1}(x) = 2^{1-n}(x^2 - 1)U'_{n-1}(x) + 2^{2-n} x U_{n-1}(x). \tag{3.22}$$

Evaluating (3.22) at $x = x^*_{n,0} = -1$ resp. at $x = x^*_{n,n} = 1$ gives $G_{n,0}(x^*_{n,0}) = -2^{2-n}U_{n-1}(-1) = (-1)^n n 2^{2-n}$ resp. $G_{n,n}(x^*_{n,n}) = 2^{2-n}U_{n-1}(1) = n2^{2-n}$, which proves (3.18). Evaluating (3.22) at $x = x^*_{n,i}, 1 \leq i \leq n-1$, gives $G_{n,i}(x^*_{n,i}) = 2^{1-n}((x^*_{n,i})^2 - 1)U'_{n-1}(x^*_{n,i})$. Invoking the identities $U'_{n-1} = T''_n/n$ and $(1-x^2)T''_n(x) - xT'_n(x) + n^2 T_n(x) = 0$ (see [23, p. 36]), gives

$$G_{n,i}(x^*_{n,i}) = \frac{2^{1-n}((x^*_{n,i})^2 - 1)(-n^2 T_n(x^*_{n,i}))}{n(1 - (x^*_{n,i})^2)} = (-1)^{n-i} n 2^{1-n},$$

which proves (3.19). □

Next, we are going to establish an explicit representation for the coefficients $\sigma_{n+1,p}$ of S_{n+1} in terms of the known coefficients $t_{n,k}$ of T_n. Note that, according to (3.20), the $\sigma_{n+1,p}$'s are, up to the sign, elementary symmetric functions associated with the zero-symmetric extremal points (1.5) of T_n, compare with [15, p. 53].

Lemma 3. *The coefficients $\sigma_{n+1,p}$ of the monic polynomial S_{n+1} are explicitly given by*

$$\sigma_{n+1,n+1} = 1, \tag{3.23}$$
$$\sigma_{n+1,p} = p^{-1}n^{-1}2^{1-n}(n^2+p-1)t_{n,p-1}, \quad \text{if } n+1-p \text{ is even and } 2 \le p \le n-1, \tag{3.24}$$
$$\sigma_{n+1,p} = 0, \quad \text{if } n-p \text{ is even and } 2 \le p \le n, \tag{3.25}$$
$$\sigma_{n+1,1} = (-1)^{\frac{n}{2}} n 2^{1-n}, \quad \text{if } n \text{ is even}, \qquad \sigma_{n+1,1} = 0, \quad \text{if } n \text{ is odd}, \tag{3.26}$$
$$\sigma_{n+1,0} = (-1)^{\frac{n+1}{2}} 2^{1-n}, \quad \text{if } n \text{ is odd} \qquad \sigma_{n+1,0} = 0, \quad \text{if } n \text{ is even}. \tag{3.27}$$

Proof. We apply the known identity $(x^2-1)U_{n-1}(x) = xT_n(x) - T_{n-1}(x)$ (see (1.2) and [23, p. 9]) to alternatively obtain $S_{n+1}(x) = 2^{1-n}(xT_n(x) - T_{n-1}(x))$. Expanding the right-hand side of this equation in powers of x and collecting coefficients of like powers immediately yields (3.23) to (3.27). □

The explicit expression in Lemma 3 for the coefficients of S_{n+1} induces an explicit expression for the coefficients of $G_{n,i} \in \Phi_n$, that is, for the numerator of $V_{i,k}$:

Lemma 4. *The coefficients $r_{i,k}(0 \le i,k \le n)$ of the monic polynomials $G_{n,i}$ (see (3.1)) are explicitly given by*

$$r_{i,n} = 1, \tag{3.28}$$
$$r_{i,n-1} = x^*_{n,i} r_{i,n} = x^*_{n,i}, \tag{3.29}$$
$$r_{i,n-2} = \sigma_{n+1,n-1} + (x^*_{n,i})^2, \tag{3.30}$$
$$r_{i,n-3} = x^*_{n,i}(\sigma_{n+1,n-1} + (x^*_{n,i})^2) = x^*_{n,i} r_{i,n-2}, \tag{3.31}$$
$$r_{i,n-2q} = \sum_{t=0}^{q} (x^*_{n,i})^{2t} \sigma_{n+1,n+1-2(q-t)}, \quad \text{if } 2 \le q \le \lfloor \tfrac{n+1}{2} \rfloor, \tag{3.32}$$
$$r_{i,n-2q-1} = x^*_{n,i} r_{i,n-2q}, \quad \text{if } 2 \le q \le \lceil \tfrac{n-2}{2} \rceil. \tag{3.33}$$

Proof. Since S_{n+1} is an even resp. odd polynomial, depending on the parity of $n+1$, we obtain recursively by synthetic polynomial division, in virtue of $G_{n,i}(x) = \frac{S_{n+1}(x)}{(x-x^*_{n,i})}$, see also [15, p. 53]:

$$
\begin{aligned}
r_{i,n} &= \sigma_{n+1,n+1} = 1, && (3.34)\\
r_{i,n-1} &= \sigma_{n+1,n} + x^*_{n,i}\sigma_{n+1,n+1} = \sigma_{n+1,n} + x^*_{n,i} r_{i,n} = x^*_{n,i} r_{i,n} = x^*_{n,i}, && (3.35)\\
r_{i,n-2} &= \sigma_{n+1,n-1} + x^*_{n,i}(\sigma_{n+1,n} + x^*_{n,i}\sigma_{n+1,n+1}) && (3.36)\\
&= \sigma_{n+1,n-1} + x^*_{n,i} r_{i,n-1} = \sigma_{n+1,n-1} + (x^*_{n,i})^2,\\
r_{i,n-3} &= \sigma_{n+1,n-2} + x^*_{n,i}(\sigma_{n+1,n-1} + x^*_{n,i}(\sigma_{n+1,n} + x^*_{n,i}\sigma_{n+1,n+1})) && (3.37)\\
&= \sigma_{n+1,n-2} + x^*_{n,i}(\sigma_{n+1,n-1} + (x^*_{n,i})^2)\\
&= x^*_{n,i}(\sigma_{n+1,n-1} + (x^*_{n,i})^2) = x^*_{n,i} r_{i,n-2},
\end{aligned}
$$

and generally, by mathematical induction, we get (3.32) and (3.33). □

From this we eventually obtain an explicit expression for the $(n+1)^2$ elements of V_n:

Lemma 5. *The elements $V_{i,k}, 0 \le i,k \le n$, of the inverse of the Vandermonde matrix* (3.5) *associated with the extremal points of T_n are given as follows:*

