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Conjectures and Results on the Multivariate Bernstein Inequality on Convex Bodies

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We survey the still unclosed research for the sharp form of a multivariate Bernstein inequality for polynomials, normalized by the condition that their maximum norm is 1 on a certain convex body $K \subset \mathbb{R}^d$. In this context, Bernstein's inequality means the estimate of the gradient DP, or of directional derivatives $D_{\mathbf{y}}P$, at interior points $\mathbf{x} \in \mathrm{int}K$. There are estimations which already give the right order of magnitude, with only a maximum $\sqrt{2}$ factor being unclarified, but some unexpected facts were revealed in the course of investigation. In particular, the paper tells the story how the two entirely different methods of Baran and Sarantopoulos were found to be equivalent.

1. How to Estimate the Derivative of a Multivariate Polynomial?

In this work let $P : \mathbb{R}^d \to \mathbb{R}$, $\deg P = n$, be a multivariate (d-variate) algebraic polynomial of total degree n; the set of all such polynomials we will denote by $\mathcal{P}_n$ or, when the degree is free, by $\mathcal{P}$. Then the gradient of P at $\mathbf{x}$ and its directional derivative at $\mathbf{x}$ in direction $\mathbf{y}$ are

$$[DP](\mathbf{x}) := DP(\mathbf{x}) := \Big(\dots, \frac{\partial P}{\partial x_j}(\mathbf{x}), \dots\Big), \qquad D_{\mathbf{y}}P(\mathbf{x}) := \langle DP(\mathbf{x}); \mathbf{y}\rangle.$$

One can also consider (bounded) polynomials on (real) normed spaces X and on their complexifications $Y = X + iX$, as is dealt with already in [17], [30] and [66]. For an introduction of polynomials in infinite dimensional normed spaces, see e.g. [29, 52] or [19, Chapter 1]. In fact, one of the main motivations for deriving Markov- and Bernstein-type inequalities comes from the possible

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use of them in the analysis of polynomials, and also of holomorphic functions, on infinite dimensional spaces [17, 19, 29, 66]. Note that for infinite dimensional normed spaces already the (homogeneous) degree one case, that is, linear functionals constitute an advanced subject, so there is no wonder that the analysis of higher degree polynomials still has much to clarify. Although in the sequel we indicate the possibilities of dealing with the case of normed spaces by the use of the more abstract notation, for this survey it is safe to read "$X = \mathbb{R}^d$" and "$Y = \mathbb{C}^d$". For more about polynomials and Markov-Bernstein inequalities in normed spaces, see also e.g. [27, 28, 29, 52, 54, 60, 62, 63].

As is well-known, in the univariate case

$$|Dp(x)| = |p'(x)| \le n\,\sqrt{\|p\|^2_{C[-1,1]} - p(x)^2}\;\omega(x), \qquad \omega(x) := \frac{1}{\sqrt{1-x^2}}. \tag{1}$$

This inequality – but with the sharper factor $\sqrt{\|p\|^2_{C[-1,1]} - p(x)^2}$ replaced just by its obvious majorant $\|p\|_{C[-1,1]}$ – was first proved by Bernstein [9]; in the closely related trigonometric case the inequality was first obtained with the precise constant by Riesz [57]; and finally the form with the improving factor $\sqrt{\|p\|^2_{C[-1,1]} - p(x)^2}$ was proved by Szegő [65] (and much later, but without noticing Szegő's solution, also by van der Corput and Schaake [18], too). For the history see e.g. [1].

We know that this form is sharp at each point $x \in I := (-1,1)$ and for each degree n. To see this, consider the n^{th} Chebyshev polynomial of the first kind, which is defined as

$$T_n(x) := \frac{1}{2}\big\{(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\big\},$$

but can equivalently be defined also as $T_n(x) := \cos(n \arccos x))$ in the interval I, and then outside of I by analytic continuation. Then indeed the equality $|T_n'(x)| = n\,\sqrt{1 - T_n^2(x)}\,\omega(x)$ is immediate from the second form of definition for any $x \in (-1,1)$.

Observe the nice separation of effects in (1) of the *degree*, of the *polynomial values* (at x and at other points of the set $I = [-1,1]$, the maximum of what is occurring), and the location of the point x within I, which we may spell out as the *geometry of the configuration* of the point and the set considered.

That motivates our next notational decision in introducing a normalization, which may rightfully be called *Bernstein-Szegő normalization*: that is, we define $B_n(P;x) := n\,\sqrt{\|P\|^2 - P(x)^2}$ and further on focus on the determination, or at least some fairly good estimation of DP and $D_{\mathbf{y}}P$ *normalized*: i.e. we care for $DP(\mathbf{x})/B_n(P;\mathbf{x})$. In what follows we will use $\mathcal{P}^* := \bigcup_{n=1}^{\infty} \mathcal{P}_n$ for the set of non-constant polynomials.

Definition 1. Let $K \subset X$ be a convex body. For any non-constant polynomial $P \in \mathcal{P}^*$ we define the *Bernstein-Szegő normalized gradient* of P as

$$GP(\mathbf{x}) := GP(K; \mathbf{x}) := \frac{DP(\mathbf{x})}{\deg P \cdot \sqrt{\|P\|^2_{C(K)} - P(x)^2}}.$$

Also we define the sets of normalized gradient vectors

$$\mathcal{G}^{(0)}(n; \mathbf{x}) := \mathcal{G}^{(0)}_K(n; \mathbf{x}) := \{GP(\mathbf{x}) : P \in \mathcal{P}_n(X)\} \quad (n \geq 1);$$
$$\mathcal{G}^{(0)}(\mathbf{x}) := \mathcal{G}^{(0)}_K(\mathbf{x}) := \{GP(\mathbf{x}) : P \in \mathcal{P}^*(X)\} = \bigcup_{n=1}^{\infty} \mathcal{G}^{(0)}_K(n; \mathbf{x}).$$

Finally, we also define the convex closure of the set of normalized gradients as

$$\mathcal{G}(\mathbf{x}) := \mathcal{G}_K(\mathbf{x}) := \operatorname{con} \mathcal{G}^{(0)}_K(\mathbf{x}) = \operatorname{con} \{GP(\mathbf{x}) : P \in \mathcal{P}^*(X)\}.$$

This way we separate the degree and the function values, which are in the same form as in dimension one, and focus on the possibly more interesting question of how the geometry of the configuration influences the description of the sets of gradient vectors and estimates of the derivative? Our primary goal is to describe the set $\mathcal{G}^{(0)}_K(\mathbf{x})$, for each fixed $\mathbf{x} \in \operatorname{int} K$, where $K \subset \mathbb{R}^d$ is a general convex body, and give as tight bounds on it as at all possible. Admitting our limited knowledge, however, we do not address here the even more precise question of the description of the sets $\mathcal{G}^{(0)}_K(n; \mathbf{x})$, for each particular degree $n \in \mathbb{N}$.

The first, immediate observation is:

Proposition 1. *For any fixed $\mathbf{x} \in \operatorname{int} K$, with $K \subset X$ a convex body, the set $\mathcal{G}^{(0)}_K(\mathbf{x})$ of normalized gradient vectors is centrally symmetric and starlike with respect to the origin.*

Proof. As for symmetry, it is quite clear, for once a polynomial P is given with $GP(\mathbf{x}) = t\mathbf{v}$, where $|\mathbf{v}| = 1$ is a unit vector, then we have $G(-P)(\mathbf{x}) = -t\mathbf{v}$ in $\mathcal{G}^{(0)}_K(\mathbf{x})$.

Now let us see that $\mathcal{G}^{(0)}_K(\mathbf{x})$ is starlike. Again, let P be given with $GP(\mathbf{x}) = t\mathbf{v}$, where $|\mathbf{v}| = 1$; then for any $0 < s < t$ we want to find some other polynomial Q with $GQ(\mathbf{x}) = s\mathbf{v}$.

We can assume $t \neq 0$, i.e DP non-null, for then there is nothing to prove. In particular, P is not a constant, for then its gradient were $\mathbf{0}$ (and, anyway, only $\deg P \geq 1$ polynomials can be considered in the definition of $\mathcal{G}^{(0)}_K(\mathbf{x})$).

If P is linear, then $P(X) = \lambda \langle \mathbf{v}, X - \mathbf{x} \rangle + P(\mathbf{x})$, for this P, considering the set of ridge polynomials $Q(X) := q(\langle \mathbf{v}, X \rangle)$, we obtain a simple question in dimension one, which can then be computed easily. After a linear substitution and restricting considerations to the essential variable $x := \langle \mathbf{v}, X \rangle$, this is the same question as to describe the full set attained by $p'(x)/(n\sqrt{\|p\|_I^2 - p^2(x)})$ for $\deg p \geq 1$ univariate polynomials, which in turn is just the interval $J_x := [-1/\sqrt{1-x^2}, 1/\sqrt{1-x^2}]$. Indeed, the endpoints are attained by $\pm T_n(x)$, see following (1); and to get values $\pm t/\sqrt{1-x^2}$, for any $t \in (-1,1)$, one can take $\pm t T_n(x) + (1-t)$. So of course for any $t \in J_x$ the set of admissible values s covers $(0,t)$ and we are done.

For all cases when $n := \deg(P) \geq 2$, let us consider the quadratically modified polynomial $Q(X) := Q_q(X) := P(X) + q|X - \mathbf{x}|^2$, depending on the constant parameter q, and put $n(q) := \deg(Q_q)$. Then there exists at most one value $q_0 \in \mathbb{R}$ for which $n(q_0) < n$; and that may occur only if $P(X) = L(X) - q_0|X - \mathbf{x}|^2$ with some L linear, in which case $n(q) = 2$ for $q \neq q_0$, and $n(q_0) = 1$. (Note that L can not be degenerate, i.e. constant, for then $DP(\mathbf{x}) = \mathbf{0}$, a case already excluded.) In all other cases $n(q) \equiv n$ for all $q \in \mathbb{R}$.

Now clearly $DQ(\mathbf{x}) = DP(\mathbf{x})$ and also $P(\mathbf{x}) = Q(\mathbf{x})$, so one finds $GQ(\mathbf{x}) = s\mathbf{v}$, with $s := s(q) = tn\sqrt{\|P\|_K^2 - P^2(\mathbf{x})}/n(q)\sqrt{\|Q\|_K^2 - P^2(\mathbf{x})}$. Observe that as q changes, almost all terms in $s(q)$ stay fixed save possibly $n(q)$ and the square-root factor $\sqrt{\|Q\|_K^2 - P^2(\mathbf{x})} = \sqrt{\|Q\|_K^2 - Q^2(\mathbf{x})}$, which clearly changes continuously in function of $q \in \mathbb{R}$; furthermore, it never becomes zero, as we have already seen that Q can at worst be linear, but not identically constant; and it tends to infinity for both $q \to \pm\infty$. So at least on one of the halflines $\mathbb{R}_+$ or $\mathbb{R}_-$, where $q \neq q_0$, we have $n(q) \equiv n$ and $s(q)$ changes continuously. Hence $s(q)$ covers all positive values between $t = s(0)$ and $\lim_{\pm\infty} s(q) = 0$, thus proving that $\mathcal{G}_K^{(0)}(\mathbf{x})$ is starlike. □

It is unclear, if the full set of normalized gradient vectors were itself convex: note the highly nonlinear normalization applied. However, for reasons apparent from the subsequent treatment, we will pass on to the convex hull $\mathcal{G}_K(\mathbf{x})$ of this set anyway. This does not change the size of the largest vector, but possibly enlarges the area or volume, what the set covers.