(a) Elements $V_{0,k}$, if $n-k$ is even, are:

$$V_{0,n} = (-1)^n n^{-1} 2^{n-2}, \tag{3.38}$$

$$V_{0,n-2q} = (-1)^n n^{-1} 2^{n-2} \sum_{t=0}^{q} \sigma_{n+1,n+1-2(q-t)}, \quad \text{if } 1 \le q \le \lceil \tfrac{n-2}{2} \rceil. \tag{3.39}$$

$$V_{0,0} = 0, \qquad \text{if } n \text{ is even.} \tag{3.40}$$

(b) Elements $V_{i,k}$ with $1 \le i \le \lfloor \frac{n-1}{2} \rfloor$, if $n-k$ is even, are:

$$V_{i,n} = (-1)^{n-i} n^{-1} 2^{n-1}, \tag{3.41}$$

$$V_{i,n-2q} = (-1)^{n-i} n^{-1} 2^{n-1} \sum_{t=0}^{q} (x^*_{n,i})^{2t} \sigma_{n+1,n+1-2(q-t)}, \tag{3.42}$$

$$\text{if } 1 \le q \le \lceil \tfrac{n-2}{2} \rceil.$$

$$V_{i,0} = 0, \qquad \text{if } n \text{ is even.} \tag{3.43}$$

(c) Elements $V_{i,k}$ in the row for $i = \frac{n}{2}$, if n is even and $n-k$ is even, are:

$$V_{\frac{n}{2},n} = (-1)^{\frac{n}{2}} n^{-1} 2^{n-1}, \tag{3.44}$$

$$V_{\frac{n}{2},n-2q} = (-1)^{\frac{n}{2}} n^{-2} (n-2q+1)^{-1} (n^2+n-2q) t_{n,n-2q}, \tag{3.45}$$

$$\text{if } 2 \le n-2q \le n-2,$$

$$V_{\frac{n}{2},0} = 1. \tag{3.46}$$

Those elements $V_{i,k}$ of V_n not covered by identities (3.38) *to* (3.46) *can be recovered from these identities by applying Lemma 1.*

Proof. The proof is a straightforward combination of the preceding Lemmas 2, 3, and 4. □

With this additional information at hand we are able to complete the proof of our Theorem. Since we treat the marginal cases $n \leq 5$ separately, we will henceforth assume that $n \geq 6$. Consider first an even n, and pick in V_n the middle row, with row number $i = \frac{n}{2}$. In that row the elements in the columns k with $n-k$ even read as given in (3.44), (3.45), and (3.46).

We set, invoking the definitions (3.13) and (3.19),

$$\begin{aligned} |V_{\frac{n}{2},n-2q}| = \frac{|r_{\frac{n}{2},n-2q}|}{|G_{n,\frac{n}{2}}(x^*_{n,\frac{n}{2}})|} &= |r_{\frac{n}{2},n-2q}|n^{-1}2^{n-1} \\ = C_q(X^2_{n,\frac{n}{2}})n^{-1}2^{n-1}, &\qquad 0 \leq q \leq \frac{n}{2}, \end{aligned} \tag{3.47}$$

and compare the ratio of two consecutive such elements:

Lemma 6. *If n is even, then*

$$\begin{aligned} \frac{|V_{\frac{n}{2},n-2q}|}{|V_{\frac{n}{2},n-2q-2}|} = \frac{|r_{\frac{n}{2},n-2q}|}{|r_{\frac{n}{2},n-2q-2}|} &= \frac{C_q(X^2_{n,\frac{n}{2}})}{C_{q+1}(X^2_{n,\frac{n}{2}})} \\ = F_n(q) &= \frac{4(n^2+n-2q)(n-q-1)(q+1)}{(n^2+n-2q-2)(n-2q+1)(n-2q)}. \end{aligned} \tag{3.48}$$

Proof. The identities follow from (1.3), (3.45) and (3.47) by straightforward calculation. □

We will consider $F_n(q)$ a real function in the variable q and with parameter n.

Lemma 7. *For n even, the function $F_n(q)$ in* (3.48) *is strictly monotone increasing for $q = 0, 1, \ldots, \frac{n}{2} - 1$ with $F_n(0) = \frac{4}{n+2} < 1$ and $F_n(\frac{n}{2} - 1) = \frac{n^2+2}{6} > 1$.*

Proof. The proof follows from (3.48) by making use of a well-known inequality for elementary symmetric functions (see [10, p. 52], [15, p. 73], [28, p. 238]), stated here with respect to the set $X^2_{n,\frac{n}{2}}$:

$$(C_q(X^2_{n,\frac{n}{2}}))^2 - C_{q-1}(X^2_{n,\frac{n}{2}})C_{q+1}(X^2_{n,\frac{n}{2}}) > 0. \tag{3.49}$$

This inequality implies $\frac{C_q(X^2_{n,\frac{n}{2}})}{C_{q+1}(X^2_{n,\frac{n}{2}})} > \frac{C_{q-1}(X^2_{n,\frac{n}{2}})}{C_q(X^2_{n,\frac{n}{2}})}$, that is, $F_n(q) > F_n(q-1)$ for all $q = 1, \ldots, \frac{n}{2} - 1$. The evaluation of the marginal cases $F_n(0)$ and $F_n(\frac{n}{2} - 1)$ is straightforward. □

The monotonicity of F_n at the integer points of the interval $[0, \frac{n}{2} - 1]$ and the values of F_n at the endpoints of that interval imply that F_n will cross the line $y = y(q) = 1$ at some interior point of the interval $[0, \frac{n}{2} - 1]$. Actually, F_n

is strictly monotone increasing on the whole interval $[0, \frac{n}{2} - 1]$, as follows from calculus since we have there $0 < F_n'(q) = \frac{\alpha(q)}{\beta(q)}$ with

$$\begin{aligned}\alpha(q) = 4\big(&10n^2 - 2n^3 - 15n^4 + n^5 + 5n^6 + n^7\\ &+ q(-24n + 28n^2 + 42n^3 - 26n^4 - 18n^5 - 2n^6)\\ &+ q^2(24 - 72n - 18n^2 + 72n^3 + 18n^4)\\ &+ q^3(48 - 48n - 48n^2) + 24q^4\big),\\ \beta(q) = (&n - 2q)^2(1 + n - 2q)^2(-2 + n + n^2 - 2q)^2.\end{aligned}$$

The cubic equation in q (see (3.48)), $F_n(q) - 1 = 0$, i.e.,

$$q^3 + \frac{q^2(-2n^2 - 6n + 5)}{4} + \frac{q(2n^3 + n^2 - 5n + 1)}{4} + \frac{(-n^4 + 2n^3 + n^2 - 2n)}{16} = 0,$$

yields with the aid of Cardano's formula a suitable solution $q = q_{\#\#}$, and in order to continue the argument with integer values, we choose $q = q^{**} = \lceil q_{\#\#} \rceil$, as is explicitly given in (2.3).

We thus have $C_q(X^2_{n,\frac{n}{2}}) = F_n(q)C_{q+1}(X^2_{n,\frac{n}{2}})$ with $F_n(q) \geq 1$ for all $q \geq q^{**}$ and this means

$$|V_{\frac{n}{2},n-2q}| = C_q(X^2_{n,\frac{n}{2}})n^{-1}2^{n-1} \geq |V_{\frac{n}{2},n-2q-2}| = C_{q+1}(X^2_{n,\frac{n}{2}})n^{-1}2^{n-1}$$

for all $q \geq q^{**}$, and hence $|V_{\frac{n}{2},n-2q^{**}}| \geq |V_{\frac{n}{2},n-2q}|$ for all $q \geq q^{**}$, i.e.,

$$|V_{\frac{n}{2},n-2q^{**}}| \geq |V_{\frac{n}{2},n-2q^{**}-2}| \geq |V_{\frac{n}{2},n-2q^{**}-4}| \geq \cdots \geq |V_{\frac{n}{2},2}|,$$

and eventually

$$|V_{\frac{n}{2},k^{**}}| \geq |V_{\frac{n}{2},k}| \geq |V_{\frac{n}{2},j}| = 0 \tag{3.50}$$

for all $j < k \leq k^{**} = n - 2q^{**}$ with $n - k$ even and $n - j$ odd.

This set of inequalities describes an ordering of elements in the middle row of V_n, where n is even. Observe that $k^{**} < n$ since $0 < q_{\#\#}$ and hence $1 \leq q^{**}$. We continue with an even n and consider now the rows of V_n with row number i where $0 \leq i \leq \frac{n}{2} - 1$. We proceed as follows, going back on the solution already found for the row number $i = \frac{n}{2}$: We compare row-wise the elements in the upper halves of two arbitrary columns with column number $k - 2 = n - 2q - 2$ and $k = n - 2q$, where $q^{**} \leq q \leq \frac{n}{2} - 2$. We will show that then $|V_{i,n-2q}| \geq |V_{i,n-2q-2}| \geq |x^*_{n,i}||V_{i,n-2q-2}| = |V_{i,n-2q-3}|$ holds for all i with $0 \leq i \leq \frac{n}{2}$, where the second inequality is trivial. Since the pair of columns with column number $k - 2$ and k is arbitrary (subject to $4 \leq k \leq k^{**} = n - 2q^{**}$), we eventually will arrive at $|V_{i,k^{**}}| \geq |V_{i,k}| \geq |V_{i,j}|$ for all indices $j < k \leq k^{**} = n - 2q^{**}$, with $n - k$ even and $n - j$ odd, and for all i.

Lemma 8. *Let $(V_{i,k-2})_{0\leq i\leq n}$ and $(V_{i,k})_{0\leq i\leq n}$, with $4 \leq k \leq k^{**}$ and $n - k$ even, denote two columns of the matrix V_n, where n is even and $k^{**} = n - 2q^{**}$*

with q^{**} *given by* (2.3). *For the elements of these columns then holds the inequality*

$$|V_{i,k-2}| \le |V_{i,k}| \qquad \textit{for } 0 \le i \le \frac{n}{2} - 1, \tag{3.51}$$

and hence (3.51) *holds for all* i*, by virtue of Lemma 1 and* (3.50).

Proof. We start with $0 \le i \le \frac{n}{2} - 2$, and with the obvious inequality $(x^*_{n,n-i-1})^2 < (x^*_{n,n-i})^2$ and multiply both sides of this inequality by the positive quantity

$$(C_q(X^2_{n,i} \backslash \{(x^*_{n,n-i-1})^2\}))^2 - C_{q+1}(X^2_{n,i} \backslash \{(x^*_{n,n-i-1})^2\}) C_{q-1}(X^2_{n,i} \backslash \{(x^*_{n,n-i-1})^2\}),$$

see [10, p. 52], [15, p. 73], [28, p. 238], where

$$X^2_{n,i} = \{(x^*_{n,\frac{n}{2}+1})^2, (x^*_{n,\frac{n}{2}+2})^2, \dots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\} \setminus \{(x^*_{n,n-i})^2\}$$

and

$$\begin{aligned} &X^2_{n,i} \setminus \{(x^*_{n,n-i-1})^2\} = X^2_{n,i+1} \setminus \{(x^*_{n,n-i})^2\} \\ &\quad = \{(x^*_{n,\frac{n}{2}+1})^2, (x^*_{n,\frac{n}{2}+2})^2, \dots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\} \setminus \{(x^*_{n,n-i-1})^2, (x^*_{n,n-i})^2\}. \end{aligned}$$

We temporarily set $X^2_{n,i} \backslash \{(x^*_{n,n-i-1})^2\} = X^2$, and thus get

$$\begin{aligned} &(x^*_{n,n-i-1})^2((C_q(X^2))^2 - C_{q+1}(X^2)C_{q-1}(X^2)) \\ &\qquad\qquad < (x^*_{n,n-i})^2((C_q(X^2))^2 - C_{q+1}(X^2)C_{q-1}(X^2)). \end{aligned}$$

Adding $C_q(X^2)C_{q+1}(X^2) + (x^*_{n,n-i-1})^2(x^*_{n,n-i})^2 C_q(X^2)C_{q-1}(X^2)$ to both sides of the latter inequality results in

$$\begin{aligned} &[C_{q+1}(X^2) + (x^*_{n,n-i-1})^2 C_q(X^2)][C_q(X^2) + (x^*_{n,n-i})^2 C_{q-1}(X^2)] \\ &\quad < [C_q(X^2) + (x^*_{n,n-i-1})^2 C_{q-1}(X^2)][C_{q+1}(X^2) + (x^*_{n,n-i})^2 C_q(X^2)]. \end{aligned}$$

It follows from the definition of elementary symmetric functions that the linear combinations within the square brackets on the left-hand side of this inequality can be replaced by $C_{q+1}(X^2_{n,i})$ and $C_q(X^2_{n,i+1})$, see also [10, p. 54], [15, p. 54] or [28, p. 235].

Analogously we get for the linear combinations within the square brackets on the right-hand side of this inequality the values $C_q(X^2_{n,i})$ and $C_{q+1}(X^2_{n,i+1})$. This amounts to the inequality $|r_{i,k-2}||r_{i+1,k}| < |r_{i,k}||r_{i+1,k-2}|$ or, equivalently, $|V_{i,k-2}||V_{i+1,k}| < |V_{i,k}||V_{i+1,k-2}|$, from which we deduce

$$\frac{|V_{i,k-2}|}{|V_{i,k}|} < \frac{|V_{i+1,k-2}|}{|V_{i+1,k}|} \qquad \text{for } 0 \le i \le \frac{n}{2} - 2. \tag{3.52}$$

We now consider the case $i = \frac{n}{2} - 1$ which will link the rows with row number $\frac{n}{2} - 1$ and $\frac{n}{2}$. Consider the elementary symmetric function $C_q = C_q(X^2_{n,\frac{n}{2}-1})$

where $X^2_{n,\frac{n}{2}-1} = X^2_{n,\frac{n}{2}} \setminus \{(x^*_{n,\frac{n}{2}+1})^2\} = \{(x^*_{n,\frac{n}{2}+2})^2, \dots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\}$. We have $(x^*_{n,\frac{n}{2}+1})^2((C_q)^2 - C_{q+1}C_{q-1}) > 0$, since both factors are positive. By adding $C_{q+1}C_{q-1} > 0$ to both sides of this inequality we obtain the inequality $C_{q+1}[C_q + (x^*_{n,\frac{n}{2}+1})^2 C_{q-1}] < C_q[C_{q+1} + (x^*_{n,\frac{n}{2}+1})^2 C_q]$. The values in square brackets are, by the same argument as above, identical with $C_q(X^2_{n,\frac{n}{2}})$ resp. $C_{q+1}(X^2_{n,\frac{n}{2}})$. We thus get the inequality $C_{q+1}C_q(X^2_{n,\frac{n}{2}}) < C_q C_{q+1}(X^2_{n,\frac{n}{2}})$, i.e., $C_{q+1}(X^2_{n,\frac{n}{2}-1})C_q(X^2_{n,\frac{n}{2}}) < C_q(X^2_{n,\frac{n}{2}-1})C_{q+1}(X^2_{n,\frac{n}{2}})$.