The gradients themselves being vectors (or, in infinite dimensional normed spaces, linear functionals), there are several ways to measure the size of $GP(\mathbf{x})$, the most important two possible means being estimations in absolute value, which equals the absolute maximum of $\langle GP(\mathbf{x}), \mathbf{y}\rangle$ over all directions $\mathbf{y} \in S_X$ – that is, the norm of $DP(\mathbf{x})$ as a linear functional acting on the space X – or *in the average*, say *volume* in $\mathbb{R}^d$. That is, we want to find

(i) the value of

$$\sup_{\mathbf{v}\in\mathcal{G}(\mathbf{x})} \|\mathbf{v}\| = \sup_{\mathbf{v}\in\{GP(\mathbf{x}):P\in\mathcal{P}^*\}} \|\mathbf{v}\| = \sup_{P\in\mathcal{P}^*(X)} \|GP(\mathbf{x})\|,$$

where $\|GP(\mathbf{x})\| := \|GP(\mathbf{x})\|_{X^*} := \sup_{\mathbf{y}\in S_X} |\langle GP(\mathbf{x}); \mathbf{y}\rangle|$;

(ii) the value of $\operatorname{vol}\mathcal{G}(\mathbf{x}) = \operatorname{vol}\{\operatorname{con}\{GP(\mathbf{x}) : P \in \mathcal{P}^*(X)\}\}$.
(This makes sense only in finite dimensional spaces, i.e. if $X = \mathbb{R}^d$.)

In this paper we first explain two methods, used in the last two decades to this end – pluripotential theory in §2 and the inscribed ellipse method in §3. Then in §6 we discuss how both of these methods are related to certain geometrical extremal problems on inscribing ellipses. The theme of §7 is then to explain how these methods are equivalent for symmetric convex bodies. Finally in §9 we give comments and hints on some possible strategies to attack the still unclarified question of sharpness of the results coming from the above methods.

2. Pluripotential Theory Approach

Pluripotential theory is a generalization, to several complex variables, of the classical univarite (logarithmic) complex potential theory of the plane, which became a substantial tool in studying univariate problems of approximation theory [59] including Bernstein type estimates and the like.

In the early 90's Miroslaw Baran found ways to derive multivariate Bernstein type inequalities in quite general settings by means of pluripotential theory. We now study that approach, and so first give a flavor of pluripotential theory itself. Although below we try to give a concise introduction, for details we refer the reader to the really nice monograph [31] of Klimek.

In pluripotential theory a key object of study is the *Siciak-Zaharjuta extremal function* of an arbitrary compact set $E \Subset Y$ (or: $\mathbb{C}^d$), defined as†

$$V_E(z) := \sup\left\{\frac{1}{\deg p}\log|p(z)| : p \in \mathcal{P}^*(Y),\ \|p\|_E \le 1\right\}, \qquad z \in Y. \quad (2)$$

It is easy to see that $V_E(z) = 0$ for $z \in E$. V_E is analogous to the one-dimensional Green function. If $E \subset X$ is real, or if it is symmetric with respect to the real subspace, then it is easy to see that here it suffices to take only real polynomials, i.e. $\mathcal{P}_n(X)$.

In general, V_E may not be continuous, and then one takes the *upper regularization* V_E^* defined as $V_E^*(z) := \limsup_{w\to z} V_E(w)$.

By the upper regularization, V_E^* is *upper semicontinuous*, moreover, V_E^* constitutes a *plurisubharmonic function* (PSH function) unless it becomes infinite throughout: $V_E^* \equiv +\infty$ (more on this dichotomy see later). PSH functions f on domains $D \subset \mathbb{C}^d$ are defined as ones which are upper semicontinuous functions on D with subharmonic restrictions on each "complex line" (one-dimensional complex subspace), meaning that $v(\zeta) := f(L(\zeta))$ is subharmonic for all fixed $a, b \in \mathbb{C}^d$ and $L(\zeta) := \zeta a + b$ on the maximal subdomain $D_L \subset \mathbb{C}$ with $L(D_L) \subset D$.

Definition of PSH functions has some flexibility, though. One plausible reason is that, in $\mathbb{C}$, and at least for smooth functions, subharmonicity is

†There are notations with $V_E(z) := \sup|p(z)|^{1/n}$, that is $\exp(V_E)$ in our present notation called as the Siciak-Zaharjuta extremal function. In all respects they are equivalent.

equivalently characterized by $\Delta f \geq 0$, which, in turn, is a property preserved under holomorphic transformations. So it is known that a function f is PSH on a domain $G \subset \mathbb{C}^d$ if and only if it is upper semicontinuous, subharmonic in the $\mathbb{R}^{2d}$ sense, and, moreover, it remains subharmonic under each holomorphic transformation of G. In view of the above invariance, also it is at least heuristically plausible that a function f being PSH implies subharmonicity of $f \circ J$ with $J : \mathbb{C} \to \mathbb{C}^d$ any holomorphic injection mapping into the domain of plurisubharmonicity G of f. In other words, f is subharmonic not only on complex lines, but also on all one-dimensional complex manifolds. We will use this in the form that $f \circ \varphi$ is subharmonic for any holomorphic mapping $\varphi : \mathbb{C} \setminus D(0,1) \to G$.

Now pluri-subharmonicity of V_E^* follows from the fact that for any polynomial p, $\log |p(z)|$ is subharmonic on complex lines. Indeed, fixing the given complex one dimensional manifold L, we have a set $\mathcal{U}$ of admissible functions, all subharmonic on L, and then taking the supremum preserves this subharmonicity property. That is a direct consequence of the characterization of one variable subharmonicity by the mean value inequality on any fixed circle $C(a,r)$ in the domain of subharmonicity, which mean value inequality, in turn, is preserved by taking supremum on an arbitrary given set $\mathcal{U}$ of functions. Indeed, if $u \in \mathcal{U}$, then $u(a)$ is majorized by its own average on $C(a,r)$; so it is even more majorized by the average on $C(a,r)$ of the sup of all functions from $\mathcal{U}$; and thus this being valid for all functions u, even the sup of $u(a)$ over all admissible functions $u \in \mathcal{U}$ is subject to this majorisation. Then taking the regularization one gets an upper semicontinuous function which is thus plurisubharmonic.

Of course, such a supremum may as well become infinite. If our set is the set of functions in (2), then, however, one thing is for sure: this sup, i.e. V_E, is identically zero on E. Now it is a property of the set E, if this still allows $V_E(z) = \infty$ somewhere else.

If, e.g., E contains a ball (of Y, the full complex space now – for normalization, let us just take the unit ball $B \subset Y \to$ then at any given complex point $z_0 \in Y$ we can draw a complex line $L := \{z \in Y : z = \zeta z_0/|z_0|,\ \zeta \in \mathbb{C}\}$ and on this line restrict any polynomial $p \in \mathcal{P}^*(Y)$ so that we arrive at a situation where on $\mathbb{C}$ a polynomial P is bounded by 1 on the unit disk $|\zeta| \leq 1$ and we look for its size at $\zeta_0 := |z_0| \in \mathbb{C}$. That size is bounded by the well-known Bernstein-Walsh inequality: $|P(\zeta_0)| \leq r_0^n$, where $r_0 := |z_0| = \zeta_0$. This leads to the fact that $\frac{1}{n} \log |p(z)|$ remains uniformly bounded – that is, $V_E(z_0)$ is finite and is bounded by the fixed bound $\log r_0$, depending on z_0 only.

Let us note in passing, that if $E = B$, then this bound is achieved already by the linear polynomial $p(z) = \sum_{j=1}^d c_j z_j$, with $c_j := \overline{z_j^{(0)}}/|z_0|$, $(j = 1, \ldots, d)$, where $z_0 = (z_1^{(0)}, \ldots, z_d^{(0)})$, so then $V_B(z) = \log_+ |z|$ throughout.

From the above it follows that *if V_E is bounded on a ball* $B(a,r) \subset \mathbb{C}^d$, by a constant say M, then anywhere in $\mathbb{C}^d$ we have

$$V_E(z) \leq \log_+ \frac{|z-a|}{r} + M, \tag{3}$$

and in particular V_E is *finite everywhere*. Indeed, by hypothesis, if a polynomial p of degree $n \geq 1$ satisfies $\|p\|_E \leq 1$, then it also satisfies $\max\limits_{B(a,r)} \frac{1}{n} \log |p| \leq M$, and thus $P(z) := e^{-Mn} p(z)$ belongs to the set of admissible polynomials for $V_{B(a,r)}$: $\|P\|_{B(a,r)} \leq 1$. But it means that for any z, $\frac{1}{n} \log |P(z)| \leq V_{B(a,r)}(z) = \log_+(|z-a|/r)$, so $\frac{1}{n} \log |p(z)| \leq \log_+(|z-a|/r) + M$. Since this holds for all p, admissible for V_E, (3) follows.

This has the immediate corollary that *either* $V_E^* \equiv +\infty$*, or* V_E *is a locally bounded function with logarithmic growth*, i.e. $V_E(z) - \log|z| = O_E(1)$. Indeed, if there is such an M, a, r, as above, then we are done; if not, then at any point $a \in \mathbb{C}^d$ we just have $V_E^*(a) = \infty$.

Let us call a set E with the property that $V_E^* \equiv \infty$, *pluripolar*. There are other, equivalent definitions to this property, but here we will not need them. In our discussion we will be restricted to assume here and henceforth that E is *not pluripolar*.

From the above we already know that in the case $\operatorname{int} E \neq \emptyset$, E is necessarily non-pluripolar. Much less is sufficient, though. As a next step, consider the case when E contains a *real neighborhood of some of its points*, say $I^d \subset \mathbb{R}^d$, with $I = [-1, 1]$. To show that then also V_E is a locally bounded function (and hence E is non-pluripolar), we combine two things. First, by a theorem of Siciak, for a set $K = K_1 \times \cdots \times K_d$, with each $K_j \Subset \mathbb{C}$, we have $V_K(z) = \max_j V_{K_j}(z_j)$, with V_{K_j} the one-dimensional extremal function of K_j for each $j = 1, \ldots, d$. Well, it is indeed clear that taking arbitrary $p(z) := p_j(z_j)$ from the definitive class for V_{K_j}, we will get $V_K(z) \geq \max_j V_{K_j}(z_j)$. The other inequality is less trivial – see e.g. [31, Theorem 5.1.8] – but at least the weaker statement that $V_K(z) \leq \sum_j V_{K_j}(z_j)$, already sufficient for us, is easy to see say inductively.