This implies $|r_{\frac{n}{2}-1,k-2}||r_{\frac{n}{2},k}| < |r_{\frac{n}{2}-1,k}||r_{\frac{n}{2},k-2}|$, or equivalently, $|V_{\frac{n}{2}-1,k-2}||V_{\frac{n}{2},k}| < |V_{\frac{n}{2}-1,k}||V_{\frac{n}{2},k-2}|$ and hence $\frac{|V_{\frac{n}{2}-1,k-2}|}{|V_{\frac{n}{2}-1,k}|} < \frac{|V_{\frac{n}{2},k-2}|}{|V_{\frac{n}{2},k}|}$. But for $i = \frac{n}{2}$ we already know from (3.50) that $\frac{|V_{\frac{n}{2},k-2}|}{|V_{\frac{n}{2},k}|} \le 1$ holds. Altogether we thus obtain the following chain of inequalities so that 3.51 holds:

$$\frac{|V_{0,k-2}|}{|V_{0,k}|} < \frac{|V_{1,k-2}|}{|V_{1,k}|} < \frac{|V_{2,k-2}|}{|V_{2,k}|} < \dots < \frac{|V_{\frac{n}{2}-1,k-2}|}{|V_{\frac{n}{2}-1,k}|} < \frac{|V_{\frac{n}{2},k-2}|}{|V_{\frac{n}{2},k}|} \le 1. \qquad \square$$

We next turn to the case of odd n. Similarly to the case of even n, we want to show that there is some $k^{**} = n - 2q^{**}$ with $\frac{|V_{i,n-2q-2}|}{|V_{i,n-2q}|} = \frac{|r_{i,n-2q-2}|}{|r_{i,n-2q}|} \le 1$ for all $k = n-2q \le k^{**}$, i.e. for all $q \ge q^{**}$, and for all i subject to $0 \le i \le (n-1)/2$, and hence for all i, in view of Lemma 1. To this end, we first observe that we have, for any $k = n - 2q$,

$$\frac{|V_{i,k-2}|}{|V_{i,k}|} < \frac{|V_{i+1,k-2}|}{|V_{i+1,k}|} \qquad \text{for } 0 \le i \le \frac{n-1}{2} - 1. \tag{3.53}$$

We skip the proof of this inequality since it is quite analogous to the proof of (3.52), however, the set $X^2_{n,i}$ is now adjusted to n odd, i.e.,

$$X^2_{n,i} = \{(x^*_{n,\frac{n+1}{2}})^2, (x^*_{n,\frac{n+1}{2}+1})^2, \dots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\} \setminus \{(x^*_{n,n-i})^2\},$$

and the row number i now ranges from 0 to $\frac{n-1}{2} - 1$, since for n odd the exceptional "middle-row" does not exist.

Consider now the ratio of two elements in row number $i = \frac{n-1}{2}$ of V_n. The elements are supposed to have column numbers $k-2$ and k, with $n-k$ even. We obtain

$$\frac{|V_{\frac{n-1}{2},k}|}{|V_{\frac{n-1}{2},k-2}|} = \frac{|V_{\frac{n-1}{2},n-2q}|}{|V_{\frac{n-1}{2},n-2q-2}|} = \frac{|r_{\frac{n-1}{2},n-2q}|}{|r_{\frac{n-1}{2},n-2q-2}|} = \frac{C_q(X^2_{n,\frac{n-1}{2}})}{C_{q+1}(X^2_{n,\frac{n-1}{2}})},$$

with $X^2_{n,\frac{n-1}{2}} = \{(x^*_{n,\frac{n+1}{2}+1})^2, \dots, (x^*_{n,n-1})^2, (x^*_{n,n})^2\}$.

By adding $C_q(X^2_{n,\frac{n-1}{2}})C_{q+1}(X^2_{n,\frac{n-1}{2}})$ to both sides of the obvious inequality

$$(x^*_{n,\frac{n+1}{2}})^2\left[(C_q(X^2_{n,\frac{n-1}{2}}))^2 - C_{q-1}(X^2_{n,\frac{n-1}{2}})C_{q+1}(X^2_{n,\frac{n-1}{2}})\right] > 0$$

we get

$$C_q(X^2_{n,\frac{n-1}{2}})[C_{q+1}(X^2_{n,\frac{n-1}{2}}) + (x^*_{n,\frac{n+1}{2}})^2 C_q(X^2_{n,\frac{n-1}{2}})]$$
$$> C_{q+1}(X^2_{n,\frac{n-1}{2}})[C_q(X^2_{n,\frac{n-1}{2}}) + (x^*_{n,\frac{n+1}{2}})^2 C_{q-1}(X^2_{n,\frac{n-1}{2}})],$$

or

$$\frac{[C_{q+1}(X^2_{n,\frac{n-1}{2}}) + (x^*_{n,\frac{n+1}{2}})^2 C_q(X^2_{n,\frac{n-1}{2}})]}{[C_q(X^2_{n,\frac{n-1}{2}}) + (x^*_{n,\frac{n+1}{2}})^2 C_{q-1}(X^2_{n,\frac{n-1}{2}})]} > \frac{C_{q+1}(X^2_{n,\frac{n-1}{2}})}{C_q(X^2_{n,\frac{n-1}{2}})}.$$

Set

$$Y^2_{n,\frac{n+1}{2}} = X^2_{n,\frac{n-1}{2}} \cup \{(x^*_{n,\frac{n+1}{2}})^2\} = \{(x^*_{n,\frac{n+1}{2}})^2, (x^*_{n,\frac{n+1}{2}+1})^2, \ldots, (x^*_{n,n})^2\}.$$

The preceding inequality then reads, taking reciprocal values and invoking the already used formula for linear combinations of elementary symmetric functions,

$$\frac{C_q(Y^2_{n,\frac{n+1}{2}})}{C_{q+1}(Y^2_{n,\frac{n+1}{2}})} < \frac{C_q(X^2_{n,\frac{n-1}{2}})}{C_{q+1}(X^2_{n,\frac{n-1}{2}})}. \tag{3.54}$$

A moment's reflection will show that the left-hand side of this inequality is identical to the ratio $\frac{|\sigma_{n+1,n+1-2q}|}{|\sigma_{n+1,n+1-2q-2}|}$ of nonzero (absolute) coefficients of the here even polynomial S_{n+1}, see (3.20). And this value we already have come across:

Lemma 9. *If n is odd, then*

$$\frac{|\sigma_{n+1,n+1-2q}|}{|\sigma_{n+1,n+1-2q-2}|} = F_n(q), \tag{3.55}$$

where $F_n(q)$ is identical with the function given in (3.48).

Proof. The proof follows from (1.3) and (3.24) by straightforward division. □

Here we are using the definition (3.48) equally for even and odd values of the parameter n.

We thus get the identical solution $q_{\#\#}$ of $F_n(q)-1=0$ as before, and hence the same integer value q_{**} as explicitly given in (2.3), so that $\frac{C_q(Y^2_{n,\frac{n+1}{2}})}{C_{q+1}(Y^2_{n,\frac{n+1}{2}})} = F_n(q) \geq 1$ for $q \geq q^{**}$ and thus $C_{q+1}(X^2_{n,\frac{n-1}{2}}) \leq C_q(X^2_{n,\frac{n-1}{2}})$ for $q \geq q^{**}$. Together with (3.53) we thus obtain the following chain of inequalities:

$$\frac{|V_{0,k-2}|}{|V_{0,k}|} < \frac{|V_{1,k-2}|}{|V_{1,k}|} < \frac{|V_{2,k-2}|}{|V_{2,k}|} < \cdots < \frac{|V_{\frac{n-1}{2},k-2}|}{|V_{\frac{n-1}{2},k}|} = \frac{C_{q+1}(X^2_{n,\frac{n-1}{2}})}{C_q(X^2_{n,\frac{n-1}{2}})} \leq 1,$$

so that eventually $|V_{i,k-2}| \leq |V_{i,k}|$ holds for $0 \leq i \leq \frac{n-1}{2}$ and for all $k = n-2q \leq k^{**} = n-2q^{**}$.

This completes the main part of the proof for the Addendum 2. We are now going to show that the function $F_n(q)$ majorizes some appropriately chosen function $f_n(q)$. From this we will then derive an alternative but weaker upper bound k^* in place of k^{**} (see Addendum 1).

Lemma 10. *Let the function $F_n(q)$ be given by* (3.48) *and the function $f_n(q)$ be defined by*

$$f_n(q) = \frac{4n(1+q)}{(n+2)(n-2q)}. \tag{3.56}$$

Then, for $n \geq 6$, we have $0 < F_n(0) = f_n(0) = \frac{4}{n+2} < 1$, and for $q \in (0, \lfloor \frac{n-1}{2} \rfloor]$ there holds $F_n(q) > f_n(q) > 0$.

Proof. We consider $f_n(q)$ a real function in the variable q and with parameter n, and we focus on the case of n even because the argument is quite analogous for n odd. The following inequalities are readily verified:

$$0 < F_n(0) = f_n(0) = \frac{4}{n+2} < 1,$$
$$f_n(\frac{n}{2} - 1) = \frac{n^2}{2+n} > 1,$$

$f_n'(q) = \frac{4n}{(n-2q)^2} > 0$ on $[0, \frac{n}{2} - 1]$, i.e., $f_n(q)$ is strictly monotone increasing and hence positive on this interval. To show that $F_n(q) > f_n(q)$ holds for $q > 0$, which implies that $f_n(q)$ crosses the line $y = y(q) = 1$ later than $F_n(q)$, we proceed as follows: Consider the auxiliary function $g_n(q)$ in the variable q and with parameter n, given by

$$g_n(q) = q(4 - 2n) + n^3 - n^2 - 6n + 4.$$

It follows by straightforward calculations that we have, for $n \geq 6$,

$$g_n(0) = n^3 - n^2 - 6n + 4 > 0,$$
$$g_n(\frac{n}{2} - 1) = n(n^2 - 2n - 2) > 0,$$
$$g_n(0) > g_n(\frac{n}{2} - 1).$$

These properties imply that the linear function $g_n(q)$ is positive on $[0, \frac{n}{2} - 1]$.

Let $A(q)$ denote the numerator and $B(q) > 0$ the denominator of the rational function $H(q)$ on $[0, \frac{n}{2} - 1]$,

$$H(q) = \frac{A(q)}{B(q)} = \frac{(2+n)(n^2+n-2q)(n-q-1)}{(1+n-2q)n(n^2+n-2q-2)}.$$

We then have $A(q) - B(q) = qg_n(q) > 0$, since $g_n(q) > 0$. Hence we get $H(q) = \frac{A(q)}{B(q)} > 1$, and this completes the proof of the last claim of Lemma 10 since $H(q) = \frac{F_n(q)}{f_n(q)}$, as is verified by straightforward division. □

The linear equation in q (see (3.56)), $f_n(q) - 1 = 0$, i. e.,

$$q(6n+4) - n^2 + 2n = 0,$$

has the solution $q = q_{\#}$, and in order to continue the argument with integer values, we choose $q = q^* = \lceil q_{\#} \rceil$ as explicitly given in (2.2). According to Lemma 10, we have $q^{**} \leq q^*$ and hence $k^* = n - 2q^* \leq k^{**} = n - 2q^{**}$, and the Table shows that $k^* < k^{**}$ holds for some n. Thus k^* is an admissible weaker upper bound for k. We still need to show that $k^* \geq k^\circ$ holds in order to finalize the proof of the Theorem. This inequality, which is a strict one for some n, is contained in the following lemma:

Lemma 11. *The upper bounds k° and k^* in the Theorem, defined by*

$$k^\circ = k^\circ(n) = \text{ largest integer } \leq \frac{2n}{3} \qquad \text{with } n - k^\circ \text{ even},$$

$$k^* = k^*(n) = n - 2q^* \qquad \text{with } q^* = \lceil q_{\#} \rceil = \lceil \frac{n^2 - 2n}{6n+4} \rceil,$$

are related to each other as follows:

$k^\circ + 2 = k^*$, if $n = 6m+1$ or $n = 6m+2$, where $m \geq 1$ is an integer, and $k^\circ = k^*$, otherwise.

Proof. Let $n = 6m+1$, then $\frac{2n}{3} = 4m + \frac{2}{3}$, so that $k^\circ = k^\circ(n) \in \{4m, 4m-1\}$, but for $n - k^\circ$ to be even we must choose $k^\circ = 4m-1$. We obtain $q_{\#} = \frac{n^2-2n}{6n+4} = m - \varepsilon$ with $\varepsilon = \frac{10m+1}{36m+10}$ and hence $0 < \varepsilon < 1$. This implies $q^* = \lceil q_{\#} \rceil = m$ and thus $k^* = n - 2q^* = 4m + 1 = k^\circ + 2$.

Similarly, for $n = 6m+2$ one gets $k^\circ = 4m$ and $q_{\#} = m - \varepsilon$ with $0 < \varepsilon = \frac{4m}{36m+16} < 1$. This implies $q^* = \lceil q_{\#} \rceil = m$ and thus $k^* = n - 2q^* = 4m + 2 = k^\circ + 2$.

Likewise, for $n = 6m$ one gets $\frac{2n}{3} = 4m$, which implies $k^\circ = 4m$. Furthermore, $q_{\#} = m - \varepsilon$ with $0 < \varepsilon = \frac{16m}{36m+4} < 1$. This implies $q^* = \lceil q_{\#} \rceil = m$ and thus $k^* = n - 2q^* = 4m = k^\circ$.