Second, we just need to know that the one dimensional extremal function of $I := [-1, 1]$ is locally bounded. In fact, a much more precise result is well-known, as $V_I(\zeta) = \log |\mathcal{H}(\zeta)|$, where in this work

$$\mathcal{J}(\zeta) := \frac{1}{2}\Big(\zeta + \frac{1}{\zeta}\Big), \qquad \mathcal{H}(\zeta) := \zeta + \sqrt{\zeta^2 - 1}$$

denote the Joukowski map and (one of) its inverse, with the square-root positive for $\zeta > 1$. (For further formulae on V_E see later.) Recall that restricting to the exterior of the unit circle $\Omega := \mathbb{C} \setminus D(0,1)$, $\mathcal{J} : \operatorname{int} \Omega \leftrightarrow \mathbb{C} \setminus I$.

Convex bodies $K \subset \mathbb{R}^d$ certainly satisfy the criteria that $\operatorname{int}_{\mathbb{R}^d} K \neq \emptyset$, so are not pluripolar. Using the analytic accessibility criteria of Plesniak (see [31, Proposition 5.3.12]), it can be seen that V_K is continuous for any convex body $K \Subset \mathbb{R}^d$. Hence at this point we may forget about upper regularization, as $V_K = V_K^*$ for convex sets.

Note that after normalization by the degree, and for any polynomial $p \in \mathcal{P}^*(Y)$, $\frac{1}{n} \log |p(z)| = \log |z| + O(1)$ whenever $z \to \infty$. So it is reasonable to consider now the *Lelong class* of *all* such functions:

$$\mathcal{L}(E) := \{u \ PSH : u|_E \leq 0,\ u(z) \leq \log|z| + O(1),\ |z| \to \infty\}$$

and to define

$$U_E(z) := \sup\{u(z) : u \in \mathcal{L}(E)\} \ . \tag{4}$$

This function may be named the *pluricomplex Green function*. The *Zaharjuta-Siciak Theorem* says that (4) and (2) are equal, at least for $E \subset \mathbb{C}^d$ compact.

We assume that E is a compact and *non-pluripolar* set. The extension of the Laplace operator is the so-called *complex Monge-Ampère operator*

$$(\partial\overline{\partial}u)^d := d!\, 4^d \det\Big[\frac{\partial^2 u}{\partial z_j \overline{\partial} z_k}(z)\Big] dV(z), \tag{5}$$

where $dV(z) = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_d \wedge dy_d$ is just the usual volume element in $\mathbb{C}^d$. Recall that for a complex variable $z = x + iy$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\overline{\partial} z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Note also that in one complex variable $4\frac{\partial^2 u}{\partial z \overline{\partial} z} = \Delta u$, and that $f = u + iv$ is regular if and only if $\frac{\partial}{\overline{\partial} z} f = 0$ (Cauchy-Riemann equations for $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$).

At first the Monge-Ampère operator (5) is applied only to smooth functions $u \in PSH \cap C^2$, say, but due to the work of Bedford and Taylor [8], the operator extends, in the appropriate sense, even to the whole set of locally bounded PSH functions (which, as shown above, covers the case of V_E^* for any non-pluripolar E).

Therefore, it makes sense to consider

$$(\partial\overline{\partial}V_E^*)^d,$$

which is then a compactly supported measure λ_E and is called the *complex equilibrium measure* of the set E.

It is shown in the theory that this measure is *normalized*, as $\lambda_E(\mathbb{C}^d) = \lambda_E(\widehat{E}) = (2\pi)^d$.

To finish with notations, for a not necessarily differentiable function F the *lower (semi)derivative* is defined as

$$D_y^+ F(a) := \liminf_{t \to 0^+} \frac{F(a + ity) - F(a)}{t}.$$

Theorem 1 (Baran, 1994 & 2004). *Let $E \subset X$ be any bounded, closed set, $x \in \operatorname{int} E$ and $0 \neq y \in X$. Then for all $p \in \mathcal{P}_n(X)$ we have*

$$|\langle Dp(x), y\rangle| \le D_y^+ V_E(x)\, n \sqrt{\|p\|_E^2 - p(x)^2}. \tag{6}$$

In other words, $|\langle Gp(x); y\rangle| \le D_y^+ V_E(x)$.

In view of this result, the following definitions are generally accepted, [12, 13, 14, 16].

Definition 2. Let $E \Subset X$ be non-pluripolar. Then the *Baran metric* of E at any point $x \in \operatorname{int} E$ – more precisely, in the relative, i.e. real interior $x \in \operatorname{int}_X E$ – is the correspondence $y \to D_y^+ V_E^*(x)$, or shortly, just $\delta_B(E; x, y) := \delta_B(x, y) := D_y^+ V_E^*(x)$.

Similarly, one defines the pointwise *Markov-Bernstein metric* or *derivative metric* of E as

$$\delta_D(E; x, y) := \delta_D(x, y) := \sup\{\langle GP(x), y\rangle : P \in \mathcal{P}^*(X)\} = \sup_{v \in \mathcal{G}_E^{(0)}(x)} \langle v, y\rangle.$$

At first sight it is a bit disturbing that in Theorem 1 we did not get estimates on $Gp(x)$ directly, but only on its inner product with the directional vectors y. To reformulate, we use standard convex geometry. Recall that the *polar*, or *dual* of a set $M \subset X$ is defined as

$$M^* := \{y \in X^* : |\langle u, y\rangle| \le 1 \ \forall u \in M\}.$$

So the result reads as $\mathcal{G}_E^{(0)}(x)^* \supseteq \{y : D_y^+ V_E^*(x) \le 1\}$ for any compact, non-pluripolar set E.

In reflexive (so surely in finite dimensional) spaces – or with definition of the bipolar set inside X and not in the bidual X^{**} – we have $M^{**} = \overline{\operatorname{con} M}$, whence we also find

$$\mathcal{G}_E(x) = \operatorname{con} \mathcal{G}_E^{(0)}(x) \subset \{y : D_y^+ V_E^*(x) \le 1\}^* = \{v : \langle v, y\rangle \le D_y^+ V_E^*(x) \ \forall y \in X^*\}.$$

Note that here is the point where consideration of the convex hull is essentially inevitable.

Theorem 2 (Baran, 1995). *Let E be a compact subset of $\mathbb{R}^d$ with non-empty interior. Then the equilibrium measure λ_E is absolutely continuous in the interior of E with respect to the Lebesgue measure of $\mathbb{R}^d$. Denote its density function by $\lambda(x)$ for all $x \in \operatorname{int} E$. Then we have*

$$\frac{1}{d!}\,\lambda(x) \ge \operatorname{vol} \mathcal{G}(x) \tag{7}$$

for a.a. $x \in \operatorname{int} E$. Moreover, if E is a symmetric convex body of $\mathbb{R}^d$, then we have $\frac{1}{d!}\,\lambda(x) = \operatorname{vol} \mathcal{G}(x)$ for all $x \in \operatorname{int} E$.

Conjecture 1 (Baran, 1995). *We have*

$$\frac{1}{d!}\,\lambda(x) = \operatorname{vol} \mathcal{G}(x)$$

even if E is a non-symmetric convex body in $\mathbb{R}^d$.

Baran also calculated a few concrete examples, the most interesting one being the case of the simplex. Based on calculations of Lundin [34], Baran [6, Example 4.8] derived the following.

Proposition 2 (Baran, 1995). *The extremal function of the standard simplex* $\Delta := \Delta_d \subset \mathbb{R}^d$ *is for any* $z = (z_1, z_2, \dots, z_d) \in \mathbb{C}^d$

$$V_\Delta(z) = \log |\mathcal{H}(|z_1| + |z_2| + \cdots + |z_d| + |1 - (z_1 + z_2 + \cdots + z_d)|)|. \qquad (8)$$

Also, for the density of the complex equilibrium measure we have (with $\omega_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$*, the volume of the unit ball)*

$$\lambda_\Delta(x) = \frac{d!\,\omega_d}{\sqrt{x_1\, x_2 \cdots x_d(1 - x_1 - \cdots - x_d)}}.$$

From (8) we can easily calculate (see [50] and also [12], p. 145).

Proposition 3 (2005). *For the standard simplex* Δ *of* $\mathbb{R}^d$ *and with any unit directional vector* $y = (y_1, \dots, y_d)$ *and any point* $x = (x_1, \dots, x_d) \in \operatorname{int} \Delta$ *we have the formula*

$$\delta_B(\Delta; x, y) := D_y^+ V_\Delta(x) = \sqrt{\frac{y_1^2}{x_1} + \cdots + \frac{y_d^2}{x_d} + \frac{(y_1 + \cdots + y_d)^2}{1 - (x_1 + \cdots + x_d)}}\ . \qquad (9)$$

3. The Inscribed Ellipse Method of Sarantopoulos

The *method of inscribed ellipses* was introduced into the subject by Y. Sarantopoulos. This works for arbitrary interior points of any sets (e.g. a possibly nonsymmetric convex body). The key of all is the next

Lemma 1 (Inscribed Ellipse Lemma, Sarantopoulos, 1991). *Let* K *be any subset in a vector space* X*. Suppose that* $\mathbf{x} \in K$ *and the ellipse*

$$\mathbf{r}(t) := \mathbf{r}_{\mathbf{a},\mathbf{x},\mathbf{y};b}(t) := \cos t\ \mathbf{a} + b \sin t\ \mathbf{y} + \mathbf{x} - \mathbf{a}, \qquad t \in [-\pi, \pi) \qquad (10)$$

lies inside K*. Then we have for any polynomial* $p \in \mathcal{P}_n(X)$ *the Bernstein type inequality*

$$|\langle Dp(\mathbf{x}), \mathbf{y}\rangle| \le \frac{n}{b}\sqrt{\|p\|_{C(K)}^2 - p^2(\mathbf{x})}. \qquad (11)$$

Proof. Let $T(t) := p(\mathbf{r}(t))$. Observe $T \in \mathcal{T}_n([-\pi, \pi))$. Indeed, we must substitute in

$$p(X) = \sum_{\substack{\mathbf{k}=k_1,\dots,k_d \\ k_1+\cdots+k_d \le n}} a_{\mathbf{k}} X_1^{k_1} \cdots X_d^{k_d}$$

the values $X_j = r_j(t) = a_j \cos t + b y_j \sin t + x_j - a_j$, $j = 1, \dots, d$.

Now let us apply the classical Bernstein-Szegő inequality for trigonometric polynomials: we get $|T'(t)| \le n\sqrt{\|T\|^2 - T^2(t)}$. Obviously $\|T\| \le \|p\|_{C(K)}$ whenever the ellipse (10) fully lies in K. Now the chain rule yields

$$T'(t) = \langle Dp(\mathbf{r}(t)), \dot{\mathbf{r}}(t)\rangle = \langle Dp(\mathbf{r}(t)), -\sin t\mathbf{a} + b \cos t\mathbf{y}\rangle.$$

Substituting $t = 0$ and noting $T(0) = p(\mathbf{r}(0)) = p(\mathbf{x})$ leads to (11). $\square$

So it is clear from the above that a way to get best possible bounds for $|\langle Gp(x), y\rangle|$ from the method, the goal is to find the largest possible b-parameter. That explains the next definition.

Definition 3. Denote, for a given compact set $E \subset X$ and $\mathbf{x} \in \operatorname{int} E$, $\mathbf{y} \in S_X$ the set of all inscribed ellipses of the form (10) by $\mathcal{E}(E; \mathbf{x}, \mathbf{y})$. Take $b^* := b^*(E; \mathbf{x}, \mathbf{y}) := \sup_{\mathcal{E}(E;\mathbf{x},\mathbf{y})} b$. An ellipse $\mathcal{E}^* \in \mathcal{E}(E; \mathbf{x}, \mathbf{y})$ with $b = b^*$ is called b-maximal, or tangent-maximal ellipse. (By compactness of E, such an ellipse must exist.) Furthermore, we define accordingly the *Sarantopoulos metric* of E by $\delta_S(E; \mathbf{x}, \mathbf{y}) := 1/b^*(E; \mathbf{x}, \mathbf{y})$, which can also be termed as the *exact yield of the inscribed ellipse method.*

Up to this point we used also the vectorial notation $\mathbf{x}$, $\mathbf{y}$, $\mathbf{r}$ etc., which is most common in $\mathbb{R}^d$. However, both in functional analysis, where most of our methods extend, and also in pluripotential theory, it is more common that the points are not typeset by boldface characters. As in this work various results from both directions will eventually merge, for uniformity we settle with the latter notation from now on.

Theorem 3 (Sarantopoulos, 1991). *Let p be any polynomial of degree at most n over the normed space X. Denote $S := S_X$ the unit sphere of X. Then we have for any unit vector $y \in S_X$ the Bernstein type inequality*

$$|\langle Dp(x), y\rangle| \leq \frac{n\sqrt{\|p\|^2 - p^2(x)}}{\sqrt{1 - \|x\|^2}}. \tag{12}$$

Equivalently,

$$\|Dp(x)\| \leq \frac{n\sqrt{\|p\|^2 - p^2(x)}}{\sqrt{1 - \|x\|^2}}. \tag{13}$$

Remark 1. It might be confusing, that in this formula several different norms occur. To better visualize their meaning, we can write e.g. (13) more precisely in the form

$$\|Dp(x)\|_{X^*} \leq \frac{n\sqrt{\|p\|^2_{C(B_X)} - p^2(x)}}{\sqrt{1 - \|x\|^2_X}}.$$

Proof. Take $r(t) := \cos t\ x + \sqrt{1 - \|x\|^2} \sin t\ y$. To check $r \subset B_X$ we use $\|y\| = 1$ and Cauchy-Schwarz inequality: $\|\cos t\ x + \sqrt{1 - \|x\|^2}\ \sin t\ y\| \leq |\cos t| \cdot \|x\| + |\sin t|\sqrt{1 - \|x\|^2} \leq \sqrt{(\cos^2 t + \sin^2 t)(\|x\|^2 + (1 - \|x\|^2))} = 1$, i.e. $\|r(t)\| \leq 1$. Thus Lemma 1 furnishes the result. $\square$

Remark 2. If we pick $y \| x$, then this ellipse is optimal, which can be seen from $\|r(t_0)\| = 1$ with $t_0 = \arccos \|x\|$. However, for generic position of $x, y \in X$ one may hope for some improvement, which, on the other hand, would depend on the geometry of x, y and the norm. That explains certain improvements, or existence of better results in the case of Hilbert spaces in particular. However, for some spaces, typically with the L^1-norm, not even general x, y may be subject of improvement. E.g. if $X = \mathbb{R}^2$, the norm being the ℓ^1 norm on two points, $e_1 := (1, 0)$ and $e_2 := (0, 1)$, then even for $x = \xi e_1$ and $y = e_2$ there is no gain in evaluating the norm of the points of the elliptic curve $r(t)$: $\|r(t_0)\| = 1$ again.

Theorem 4 (Sarantopoulos, 1991). *Let K be a symmetric convex body and y a unit vector in the normed space X. Let p be any polynomial of degree at most n. We have*

$$|\langle Dp(x), y \rangle| \leq \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{\tau(K, y)\sqrt{1 - \varphi^2(K, x)}} ,$$

where $\varphi(K, x)$ is the Minkowski functional (norm) with respect to K, and $\tau(K, y)$ is the maximal chord in direction of y. In particular, we have

$$\|Dp(x)\| \leq \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1 - \varphi^2(K, x)}} ,$$

where $w(K)$ stands for the minimal width of K.

Proof. $\varphi(K, y) = 2/\tau(K, y)$, so passing to $\| \cdot \|_K = \varphi(K, \cdot)$ and normalizing accordingly, (12) yields the result. □

This is quite satisfactory as regards (centrally) *symmetric* convex bodies. The results are sharp – at least when specializing to the base case of dimension 1, we get back the known sharp form of the Bernstein-Szegő inequality (13).

What to do with *non-symmetric* convex bodies?

An immediate obstacle to a straightforward extension is that $\varphi(K; x)$ is not defined‡ for non-symmetric convex sets. However, this can be removed from our way, as there exists an extension of the Minkowski gauge or norm $\varphi(K, x)$. Namely, we have the so-called *generalized Minkowski functional*, which was introduced and used by Minkowski [39] and Radon [55] already a century ago. It then resurfaced several times and in several forms, but the surprisingly rich variation of its equivalent definitions were described only recently [54]. The notion appeared first in approximation theory in the paper of Rivlin and Shapiro [58]. As for its definition, first let

$$\gamma(K, x) := \inf \left\{ 2\frac{\sqrt{\|x - a\| \, \|x - b\|}}{\|a - b\|} : a, b \in \partial K, \text{ s.t. } x \in [a, b] \right\}.$$

‡More precisely, there is an extended definition of the Minkowski gauge *with respect to an arbitrary, fixed point* $x_0 \in \operatorname{int} K$, namely $\varphi(K, x_0; x) := \min\{\lambda : x \in \lambda(K - x_0)\}$, but this is quite useless in our topic.

Then we can set

$$\alpha(K,x) := \sqrt{1-\gamma^2(K,x)}.$$

Note the striking resemblance of $\gamma(K,x) = \sqrt{1-\alpha^2(K,x)}$ to $1/\omega(x)$ with the harmonic measure $\omega(x)$. When $K = -K$, we have $\alpha(K,x) = \varphi(K,x)$, so it is indeed a generalization of the usual Minkowski gauge.

Also, $\alpha(K,x)$ can be introduced in the following manner. Let $\alpha(K,x) := \min\{\lambda : x \in K_\lambda\}$, where $K_\lambda := \cap_{y \in S_{X^*}} S(K,y,\lambda)$ with $S(K,y,\lambda)$ being the λ-homothetic copy, about any of its own symmetry centers, of the minimal strip, orthogonal to $y \in S_{X^*}$ and covering K. That is, $S(K,y,\lambda)$ is just

$$\begin{aligned} S(K,y,\lambda) := \big\{ z \in X : (1+\lambda) \inf_{x\in K} \langle y,x\rangle + (1-\lambda) \sup_{x\in K} \langle y,x\rangle \\ \le 2\langle y,z\rangle \le (1-\lambda) \inf_{x\in K} \langle y,x\rangle + (1+\lambda) \sup_{x\in K} \langle y,x\rangle \big\}. \end{aligned}$$

There are many equivalent definitions of the generalized Minkowski functional, (some of them due to Minkowski himself). For more see §4 and [54].

Theorem 5 (Kroó-Révész, 1998). *Let K be an arbitrary convex body, $x \in \operatorname{int} K$ and $\|y\| = 1$, where X can be an arbitrary normed space. Then we have*

$$|\langle Dp(x), y\rangle| \le \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{\tau(K,y)\sqrt{1-\alpha(K,x)}}, \tag{14}$$

for any polynomial p of degree at most n. Moreover, we also have

$$\|Dp(x)\| \le \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha(K,x)}} \le \frac{2\sqrt{2}\,n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2(K,x)}}.$$

Note that in [33] the best ellipse is not found; for most cases, the construction there gives only a good estimate, but not an exact value of b^*.

On the other hand specialization to dimension one immediately shows that the estimate is within a constant $\sqrt{2}$ factor of the best possible.

Note that the generalized Minkowski functional was found to be the exact thing to consider for the *Chebyshev problem of polynomial growth* in the multivariate setting, as we will explain briefly also in §4. A decade ago it seemed that also in the Bernstein problem the right thing to consider is the generalized Minkowski functional. So it was natural to pose the following conjecture in [54].

Conjecture 2 (Révész–Sarantopoulos, 2001). *Let X be a normed vector space, and K be a convex body in X. For every point $x \in \operatorname{int} K$ and every (bounded) polynomial p of degree at most n over X we have*

$$\|Dp(x)\| \le 2\,\frac{n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2(K,x)}},$$

where $w(K)$ stands for the minimal width of K.

But a few years later, calculating a certain concrete case, I encountered the following surprising fact [50].

Proposition 4 (2005). *Either Conjecture 1, or Conjecture 2 must fail.*

So I became suspicious about my own conjecture and proposed the counter-conjecture to Nikola Naidenov.

Theorem 6 (Naidenov, 2006). *Conjecture 2 is false!*

More precisely, N. Naidenov exhibited counterexamples with e.g.

$$|Dp(x)| \approx 2.015... \cdot \frac{n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2(K,x)}}.$$

How to find these counterexamples? The paper [47] is rather concise[§]. So, it might be a good idea to look at [48] for the actual construction and algorithm.

Well, whatever is the situation with the generalized Minkowski functional, but there has to be a largest b parameter, defined above as b^*, and that gives an estimate, which is perhaps sharp. But what is this b^*, the exact yield of the inscribed ellipse method?

We managed to compute this with an instructive, important example – the standard simplex Δ of $\mathbb{R}^d$.

Theorem 7 (Milev-Révész, 2003). *$b^*(\Delta, x, y)$ has the precise value*

$$b^*(\Delta, x, y) = \Big(\frac{y_1^2}{x_1} + \cdots + \frac{y_d^2}{x_d} + \frac{(y_1 + \cdots + y_d)^2}{1 - x_1 - \cdots - x_d}\Big)^{-1/2}. \tag{15}$$

In other words, $\delta_S(\Delta; x, y) = \sqrt{\frac{y_1^2}{x_1} + \cdots + \frac{y_d^2}{x_d} + \frac{(y_1+\cdots+y_d)^2}{1-x_1-\cdots-x_d}}$.