Likewise, for $n = 6m+3$ one gets $\frac{2n}{3} = 4m+2$, which implies $k^\circ = 4m+1$. Furthermore, $q_{\#} = m + \varepsilon$ with $0 < \varepsilon = \frac{2m+3}{36m+22} < 1$. This implies $q^* = \lceil q_{\#} \rceil = m+1$ and thus $k^* = n - 2q^* = 4m+1 = k^\circ$.

The proof is also quite analogous for the remaining cases:

$n = 6m+4$ with $k^\circ = 4m+2$, $q_{\#} = m + \frac{8m+8}{36m+28}$, $q^* = m+1$, and $k^* = 4m+2$, and

$n = 6m+5$ with $k^\circ = 4m+3$, $q_{\#} = m + \frac{14m+5}{36m+34}$, $q^* = m+1$, and $k^* = 4m+3$, which exhaust all possible cases. □

3.2. Proof of the Inequalities for the Marginal Cases $3 \leq n \leq 5$

We now turn to a proof of (2.8) and start with the lowest polynomial degree, $n = 3$, so that we have to show that $|a_0| + |a_3| \leq |t_{3,3}| = 4$ holds for the coefficients a_0 and a_3 of any polynomial $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ from class B_3 or C_3. According to (3.17), we evaluate the right-hand side of the inequality $|a_j + a_k| \le 2 \sum_{i=0}^{\frac{n-1}{2}} \max\{|V_{i,j}|, |V_{i,k}|\}$, where we have to set here $j = 0$ and $k = n = 3$, i.e. we evaluate the upper bound $2(\max\{|V_{0,0}|, |V_{0,3}|\} + \max\{|V_{1,0}|, |V_{1,3}|\})$, and the goal is to verify that this expression indeed equals 4. We obtain by elementary calculations according to Lemmas 1, 3 and 5:

$$|V_{0,0}| = |V_{0,1}| = \frac{1}{6}, \;\; |V_{0,3}| = \frac{2}{3}, \;\; |V_{1,0}| = |x^*_{3,1}||V_{1,1}| = \frac{1}{2}\,|V_{1,1}| = \frac{2}{3}, \;\; |V_{1,3}| = \frac{4}{3}.$$

Hence we effectively have

$$|a_0 + a_3| \le 2\big(\max\{|V_{0,1}|, |V_{0,3}|\} + \max\{|x^*_{3,1}||V_{1,1}|, |V_{1,3}|\}\big) = 4.$$

The same upper bound applies to $|a_0 - a_3|$ since the upper bound depends on the grid (1.5) but not on the values of P_3 resp. of P_3° on that grid.

The verification of the inequalities for the remaining polynomial degrees $n = 4$ and $n = 5$ follows the same pattern as for the case $n = 3$. The absolute values of the affected elements of V_4 are, suppressing their elementary derivation from Lemma 5,

$$|V_{0,2}| = \frac{1}{2}, \;\; |V_{1,2}| = 2, \;\; |V_{2,2}| = 3, \;\; |V_{0,4}| = 1, \;\; |V_{1,4}| = 2, \;\; |V_{2,4}| = 2.$$

To show that $|a_1|+|a_4| \le |t_{4,4}| = 8$ holds, we evaluate only the corresponding upper bound (3.16), where $|x^*_{4,1}| = \frac{1}{\sqrt{2}}$, and get, as required:

$$|a_1 + a_4| \le 2\big(\max\{|V_{0,2}|, |V_{0,4}|\} + \max\{|x^*_{4,1}||V_{1,2}|, |V_{1,4}|\}\big) + |V_{2,4}| = 8.$$

The absolute values of the affected elements of V_5 are, again suppressing their derivation from Lemma 5,

$$|V_{0,1}| = \frac{1}{10}, \;\; |V_{1,1}| = \frac{2}{5}(3-\sqrt{5}), \;\; |V_{2,1}| = \frac{2}{5}(3+\sqrt{5}), \;\; |V_{0,3}| = \frac{6}{5},$$
$$|V_{1,3}| = |V_{1,1}| + \frac{16}{5}, \;\; |V_{2,3}| = |V_{2,1}| + \frac{16}{5}, \;\; |V_{0,5}| = \frac{8}{5}, \;\; |V_{1,5}| = \frac{16}{5}, \;\; |V_{2,5}| = \frac{16}{5}.$$

To show that $|a_0|+|a_3| \le |t_{5,3}| = 20$ holds, we evaluate only the corresponding upper bound (3.17), with $|x^*_{5,1}| = \frac{1}{2}\sqrt{\frac{3+\sqrt{5}}{2}}$, $|x^*_{5,2}| = \frac{1}{2}\sqrt{\frac{3-\sqrt{5}}{2}}$, and get:

$$|a_0 + a_3| \le 2\big[\max\{|V_{0,1}|, |V_{0,3}|\} + \max\{|x^*_{5,1}||V_{1,1}|, |V_{1,3}|\} + \max\{|x^*_{5,2}||V_{2,1}|, |V_{2,3}|\}\big] = 20.$$

To show that $|a_0|+|a_5| \le |t_{5,5}| = 16$ holds, we evaluate only the corresponding upper bound (3.17) and get:

$$|a_0 + a_5| \le 2\big[\max\{|V_{0,1}|, |V_{0,5}|\} + \max\{|x^*_{5,1}||V_{1,1}|, |V_{1,5}|\} + \max\{|x^*_{5,2}||V_{2,1}|, |V_{2,5}|\}\big] = 16.$$

Finally, to show that $|a_2| + |a_5| \leq |t_{5,5}| = 16$ holds, we evaluate only the corresponding upper bound (3.17) and likewise get, as required:

$$|a_2 + a_5| \leq 2\big[\max\{|V_{0,3}|, |V_{0,5}|\} + \max\{|x^*_{5,1}||V_{1,3}|, |V_{1,5}|\} \\ + \max\{|x^*_{5,2}||V_{2,3}|, |V_{2,5}|\}\big] = 16.$$

3.3. Proof of the Counterexample

We at last turn to a proof of (2.9). From (3.39) and (3.42) we deduce

$$V_{0,n-2} = \frac{2^{n-2}}{n}\Big[(-1)^n + (-1)^{n+1}\big(\frac{n}{4} + \frac{1}{2}\big)\Big],$$

and, for $1 \leq i \leq \lfloor\frac{n-1}{2}\rfloor$,

$$V_{i,n-2} = \frac{2^{n-1}}{n}\Big[(-1)^{n-i}(x^*_{n,i})^2 + (-1)^{n+1-i}\big(\frac{n}{4} + \frac{1}{2}\big)\Big].$$

Hence,

$$\begin{aligned}|c_{n-1,n-3}| &= \sum_{i=0}^{n}|V_{i,n-3}| = \sum_{i=0}^{n}|x^*_{n,i}||V_{i,n-2}| = 2|V_{0,n-2}| + 2\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}|x^*_{n,i}||V_{i,n-2}| \\ &= \frac{2^{n-1}}{n}\Big|(-1)^n + (-1)^{n+1}\big(\frac{n}{4} + \frac{1}{2}\big)\Big| \\ &\quad + \frac{2^n}{n}\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}|x^*_{n,i}|\Big|(-1)^{n-i}(x^*_{n,i})^2 + (-1)^{n+1-i}\big(\frac{n}{4} + \frac{1}{2}\big)\Big| \\ &= \frac{2^{n-1}}{n}\big(\frac{n}{4} - \frac{1}{2}\big) + \frac{2^n}{n}\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}|x^*_{n,i}|\Big[\big(\frac{n}{4} + \frac{1}{2}\big) - (x^*_{n,i})^2\Big].\end{aligned}$$

Replacing $(x^*_{n,i})^2$ by 1 yields the estimate

$$\begin{aligned}|c_{n-1,n-3}| &> \frac{2^{n-1}}{n}\big(\frac{n}{4} - \frac{1}{2}\big) + \frac{2^n}{n}\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}|x^*_{n,i}|\big(\frac{n}{4} - \frac{1}{2}\big) \\ &= \frac{2^{n-1}}{n}\big(\frac{n}{4} - \frac{1}{2}\big)\Big[1 + 2\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}|x^*_{n,i}|\Big].\end{aligned}$$

We proceed to show that the product of the last two factors is larger than n for all $n \geq 9$, which implies $|c_{n-1,n-3}| > 2^{n-1}$. That $|c_{n-1,n-3}| > 2^{n-1}$ holds for $n = 7$ and $n = 8$, too, can be verified through straightforward computation of $c_{6,4}$ and $c_{7,5}$ with reference to the interpolatory condition (1.13) resp. (1.12): $|c_{6,4}| = 65.297\cdots > 64 = t_{7,7}$ and $|c_{7,5}| = 146.751\cdots > 128 = t_{8,8}$.

We shall need the two identities for the sum of the moduli of the extremal points of $\pm T_n$ as given in Lemma 12 below. With the aid of these identities, which are not without interest in themselves, it is then readily verified by methods of calculus that we indeed have

$$\Big(\frac{n}{4}-\frac{1}{2}\Big)\Big[1+2\sum_{i=1}^{\frac{n-1}{2}}|x^*_{n,i}|\Big]=\Big(\frac{n}{4}-\frac{1}{2}\Big)\frac{1}{\sin\frac{\pi}{2n}}>n \qquad \text{for all odd } n\geq 9,$$

$$\Big(\frac{n}{4}-\frac{1}{2}\Big)\Big[1+2\sum_{i=1}^{\frac{n}{2}-1}|x^*_{n,i}|\Big]=\Big(\frac{n}{4}-\frac{1}{2}\Big)\frac{1}{\tan\frac{\pi}{2n}}>n \qquad \text{for all even } n\geq 10.$$

Lemma 12.

$$\sum_{i=0}^{n}|x^*_{n,i}|=2+2\sum_{i=1}^{\frac{n-1}{2}}|x^*_{n,i}|=1+\frac{1}{\sin\frac{\pi}{2n}}, \qquad \textit{if } n \textit{ is odd},$$

$$\sum_{i=0}^{n}|x^*_{n,i}|=2+2\sum_{i=1}^{\frac{n}{2}-1}|x^*_{n,i}|=1+\frac{1}{\tan\frac{\pi}{2n}}, \qquad \textit{if } n \textit{ is even}.$$

Proof. Dirichlet's kernel identity [15, p. 148] reads

$$D_N(x)=1+2\sum_{m=1}^{N}\cos mx=\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}, \qquad \text{if } x\neq 2N\pi. \tag{3.57}$$

Since $\sum_{i=0}^{n}|x^*_{n,i}| = 2+\sum_{i=1}^{n-1}|x^*_{n,i}| = 2+\sum_{i=1}^{n-1}|\cos\frac{(n-i)\pi}{n}|$, we get, in view of (1.4),

$$\sum_{i=0}^{n}|x^*_{n,i}|=2+2\sum_{m=1}^{\frac{n-1}{2}}\cos\frac{m\pi}{n}, \qquad \text{if } n \text{ is odd},$$

$$\sum_{i=0}^{n}|x^*_{n,i}|=2+2\sum_{m=1}^{\frac{n}{2}-1}\cos\frac{m\pi}{n}, \qquad \text{if } n \text{ is even}.$$

Applying identity (3.57) for $D_N(\frac{\pi}{n})$ to the first of these equations (n odd) gives:

$$\sum_{i=0}^{n}|x^*_{n,i}|=1+\Big(1+2\sum_{m=1}^{\frac{n-1}{2}}\cos\frac{m\pi}{n}\Big)=1+\frac{1}{\sin\frac{\pi}{2n}}.$$

Similarly, applying the identity for $D_N(\frac{\pi}{n})$ to the second of these equations (n even) gives:

$$\sum_{i=0}^{n}|x^*_{n,i}|=1+\Big(1+2\sum_{m=1}^{\frac{n}{2}-1}\cos\frac{m\pi}{n}\Big)=1+\frac{1}{\tan\frac{\pi}{2n}},$$

as required. ∎

4. Concluding Remarks

Remark 1. As mentioned before, for a given polynomial degree n the upper bound k^{**} in Addendum 2 of our Theorem needs not to be optimal, i.e. largest possible, see also Remark 7 below. A numerical calculation of the optimal upper bound $k_{opt} = k_{opt}(n)$ for $6 \le n \le 30$ shows that either $k^{**} = k_{opt}$ or $k^{**} + 2 = k_{opt}$ holds, see the Table, so that k^{**} is rather close to k_{opt}.

And this conclusion actually holds for all $n \ge 6$, in the following sense: The magnitude of k^{**} can be sketched by means of the inequality $|k^{**}+1-\lceil \frac{n}{\sqrt{2}} \rceil| \le 1$, where $\frac{1}{\sqrt{2}} = 0.7071\ldots$, which we have verified numerically utilizing (2.3). Incidentally, the magnitude of the index $k^\bullet = k^\bullet(n)$ which determines the height $H(T_n)$ of T_n, i.e. $H(T_n) = \max_{0\le k\le n} |t_{n,k}| = |t_{n,k^\bullet}|$, can be sketched by means of the similar inequality $|k^\bullet - \lceil \frac{n}{\sqrt{2}} \rceil| \le 1$, which we have verified numerically utilizing (5.27) in [21]. A further numerical calculation involving Lemma 5 indicates that likewise the optimal upper bound k_{opt} can be sketched by means of the similar inequality $|k_{opt} - \lceil \frac{n}{\sqrt{2}} \rceil| \le 1$. This means that we approximately have, with an error not greater than 2, $k^{**} + 1 \approx k^\bullet \approx k_{opt} \approx \lceil \frac{n}{\sqrt{2}} \rceil$. For example, for $n = 36$ we get $k^{**}(36) = k^\bullet(36) = k_{opt}(36) = \lceil \frac{36}{\sqrt{2}} \rceil = 26$. The connection of k^{**} with the index $k^\bullet$ manifests itself in the asymptotic equation $q_{\#\#} - q_H = \frac{1}{4}$ for $n \to \infty$, where $q_{\#\#} = q_{\#\#}(n)$ is defined in (2.3) by means of $k^{**} = n - 2q^{**}$ with $q^{**} = \lceil q_{\#\#} \rceil$. The value $q_H = q_H(n)$ is defined in [21, p. 69] by means of $k^\bullet = n - 2\lceil q_H \rceil$ with $q_H = \frac{n}{2} - \frac{5}{8} - \frac{1}{8}\sqrt{8n^2 - 7}$. The question of uniqueness of $k^\bullet$ is settled in [21].