To get this we wrote out the explicit linear conditions of the ellipse being to the same side of each side polygon then the remaining vertex; the conditions regarding x, y are also linear restraint, furthermore, under these conditions the b-parameter of the ellipse is a linear goal function and via the Kuhn-Tucker Theorem one can solve the maximization problem.

As an immediate corollary, for Δ we obtain the best possible estimate, *available via the inscribed ellipse method*: we have with the value in (15) that

$$|D_y p(x)| \leq \frac{n\sqrt{\|p\|^2_{C(\Delta)} - p^2(x)}}{b^*(\Delta, x, y)}. \tag{16}$$

Comparing with Proposition 3, we were thus led to the following surprising corollary [50].

[§]In particular, about certain counterexamples Naidenov simply wrote: "The counterexample was found by a computer". Well, that is an explanation, but the odd thing about it is that somehow my computer did never find any.

Corollary 1. *The pluripotential theoretical estimate* (6) *of Baran, calculated for the standard simplex of* $\mathbb{R}^d$ *in* (9)*, gives the result exactly identical to* (16)*, obtained from the inscribed ellipse method. In other words,* $\delta_B(\Delta; x, y) = \delta_S(\Delta; x, y)$.

4. Lines, Ellipses and "Bojanov's Principle"

It is appropriate here and now to recall Borislav Bojanov's frequent remark, that in fact we do not have real multidimensional results. This he pointed out several times, even if he himself had very good multivariate results of his own. But he liked to have the point, that our heuristics, our insight fails badly in higher dimensional questions, so either things work the same way, as for low dimensions, and then our result is basically a one-dimensional thing, which by chance remains unchanged for higher dimensions, or all what we can do is desperately look for some one-dimensional steps, induction, restriction, to be able to analyze the question and do something with it with our poor one-dimensional armory.

The topic of the present survey is really multivariate, and the higher dimensional geometry is playing a decisive role. Nevertheless, working on this topic, often discussing it with Borislav Bojanov, lecturing on newer and newer results, I always felt more and more, how much he is right. Now achieving a semi-complete state of the matter, it is even more apparent, that all what we did is nothing much more than an elaboration on the above principle.

So instead of hopelessly trying to hide it, I decided to make it a point. In this and the next sections I give ample explanations, heuristical background and related geometrical and analytical facts, which show the reader how we worked exactly according to this principle both in the real and in the complex analysis approaches. Perhaps, the art is only this: to do it so well, that, in spite of lacking the real multivariate insight, we can still squeeze out some results from our crude methods and univariate formulas.

As a first example, how univariate approaches may yield nontrivial results, let us recall Wilhelmsen's Markov type estimate [71].

Theorem 8 (Wilhelmsen, 1974). *Let* $K \subset \mathbb{R}^d$ *be a convex body and denote by* $w(K)$ *the minimal width of* K*. Then at any point* $x \in K$ *and* $P \in \mathcal{P}_n$ *we have*

$$|DP(x)| \le \frac{4n^2 \|P\|_K}{w(K)}.$$

Proof. Denote the direction of the gradient at the arbitrary point $x \in K$ as $v \in S_X$; then we can write $DP(x) = tv$ with $t = |DP(x)|$, to be estimated. Consider now the maximal chord of K in the direction of v, that is, a chord between $a, b \in K$ such that $b - a = \tau(K, v)v$. Also, project x to this line segment: let the projection be $y \in \ell$, where ℓ is the line through a and b. By convexity of K, both segments $[x,a],[x,b]\subset K$. Observe that at least one of the distances $|y-a|$ and $|y-b|$ must exceed $\tau(K,v)/2 \geq w(K)/2$, so it is possible to select an endpoint, say a, with $|y-a|\geq w(K)/2$.

Consider now the directional vector $u := (a-x)/|a-x|$, and the directional derivative $D_uP(x) = \langle DP(x), u\rangle = t\langle v,u\rangle$. The directional derivative is in fact the derivative of the univariate polynomial $p(s) := P(x+su)$, so the well-known *univariate Markov inequality applies*, furnishing the estimate

$$|D_uP(x)| \leq 2n^2 \max_{0\leq s\leq |a-x|} |p(s)|/|a-x| \leq 2n^2 \max_K |P|/|a-x|.$$

On the other hand by the properties of orthogonal projection, $|a-y| = |\langle v,u\rangle|\cdot |a-x|$, so collecting these we find

$$t = \left|\frac{D_uP(x)}{\langle v,u\rangle}\right| \leq \frac{2n^2\|P\|_K}{|\langle v,u\rangle|\,|a-x|} = \frac{2n^2\|P\|_K}{|a-y|} \leq \frac{4n^2\|P\|_K}{w(K)}.$$

□

Even if the inscribed line segment seems to be a rough approach, the result is almost sharp: specializing to one variable gives back the known sharp Markov inequality apart from a factor 2. It was a question for long, how sharp is the result: a natural conjecture was that similarly to the symmetric case, even here the constant should perhaps be 2, not 4. However, by a definite construction this was disproved by Goethgeluck and Białas-Ciez [10]. Finally, the best possible constant – asymptotically $4/\pi$ – was found by Skalyga [63], and it seems that independently also by Subbotin and Vasil'ev [64].

In this respect the inscribed ellipse method is nothing but a variant of the approach of Wilhelmsen: the only difference is that now we inscribe one dimensional *curves*, not necessarily lines, but of course only very special ones such that restriction of P on them will again be univariate polynomials – the only difference is that now univariate *trigonometric polynomials* – and then apply again the *univariate Bernstein inequality.* Theorem 5 squeezes out again a result almost sharp apart from a $\sqrt{2}$ factor. Of course we aim at finding the best ellipse, but as that is impossible to compute explicitly in generic situation, the argument in [33] runs along rather similar lines than Wilhelmsen's argument: we again draw the maximal chord in direction of y, and then connect it with x – now by an ellipse of the admissible class considered in the definition of $b^*(K,x,y)$. A concrete calculation of the ellipse and its b parameter, what we can achieve with this construction, results in Theorem 5.

Note how well the story of the Markov inequality repeats itself with the Bernstein inequality, at least as for the slight loss in constant, which encourages to conjecture that even the full symmetric case result may go through, but finally it is found to fail.

Before showing how even the pluripotential theoretic approach builds up from one dimensional considerations, let us discuss a geometric issue, the question of diagonals and generalization of polar coordinates with respect to a convex body. Even if this might seem a detour, we will see how relevant it is in several respects.

As an instructive example, we may start from the case of the unit disk (or ball). Clearly diameters are straight lines, which intersect only inside, have a certain maximal part within the convex body, and then leave it and "out there" they do not intersect. Moreover, to each exterior point there is a unique diameter, which contains it, and diameters can be parameterized in a continuous way so that the exterior of the circle is covered uniquely by the continuous family of these diagonals. In fact, polar coordinate system of the plane (space) is the realization of this. Half a century ago, Hammer [24] started to analyze the issue, what can be done analogously with general convex bodies. If our convex body is centrally symmetric, then in any direction some maximal chord goes through the center and so it is easy to see that the analogous diagonal coverage can be constructed. For a square, however, there is already some ambiguity – there are parallel maximal chords, and the above selection of the central diagonal is a bit arbitrary. For nonsymmetric convex bodies, however, there is no symmetry center, and we do not see any obvious way to define the "right diagonal system". Hammer treated this question thoroughly on the plane, and came up with a positive solution: to all planar convex bodies there exists a (sub)set of diagonals such that each exterior point is covered once, and only once, and each interior point is also covered (but diagonals usually mesh in K).

I do not know if the above investigation of Hammer has been extended later to higher dimensional diagonal coverage systems. It seems not, although finding diagonal covering systems in $\mathbb{R}^d$ could have been useful, and is certainly interesting for its own sake, too. However, for our present, mostly illustrative purposes the above is enough. But let us note that regarding the existence and unicity of some generalized center, and concerning the so-called "associated bodies" (our K_λ's in the above introduction of $\alpha(K, x)$, the generalized Minkowski functional) Hammer mentions a forthcoming, unpublished note with Sobczyk, (which seems having remained unpublished, though) and that the existence, but non-uniqueness of the generalized centers have been known among geometers. On the other hand it is of interest that the relation to the generalized Minkowski functional, as introduced by Minkowski and used by e.g. Radon, seems to have been unnoticed until [54].

Returning to diameter coverage, we see that there are two more or less independent questions arising: whether one can construct a precise coverage (no points are covered by two diagonals, all exterior points are covered, in all directions there is a diagonal in the covering system), and whether there is a *continuous* coverage. Continuity does not play much role in the real approximation problems which we discuss here, but it will have a decisive role in the complex generalization later.

Let us consider here a genuine approximation theory problem, which gives rise to a wonderful illustration both to the use of covering by diagonals and to the "Bojanov principle" I am talking about. Consider the *Chebyshev problem* *of polynomial growth* in $\mathbb{R}^d$. Here we ask that if a polynomial $p(x)$ is bounded by 1 on K, then how large it can possibly become at an exterior point P? Formalizing, we denote the extremal value for all polynomials of degree n by $C_n(K,P)$. This was first determined by Rivlin and Shapiro [58], and their seminal paper seems to be the very first place where the generalized Minkowski functional surfaced in approximation theory (although in a somewhat disguised form). Let us describe briefly the solution to this question.

An attempt to get upper estimation to this extremal quantity is obvious through restriction of polynomials $p \in \mathcal{P}_n$ on straight lines ℓ, for we already know that in dimension one the extremum is the Chebyshev polynomial T_n. We are given $P \in \mathrm{ext} K$, so let us consider all directions $v \in S_X$, draw the line $\ell = \ell(P,v)$ through P in the given direction v, and take $[A,B] := \ell \cap K$, with say A being the farther from P; denote $t := t(K,P,v) := |P-A|/|A-B|$. Then if $p_0 := P|_\ell$, then $\|p_0\|_{[A,B]} \le \|P\|_K \le 1$, and a moment's computation gives $|p(P)| \le T_n(2t-1)$.

This one dimensional estimate can then be taken into account for all possible directions. Clearly, the smaller the value of t, the smaller the obtained upper estimate is: so this yields $C_n(K,P) \le T_n(\mu(K,P))$ with

$$\mu(K,P) := 2\min\left\{\frac{|\mathrm{con}(A,B,P)|}{|A-B|} : \exists \ell,\ P \in \ell,\ \ell \cap K = [A,B]\right\} - 1\,. \qquad (17)$$

Which is the extremal line? It is not difficult to show that once $\ell = \ell(P,v)$ is extremal in the sense of (17), then the boundary points $A, B \in \partial K$ support parallel supporting hyperplanes of K. But then no lines, parallel to v, can intersect K in any longer segment than $[A,B]$. That is, $\tau(K,v) = |A-B| = |\ell \cap K|$, and ℓ is just a diagonal of K. Therefore, we found that the best upper bound was furnished precisely by some diagonal through P.

Having an upper estimation, we now consider lower estimates. One works along the Bojanov principle once again, though a different way. We want to find a multivariate polynomial, bounded by 1 on K and as large as possible at P. If we have not much fantasy about some really multivariate constructions, we try with essentially univariate polynomials, that is, with ridge polynomials of the form $p(x) = p_0(\langle x,v\rangle)$, where $v \in S_X$ is a unit vector and p_0 is a univariate polynomial. Now with ridge polynomials we must select p_0 so that it is bounded by 1 on K, hence on the whole strip where $\langle x,v\rangle$ agrees with the same inner product with some point of K. The strip in question is the union of hyperplanes, orthogonal to v, and passing through some points of K. So there will be two bounds, $a = \min_{x\in K}\langle x,v\rangle = \langle A,v\rangle$ with $A \in K$, say, and $b := \max_{x\in K}\langle x,v\rangle = \langle B,v\rangle$ $(B \in K)$, between which $|p_0(t)| \le 1$, and otherwise we want p_0 grow as fast as it can. So $p_0(t)$ should again be chosen the Chebyshev polynomial, adjusted to interval $[a,b]$. This way we can construct as large a value as $T_n(2\langle P-A,v\rangle/\langle B-A,v\rangle - 1)$. It remains to look for as large a ratio $\langle P-A,v\rangle/\langle B-A,v\rangle$ in this construction, as possible. Formally we can define this as an extremal quantity $\sigma(K,P) := 2\max_{v\in S_X}\{\langle P-A,v\rangle/\langle B-A,v\rangle : \dots\}$, and then write that $C_n(K,P) \ge T_n(\sigma(K,P)) - 1$.

Can such rough one–dimensional approaches give a precise answer? The nice thing is that the two different one dimensional arguments indeed meet. To fill the gap in the argument we need only to show that $\mu(K,P) = \sigma(K,P) = \alpha(K,P)$, a geometric connection, in fact equivalent formulations of the definition of the generalized Minkowski functional. For more, (in particular as regards infinite dimensional spaces where some of the above arguments, relying on compactness, fail) see [54]. Here we wanted only to point out that the generalized Minkowski functional is indeed the right thing to consider at least in the Chebyshev problem, and that essentially one dimensional estimates, in particular univariate analysis along diagonals, can be rather useful.

5. Foliations – "Bojanov's Principle" at Work in the Complex Variables

The main aim of this section is to explain, how much the moral of the last section is true even in complex variables and in the pluripotential theoretic approach. It is striking, how researchers of several complex variables and pluripotential theory, (even if totally unaware of Hammer's instructive real geometry work), did essentially the same thing in $\mathbb{C}^d$. What became a standard notion in $\mathbb{C}^d$, analogously to diagonals and diagonal coverage, are the notions of *leafs* and *foliations*. We give the formal definition.

Definition 4. Let $E \subset \mathbb{C}^d$. The family $\mathcal{F}$ is a (one-dimensional) *foliation* of the exterior of E, if

(i) $\mathcal{F}$ is a family of complex holomorphic maps $f : \mathbb{C} \setminus \overline{D} \to \mathbb{C}^d \setminus E$;

(ii) each point $z \notin E$ belongs to exactly one image $L = L(f) := \mathcal{R}f$ (the range of f) of some $f \in \mathcal{F}$. Elements $f \in \mathcal{F}$ or, loosely speaking, just the ranges $L(f)$ for all $f \in \mathcal{F}$ are called the *leafs* of the foliation.

Moreover, the foliation is *continuous*, if for each $z \in \mathbb{C} \setminus E$ there is a neighborhood U and a continuous map $\varphi = (\varphi_1, \varphi_2) : U \to \mathbb{C} \times \mathbb{C}^{d-1}$, so that for any leaf $L = \mathcal{R}f$ the set $L \cap U$ can be represented as φ_2 =const.

The definition of leafs and foliations itself does not refer to any maximality property of the leafs, as opposed to Hammer's treatise of *diagonal* coverage systems. Even in $\mathbb{R}^d$ one may consider more general covering systems of lines, but it seems that the most useful covering systems are those which have at least some extremal properties. For more on this see [25, 26]. Above we saw in the Chebyshev problem, how our lines, drawn arbitrarily at the outset, turned out to be diagonals when chosen to exhibit the best available bounds. Something similar is happening in the complex settings.

Before proceeding, let us recall how in (complex) dimension one the complex Green function can be used for obtaining Bernstein-type inequalities. In dimension one the definition of the Green function $g(z) := g_\Omega(z, \infty)$ (where Ω is the (infinite component of) the complement $\mathbb{C} \setminus E$) can be obtained in several equivalent ways, from conformal mapping properties to analogous constructions to the Siciak-Zaharjuta extremal function. The main properties of g are that it is subharmonic, moreover harmonic on Ω, it has logarithmic growth, it is identically zero on E (if $E \Subset \mathbb{R}$ is not of zero capacity – analogy of pluripolarity – what we always assume) and it controls the growth of polynomials the same way as $\exp(V_K)$. Namely, for any complex point $z = x + iy \in \Omega$, we have $|p(z)| \le \max_E |p| \cdot e^{\deg p \cdot g(z,\infty)}$ by the Bernstein-Walsh inequality [70, p. 77]. This corresponds to the fact that $g(z, \infty) = V_E^*(z)$, a relation justifying the name given to $U_E = V_E$ above.

Recall that in one variable for a compact set $E \subset \mathbb{R}$, (or in some already described cases even for $E \subset \mathbb{C}$, see e.g. [45], [46], [67]) a Bernstein type estimate of the derivative at a given point $x \in E$ (or a Markov type uniform estimate on $\|p'\|_E$) can generally be obtained by means of the imaginary direction directional derivative (or the outer normal directional derivative) of the Green function. We indeed have even for general compact sets $E \subset \mathbb{R}$

$$|p'(x)| \le g'_{\pm i}(x, \infty) \deg p \sqrt{\|p\|_E^2 - p^2(x)} \qquad (x \in \operatorname{int} E),$$

where here $g(x, \infty) = V_E(x)$ shows how Baran's estimate is the generalization of the customary one variable formulation.

One would think that the one variable case is an easy-to-derive, classical thing, well explained in many textbooks. However, this is not the case, and the proof, already for sets $E \subset \mathbb{R}$ consisting of finitely many intervals, is surprisingly difficult (see [68, Theorem 3.2]). Also, most formulations usually involve the *density* ω_E of the *equilibrium measure* μ_E instead of reference to the Green (or the Siciak-Zaharjuta extremal) function of the (unbounded component of) the complementary domain Ω, and to look up the relation between g_Ω and ω_E complicates matters. Nevertheless, it is straightforward to see – and actually provides an equivalent introduction to Green's function – that $g_\Omega(z, \infty) = U^{\mu_E}(z) - I(\mu_E)$, the *potential* of μ_E translated by a constant, the *energy* of μ_E, c.f. [56, page 107].

In this respect let us note that if $E \Subset \mathbb{C}$ and $z_0 \in \operatorname{int} E$ (interior now interpreted in the sense of plane topology), then for a measure μ, supported on E, and for the corresponding logarithmic potential $U^\mu(z) := \int_E \log|z-w|\, d\mu(w)$, we have the Poisson equation (which is an extension of the usual Laplace equation) $\Delta U^\mu(z_0) = 2\pi\rho(z)$, where ρ is the density function of μ, at points $z \in \operatorname{int} E$ of absolute continuity of the measure μ at least when ρ is Lipschitz continuous, c.f. [59, II. Theorem 1.3]. However, when $E \Subset \mathbb{R}$, and a measure is supported on E, then even at points $x_0 \in \operatorname{int} E$, μ cannot be absolutely continuous in the planar sense (as then $\mu(E)$ would have to be null), and then correspondingly one derivative in the y, i.e. imaginary direction should be dropped in correspondence to the loss in smoothness in that direction. On the other hand the x-derivative may completely vanish, if we consider g, which identically vanishes on the real segment belonging to the interior of E around x_0. So using $g(z) = U^{\mu_E}(z) - I(\mu_E)$ we arrive at a formula $\pi\omega_E(x) = \frac{1}{2}\left(g'_{+i}(x,\infty) + g'_{-i}(x,\infty)\right)$, (valid more generally in the analogous situation when E is a suitably nice curve, see [59, II. Theorem 1.5]). By the reflection principle here, when $E \subset \mathbb{R}$, it is clear that $g'_{+i}(x,\infty) = g'_{-i}(x,\infty)$, so we get in fact $\pi\omega_E(x) = g'_{\pm i}(x,\infty)$.

Let us note in passing that even if such connections are well-explored in dimension one, apart from the mere definition of the equilibrium measure λ_E and establishment of its absolute continuity in $\operatorname{int} E$, Baran's results are the only relations we know of in several dimensions. In particular, it is not clear how λ_E can be interpreted in terms of potential theoretic equilibrium and in what sense V_E is to correspond to some potential of λ_E.

To construct foliations is even less easy than finding diagonal covering systems in $\mathbb{R}^d$. Below we will briefly describe, how these foliations are constructed: the description is taken from our paper [16]. To understand it, first we recall the notion of the projective space $\mathbb{P}^d$, which can be visualized as the set of complex lines in $\mathbb{C}^{d+1}$, drawn from the point $(0,0,\dots,0,-1)$, say. If the complex line meets the hyperplane $\mathbb{C}^d \times \{0\}$, then through the respective intersection point the given element of $\mathbb{P}^d$ can be identified to a point of $\mathbb{C}^d$, while in case our complex line L is parallel to $\mathbb{C}^d \times \{0\}$, i.e. consists of points of $\mathbb{C}^d \times \{-1\}$ only, then we can identify with the hyperplane $\mathbb{H}$ of directions, since $L = \{\zeta z : \zeta \in \mathbb{C}\}$, where $z \in S_{\mathbb{C}^d}$, is a complex unit vector, i.e. a complex direction.