Remark 2. Szegő's coefficient inequality (Theorem C) can be extended as follows: In place of $P_n \in B_n$ or $P_n \in C_n$ consider $P_n \in D_n$ where $D_n = \{P_n \in \Phi_n : |P_n(x_{n,i})| \le M_i \text{ for } 0 \le \text{i} \le \text{n}\}$. Here the points $x_{n,i}$ belong to some arbitrary zero-symmetric partition of I, i.e. $-1 = x_{n,0} < x_{n,1} < \cdots < x_{n,n-1} < x_{n,n} = 1$ with $x_{n,i} + x_{n,n-i} = 0$, and the upper bounds M_i are given nonnegative real numbers, not all zero, which are symmetric in the sense that $M_i = M_{n-i}$. Obviously, C_n is a special case of D_n. The set D_n has been utilized e.g. in [5], [19], [24] to extend coefficient inequalities originally stated for $P_n \in B_n$ or $P_n \in C_n$, and it has been shown in [5, pp. 2744–2745] (see also [21, p. 49]) that a Szegő-like coefficient inequality analogous to (1.10) holds for $P_n \in D_n$. In place of T_n the extremal polynomial is now R_n, given by $R_n(x_{n,i}) = (-1)^{n-i} M_i$ for $0 \le i \le n$. An extension of our present Theorem to class D_n can also be derived, but we leave the details to the reader. Multivariate extensions of Theorem C are provided in [20], [21].

Remark 3. In (1.10) and (2.1) sharp majorants, in terms of the nonzero coefficients of T_n, are provided for pairs of coefficients $|a_j| + |a_k|$ of $P_n \in B_n$ or $P_n \in C_n$ under the assumption $j < k$. The natural question arises under which conditions the Chebyshev polynomial will likewise be extremal for pairs of coefficients $|a_k| + |a_j|$ with $k < j$ (where $n-k$ even and $n-j$ odd). A partial answer for the case of a_k and its successor coefficient a_{k+1}, i.e. $j = k + 1$, has already been announced in [20, pp. 326–327]. The posed question is explored in an unpublished manuscript of the present author.

Remark 4. In [18, Theorem 2] the authors have obtained an interesting generalization of Chebyshev's inequality, see (1.11), for the leading coefficient of P_n, not covered by (1.10) and (2.1):

If $P_n \in B_n$ or $P_n \in C_n$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$, then

$$|a_0| + |a_n| \leq 2^{n-1}, \qquad \text{if } n \text{ is odd,} \tag{iii}$$

with equality if $P_n = \pm T_n$.

It provides another example where the Chebyshev polynomial is extremal for a pair of coefficients. Estimates for the magnitude of the leading coefficients of bounded polynomials are explored in an unpublished manuscript of the present author. We announce here a partial result which for $n \geq 7$ further extends the above generalization (iii) of Chebyshev's coefficient inequality and also generalizes Chebyshev's coefficient inequality in case n is even:

If $P_n \in B_n$ or $P_n \in C_n$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$, then

$$\begin{aligned} &|a_0| + |a_n| \leq 2^{n-1} && \text{as well as} && \text{(iv)} \\ &|a_2| + |a_n| \leq 2^{n-1}, && \text{if } n \geq 7 \text{ is odd,} \\ &|a_1| + |a_n| \leq 2^{n-1} && \text{as well as} && \text{(v)} \\ &|a_3| + |a_n| \leq 2^{n-1}, && \text{if } n \geq 8 \text{ is even,} \end{aligned}$$

with equality if $P_n = \pm T_n$.

Actually, inequality (iii) is stated in [18] for $P_n \in D_n$, and the inequalities (iv) and (v) can also be lifted to $P_n \in D_n$ with the extremal polynomial R_n replacing T_n, see Remark 2 above.

Remark 5. Majorants for pairs of coefficients of complex polynomials which are bounded by 1 in the unit disc are given in [3], [15, pp. 125–130], [22, pp. 637–641].

Remark 6. There is an interesting connection of the polynomial S_{n+1} (see (3.20)) with the snake polynomial $M_{n+1,0} = (T_{n+1} - T_{n-1})/2$ (with respect to the circular majorant $\varphi_0(x) = \sqrt{1 - x^2}$), whose graph on I lies entirely in the unit disc, i.e. $|M_{n+1,0}(x)| \leq \varphi_0(x)$ for all $x \in I$, and alternately touches there $\pm\varphi_0(x)$, compare with [15, p. 546], [21, Section 2.2 and 5.3], [23, p. 145]. In view of (1.2) and the proof of Lemma 3, S_{n+1} is a scalar multiple of $M_{n+1,0}$:

$$S_{n+1} = 2^{1-n} M_{n+1,0} \qquad \text{for } n \geq 2.$$

and hence the properties of $M_{n+1,0}$ and of its coefficients, as described in [21], can be easily carried over to S_{n+1}.

Remark 7. A column of V_n which determines the height $H(T_n)$ of the Chebyshev polynomial T_n (compare [21]),

$$H(T_n) = \max_{0\leq k\leq n} |t_{n,k}| = \max_{0\leq k\leq n} \sum_{i=0}^{n} |V_{i,k}| = \|V_n\|_1 \qquad \text{(column-sum norm)},$$

needs not to coincide with a column of V_n that contains the maximal element of V_n in absolute value, and hence determines the matrix norm $\|V_n\|_{\max} = (n+1)\max_{0\leq i,k\leq n} |V_{i,k}|$.

For example, for $n = 10$ we obtain from (1.3) and Lemma 5: $H(T_{10}) = \|V_{10}\|_1 = \sum_{i=0}^{10} |V_{i,8}| = |t_{10,8}| = 1280$, but on the other hand $\max_{0\leq i,k\leq 10} |V_{i,k}| = |V_{5,6}| = \frac{1696}{10}$, which gives $\|V_{10}\|_{\max} = 1865.6$. By the way, the largest element, in absolute value, in column $(V_{i,8})_{0\leq i\leq 10}$ is $|V_{5,8}| = \frac{1536}{10} < |V_{5,6}|$.

This example also shows that although the inequality $|V_{5,6}| \leq |V_{5,8}|$ is false, we nevertheless have $0 = |V_{5,5}| \leq |V_{5,8}|$ and in fact $|V_{i,5}| \leq |V_{i,8}|$ for all i, and hence $|a_5| + |a_8| \leq |t_{10,8}|$ for all $P_{10}(x) = \sum_{k=0}^{10} a_k x^k$ with $P_{10} \in C_{10}$. This is why for $n = 10$ we have $k_{opt} = 8 > k^{**} = 6$, see the Table.

In the present example ($n = 10$) there is the coincidence that $|t_{10,k^\bullet}| = H(T_{10})$ for $k^\bullet = k_{opt} = 8$. But for $n = 12$ we have $|t_{12,k^\bullet}| = H(T_{12}) = 6912$ for $k^\bullet = 8 \neq k_{opt} = 10$, see the Table and Remark 1 above.

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