Let $K \subset \mathbb{R}^d \subset \mathbb{C}^d$ be a convex body, and consider $\mathbb{C}^d \subset \mathbb{P}^d$, the complex projective space with $\mathbb{H} := \mathbb{P}^d \setminus \mathbb{C}^d$ the hyperplane at infinity. Let $\sigma : \mathbb{P}^d \to \mathbb{P}^d$ be the anti-holomorphic map of complex conjugation, which preserves $\mathbb{C}^d$ and $\mathbb{H}$, and is the identity on $\mathbb{R}^d$. Let $\mathbb{H}_\mathbb{R}$ denote the real points of $\mathbb{H}$ (fixed points of σ in $\mathbb{H}$). For any non-zero vector $c \in \mathbb{C}^d$, let $\sigma(c) = \bar{c}$, and $[c] \in \mathbb{H}$ the point in $\mathbb{H}$ given by the direction of c. If $[c] \neq [\bar{c}]$, then $c, \bar{c}$ span a complex subspace $V \subset \mathbb{C}^d$ of dimension two which is invariant under σ; hence V is the complexification of a two-dimensional real subspace $V_0 \subset \mathbb{R}^d$. If we translate V by a vector $A \in \mathbb{R}^d$, we get a complex affine plane $V + A$ invariant by σ and containing the real form $V_0 + A$, the fixed points of σ in $V + A$. Associated to the point $[c] \in \mathbb{H}$, we consider holomorphic maps $f : D \to \mathbb{P}^d$, D being the unit disk in $\mathbb{C}$, such that $f(0) = [c]$, and $f(\partial D) \subset K$. Such maps can be extended by Schwarz reflection to maps (still denoted by) $f : \mathbb{P}^1 \to \mathbb{P}^d$ by the formula

$$f(\tau(\zeta)) = \sigma(f(\zeta)) \in \mathbb{P}^d$$

where $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is the inversion $\tau(\zeta) = 1/\bar{\zeta}$. In particular, such maps have the form

$$f(\zeta) = \rho\,\frac{C}{\zeta} + A + \rho\,\bar{C}\zeta, \tag{18}$$

where $[c] = [C]$, i.e., $C = \lambda c$, for some $\lambda \in \mathbb{C}$, $A \in \mathbb{R}^d$, and $\rho > 0$. Then $f(\mathbb{P}^1) \subset \mathbb{P}^d$ is a quadratic curve, and restricted to ∂D, the unit circle in $\mathbb{C}, f$ gives a parametrization of a real ellipse inside the planar convex set $K \cap \{V_0 + A\}$, with center at A. According to [15], the extremal function V_K is harmonic on the holomorphic curve $f(D \setminus \{0\}) \subset \mathbb{C}^d \setminus K$ if and only if the area of the ellipse bounded by $f(\partial D)$ is *maximal* among all those of the form (18).

For a fixed, normalized C, this is equivalent to varying $A \in \mathbb{R}^d$ and $\rho > 0$ among the maps in (18) with $\mathcal{E} = f(\partial D) \subset K$ in order to maximize ρ. Fixing C amounts to prescribing the orientation (major and minor axis) and eccentricity of a family of inscribed ellipses in K. So this results in the next definition.

Definition 5. Given $C \in \mathbb{H}$, i.e. $C \in \mathbb{C}^d$ with $|C| = 1$, among all ellipses of the form (18) and satisfying $f(\partial D) \subset K$ we chose one with maximal ρ, i.e. with maximal area among all ellipses of the class. We will call such an extremal ellipse $\mathcal{E}$ a *maximal area ellipse*, or simply *a-maximal*.

In the case where ∂K contains no parallel faces, for each $[c] \in \mathbb{H}$ there is a unique a-maximal ellipse (Theorem 7.1, [15]); we denote the corresponding map by f_c. In this situation, the collection of complex ellipses $\{f_c(D \setminus \{0\}) : [c] \in \mathbb{H}\}$ forms a continuous foliation of $\mathbb{C}^d \setminus K$. In simple terms, this means that if z, z' are distinct points in $\mathbb{C}^d \setminus K$, with $|z - z'|$ small, lying on leaves $L(z) := f_c(D \setminus \{0\})$ and $L'(z) := f_{c'}(D \setminus \{0\})$, then the corresponding leaf parameters in (18) are close; i.e., $C \sim C'$, $\rho \sim \rho'$ and $A \sim A'$ (and of course $|\zeta_z| \sim |\zeta_{z'}|$ where $f_c(\zeta_z) = z$ and $f_{c'}(\zeta_{z'}) = z'$). Any convex body in $\mathbb{R}^2$ admits a continuous foliation; this follows using ideas in [15]. Moreover, if we let $\mathcal{C}$ denote the set of all convex bodies $K \subset \mathbb{R}^d$ admitting a continuous foliation, then $\mathcal{C}$ is dense in the Hausdorff metric in the set $\mathcal{K}$ of all convex bodies $K \subset \mathbb{R}^d$. This follows, for example, from the fact that strictly convex bodies K belong to $\mathcal{C}$ (cf., Theorem 7.1 of [15]). In addition, all symmetric convex bodies admit a continuous foliation.

For convenience, instead of using the holomorphic curves $f(D \setminus \{0\})$ we will work with the holomorphic curve $f(\mathbb{C} \setminus \bar{D})$; thus V_K being harmonic on this curve means that

$$V_K(f(\zeta)) = \log |\zeta| \quad \text{for} \quad |\zeta| \geq 1. \tag{19}$$

This identity will play a key role in our forthcoming argument, so it is important to have a feeling of it. The $\leq$ inequality is indeed clear for any holomorphic mapping f mapping ∂D into K; the converse follows from the nontrivial considerations of [14] and [15].

This property makes it clear why we look for a-maximal ellipses, i.e. maximal leaf mappings: only for them we can guarantee this simple and useful formula, which in turn will be a crucial ingredient in our analysis. Therefore, although not required at the outset, we are back to the extremal property of the leafs: like diagonals are extensions of the maximal chords in $\mathbb{R}^d$, also in $\mathbb{C}^d$ the leafs will be complex extensions of a-maximal ellipses.

Furthermore, as in case of the generalized Minkowski functional and the Chebyshev problem, characterization of extremal leaf mappings, i.e. extremal ellipses occur in different forms from different approaches – not even necessarily related to the given approximation theory question or the concrete method we pursue – and it is important and potentially rather useful to understand the equivalent forms of this extremality. That we touch upon next.

6. Methods Translated to Extremal Ellipses

We have thus seen above, that even Baran's pluripotential theoretic estimates boil down to the search of certain extremal ellipses. We have thus two families of extremal ellipses: the a-maximal ellipses of Defintion 5, and the class of b-maximal ellipses of Definition 3.

The following was shown in [16, Proposition 3.2].

Proposition 5 (Burns, Levenberg, Ma'u, Révész, 2007). *For any convex body K, a b-maximal ellipse $\mathcal{E}$ is also an a-maximal ellipse.*

Proof. First observe that an a-maximal ellipse $\mathcal{E}$ is characterized by the property that *no translate of $\mathcal{E}$ lies entirely in the interior K^o of K.* For if $\mathcal{E}+v \subset K^o$ for some v, then one can dilate $\mathcal{E}+v$ still within K, to get an ellipse with the same orientation and eccentricity as $\mathcal{E}$ but having larger area. Conversely, if $\mathcal{E}$ has area A but is *not* an a-maximal ellipse, then one can find an ellipse $\mathcal{E}'$ with the same orientation and eccentricity as $\mathcal{E}$ which lies in K but has area $A' > A$; and then the $\sqrt{A/A'}$ dilation of $\mathcal{E}'$, about any interior point of K will be fully in K^o, homothetic to $\mathcal{E}'$ and of area A, that is, congruent to $\mathcal{E}$.

Moreover: $\mathcal{E}$ is *not* an a-maximal ellipse if and only if there is a unit vector v and $\delta > 0$ such that $\mathcal{E}+sv \subset K^o$ for $0 < s < \delta$. This follows, since if K is a convex body, $u \in K$ and $u+w \in K^o$, then the entire half-open segment $(u, u+w]$ lies in K^o.

Now suppose that $\mathcal{E}$ given by $\theta \to a\cos\theta + by\sin\theta + (x-a)$ is a b-maximal ellipse for x, y. For the sake of obtaining a contradiction, we assume that $\mathcal{E}$ is *not* an a-maximal ellipse. So, $\exists v \neq 0$ and $\delta > 0$ such that $\mathcal{E}_s := \mathcal{E}+sv \subset K^o$ for $0 < s < \delta$.

For $0 < \epsilon < \delta/2$, consider the ellipse $\tilde{\mathcal{E}}(\epsilon)$ given by

$$\begin{aligned} r_\epsilon(\theta) &= (a-\epsilon v)\cos\theta + by\sin\theta + x - (a-\epsilon v) \\ &= a\cos\theta + by\sin\theta + (x-a) + \epsilon v(1-\cos\theta). \end{aligned}$$

Observe that the point $r_\epsilon(\theta) \in \tilde{\mathcal{E}}(\epsilon)$ lies on the ellipse $\mathcal{E}_{s_\theta} := \mathcal{E} + \epsilon(1-\cos\theta)v$ where $s_\theta = \epsilon(1-\cos\theta) \leq 2\epsilon < \delta$. Thus $\tilde{\mathcal{E}}(\epsilon) \subset K^o$.

Note that $r_\epsilon(0) = x \in \tilde{\mathcal{E}}(\epsilon)$ and $r'_\epsilon(0) = by$; in particular, the "b" for $\tilde{\mathcal{E}}(\epsilon)$ is the same as the "b" for $\mathcal{E}$.

But because $\tilde{\mathcal{E}}(\epsilon) \subset K^o$, some dilation at x yields a larger b, so $\mathcal{E}$ was not a b-maximal ellipse for x, y. Contradiction: whence $\mathcal{E}$ must have been a-maximal, too. $\square$

This is used in our proof of the next Theorem 9. From that result, however, a posteriori, we find even the following geometry conclusion.

Corollary 2. *For any convex body K, an ellipse $\mathcal{E} \subset K$ is a-maximal if and only if it is b-maximal for all $x \in \mathcal{E} \cap K^o$ and $y \in T_x\mathcal{E}$.*

I still don't know any direct, geometrical proof of this fact, although the assertion is purely geometrical, and one direction was proved relatively easily in Proposition 5 above. Nevertheless, it shows a further geometrical characterization of maximal leafs, a further analogy to the situation with the generalized Minkowski functional and diagonal coverings in $\mathbb{R}^2$.

7. A Unifying Result

For any compact set $K \subset \mathbb{R}^d$ with non-empty interior, take $x \in K^o$ and $y \in \mathbb{R}^d \setminus \{0\}$. According to the above, we have the pointwise inequalities

$$\delta_D(x,y) := \sup_{\mathbf{0} \neq P \in \mathcal{P}_n} \langle GP(x), y \rangle \leq \begin{cases} \delta_B(x,y) := \liminf\limits_{t \to 0^+} \frac{V_K(x+ity)}{t}, \\ \delta_S(x,y) := \frac{1}{b^*(x,y)}. \end{cases}$$

These quantities are the so-called *metrics* (Finsler metrics) in the differential geometry sense. The assertion that e.g. the inscribed ellipse method is optimal, that is it gives sharp estimates, would be to say that $\delta_D(x,y) = \delta_S(x,y)$. The above Corollary 1 says that $\delta_S(x,y) = \delta_B(x,y)$ in case of Δ. Realizing this in 2005 I conjectured that it may hold in general, see [50, Hypothesis A]. And it indeed does, as was shown recently [16].

Theorem 9 (Burns, Levenberg, Ma'u, Révész, 2007). *Let K be a convex body in $\mathbb{R}^d$. Then the limit in the definition of the directional derivative (in the imaginary direction iy) of V_K exists and equals $\frac{1}{b^*(x,y)}$:*

$$\delta_B(x,y) := \lim_{t \to 0^+} \frac{V_K(x+ity)}{t} = \frac{1}{b^*(x,y)} =: \delta_S(x,y). \tag{20}$$

Proof. Let K be an arbitrary convex body in $\mathbb{R}^d$. Fix $x \in K^o$ and $y \in S_{\mathbb{R}^d}$. Take a b-maximal ellipse $\mathcal{E}$ through x with tangent direction y at x; it will be convenient to have the center written as a instead of $x - a$, so we write

$$\theta \to r(\theta) = (x-a)\cos\theta + b^*(x,y) y \sin\theta + a, \qquad \theta \in [0, 2\pi].$$

This is an a-maximal ellipse $\mathcal{E}$ by Proposition 5; i.e., $\mathcal{E}$ forms the real points of a leaf L

$$f(\zeta) = (x-a)\Big[\frac{1}{2}(\zeta + 1/\zeta)\Big] + b^*(x,y)\,y\Big[\frac{i}{2}(\zeta - 1/\zeta)\Big] + a, \qquad |\zeta| \geq 1 \quad (21)$$

of our foliation for the extremal function V_K. We can compare this "b-maximal" form of the leaf with its a-maximal form (18):

$$f(\zeta) = A + c\zeta + \bar{c}/\zeta, \qquad |\zeta| \geq 1, \quad (22)$$

where, for simplicity, we write $c := \rho C$ in (18). Thus, from (19), $V_K(f(\zeta)) = \log|\zeta|$ for $|\zeta| \geq 1$.

We first show that

$$\lim_{r\to 1^+} \frac{f(r) - f(1)}{r-1} = ib^*(x,y)y.$$

This follows from the calculation

$$f(r) - f(1) = (x-a)\big(\frac{(r-1)^2}{2r}\big) + ib^*(x,y)y\frac{(r-1)(r+1)}{2r}.$$

Thus the real tangent vector to the real curve $r \to f(r)$, $r \geq 1$ as $r \to 1^+$ is in the direction $ib^*(x,y)y$. Now $f(1) = x$ and $x \in K$ so $V_K(f(1)) = V_K(x) = 0$; and, since f is a leaf of our foliation, $V_K(f(r)) = \log r$. Hence

$$\frac{V_K(f(r)) - V_K(f(1))}{r-1} = \frac{\log r}{r-1}.$$

This elementary calculation shows that for any convex body $K \subset \mathbb{R}^d$,

$$\lim_{r\to 1^+} \frac{V_K(f(r)) - V_K(f(1))}{b^*(x,y)(r-1)} = \frac{1}{b^*(x,y)}; \quad (23)$$

i.e., the curvilinear limit along the curve $f(r)$ in the direction of iy at x exists and equals $\frac{1}{b^*(x,y)}$. Note that

$$f(r) - x = f(r) - f(1) = ib^*(x,y)y(r-1) + O((r-1)^2),$$

so that the point $x + ib^*(x,y)y(r-1)$ is $O((r-1)^2)$ close to the point $f(r)$. We use the explicit form (21) of the leaf to verify the existence of the limit in the directional derivative $\delta_B(x,y)$.

If we can show that

$$\lim_{r\to 1^+} \frac{V_K(f(r)) - V_K(x + ib^*(x,y)y(r-1))}{b^*(x,y)(r-1)} = 0, \quad (24)$$

then using (23) and the preceding discussion, we will have the result.

We first consider the case when K admits a *continuous foliation*; so as above, let the family of all such convex bodies of $\mathbb{R}^d$ be denoted by $\mathcal{C}$, and assume $K \in \mathcal{C}$. Consider a fixed point $w := x + ib^*(x,y)y(r-1) \in \mathbb{C}^d$. This belongs to some foliation leaf M which we write in the form (22):

$$g(\zeta) = \alpha + \gamma\zeta + \bar{\gamma}/\zeta \ : \ \mathbb{C} \setminus D \to M \subset \mathbb{C}^n.$$

We need to use the facts that when $r \to 1^+$, then $w \to x \in L$, and, by continuity of the foliation, the leaf parameters for (g, M) should converge to those of (f, L); i.e., $\alpha \to A$ and $\gamma \to c$. We remark that if we compare (21) and (22), writing $b := b^*(x,y)$ we have the relations

$$A = a \quad \text{and} \quad c = \frac{1}{2}(x - a + iby). \tag{25}$$

Here we suppress a rotational invariance: the substitution $\zeta' := \zeta e^{i\varphi}$ for any fixed constant φ describes the same leaf with a different parametrization; thus we fix its value so that

$$\xi := g(1) = 2\,\mathrm{Re}\,\gamma + \alpha$$

is closest to $x := f(1) = 2\,\mathrm{Re}\,c + A$, i.e., $|g(1) - f(1)| \le |g(e^{i\theta}) - f(1)|$ for all θ. To emphasize, we write the leaf (f, L) in b-maximal form (21),

$$f(\zeta) = (x-a)\frac{1}{2}\Big(\zeta + \frac{1}{\zeta}\Big) + by\frac{i}{2}\Big(\zeta - \frac{1}{\zeta}\Big) + a$$

where, from (25) and the fact that y is a unit vector, $b := 2\,|\mathrm{Im}\,c| > 0$ and $y := \frac{2}{b}\,\mathrm{Im}\,c \in \mathbb{R}^d$. Now, apriori, we do not know if (g, M) is b-maximal (aposteriori, it is: see Corollary 2). However, we may still write this leaf in the form

$$g(\zeta) = (\xi - \alpha)\frac{1}{2}\Big(\zeta + \frac{1}{\zeta}\Big) + \beta\eta\frac{i}{2}\Big(\zeta - \frac{1}{\zeta}\Big) + \alpha$$

with $\beta := 2\,|\mathrm{Im}\,\gamma| > 0$ and $\eta := \frac{2}{\beta}\,\mathrm{Im}\,\gamma \in \mathbb{R}^n$. Note that continuity of the foliation implies $\beta > 0$ since $b > 0$; indeed, $\beta \sim b$, $\xi \sim x$, $\eta \sim y$, $\alpha \sim a$, and $\gamma \sim c$.

Since $w \in M$, there is a point $\omega \in \mathbb{C} \setminus D$ with $g(\omega) = w$. Our task is to calculate $V_K(w) = V_K(g(\omega))$. On a leaf of the foliation we have the formula $V_K(g(\omega)) = \log|\omega|$, so it suffices to compute $\log|\omega|$. The representation of w as $g(\omega)$ means that coordinatewise, i.e. for $j = 1, \ldots, d$,

$$x_j + iby_j(r-1) = w_j = g_j(\omega) = (\xi_j - \alpha_j)\frac{1}{2}\Big(\omega + \frac{1}{\omega}\Big) + \beta\eta_j\frac{i}{2}\Big(\omega - \frac{1}{\omega}\Big) + \alpha_j.$$

Since y and η are unit vectors which are close to each other, we can choose a coordinate j with $y_j \neq 0$, $\eta_j \neq 0$. For this coordinate j, the previous displayed equation gives

$$\frac{1}{2}(\xi_j - \alpha_j + i\beta\eta_j)\omega^2 + (\alpha_j - x_j - iby_j(r-1))\omega + \frac{1}{2}(\xi_j - \alpha_j - i\beta\eta_j) = 0,$$

a quadratic equation in ω. Corresponding to the double mapping properties of the Joukowski map $\frac{1}{2}(\zeta + 1/\zeta)$, there are two roots, one in $|\zeta| < 1$ and one in $|\zeta| > 1$, the latter being our ω as we parameterized leafs on $\mathbb{C} \setminus D$. For convenience, put $\rho := b(r-1)y_j$. Since $by_j \neq 0$, $\rho \asymp r-1$. By the quadratic formula,

$$\omega_{1,2} = \frac{x_j - \alpha_j + i\rho \pm \sqrt{(\alpha_j - x_j - i\rho)^2 - (\xi_j - \alpha_j)^2 - \beta^2\eta_j^2}}{\xi_j - \alpha_j + i\beta\eta_j}.$$

Set $Q := \beta^2\eta_j^2 + (\xi_j - \alpha_j)^2 - (x_j - \alpha_j)^2 \sim b^2 y_j^2 > 0$ by continuity of the leaf parameters and choice of j. Using this and the simple formula $\sqrt{A+2B} = \sqrt{A} + B/\sqrt{A} + O(B^2/A^{3/2})$, valid uniformly for $|B| < A/3$, say, we can rewrite the square root as

$$\pm\sqrt{(x_j - \alpha_j)^2 - i2(\alpha_j - x_j)\rho - \rho^2 - (\xi_j - \alpha_j)^2 - \beta^2\eta_j^2} \\ = \pm\Big\{\frac{(x_j - \alpha_j)\rho}{\sqrt{Q}} + i\sqrt{Q} + O(\rho^2)\Big\}.$$

Put $P := \xi_j - \alpha_j + i\beta\eta_j$. Then

$$\begin{aligned} |\omega_{1,2}P|^2 &= \big|(x_j - \alpha_j)(1 \pm \frac{\rho}{\sqrt{Q}}) + i(\pm\sqrt{Q} + \rho) + O(\rho^2)\big|^2 \\ &= (\alpha_j - x_j)^2 + Q \pm \frac{2\rho}{\sqrt{Q}}\big(Q + (\alpha_j - x_j)^2\big) + O(\rho^2). \end{aligned}$$

We have the identity $|P|^2 = (\alpha_j - x_j)^2 + Q$; dividing by this quantity on both sides yields

$$|\omega_{1,2}|^2 = 1 \pm \frac{2\rho}{\sqrt{Q}} + O(\rho^2).$$

Fixing the branch of the square-root with $\sqrt{Q} > 0$, it is clear that among the choice of signs in $\pm$ the one equal to the sign of y_j leads to the larger absolute value (and the one with $|\omega|$ exceeding 1); hence for such ω with $g(\omega) = w$,

$$|\omega|^2 = 1 + \frac{2|\rho|}{\sqrt{Q}} + O(\rho^2) = 1 + \frac{2b|y_j|(r-1)}{\sqrt{Q}} + O((r-1)^2),$$

so that

$$\log|\omega|^2 = \frac{2b|y_j|(r-1)}{\sqrt{Q}} + O((r-1)^2).$$

Hence

$$\frac{V_K(x + iby(r-1))}{r-1} = \frac{V_K(g(\omega))}{r-1} = \frac{\log|\omega|}{r-1} = \frac{\log|\omega|^2}{2(r-1)} = \frac{b|y_j|}{\sqrt{Q}} + O(r-1).$$

From the continuity of the foliation, as $r \to 1$ we have $\xi_j \to x_j$, $\alpha_j \to a_j$, $\beta \to b$, and thus $\sqrt{Q} \to b|y_j|$. Hence

$$\lim_{r\to 1^+} \frac{V_K(x + iby(r-1))}{r-1} = 1 = \lim_{r\to 1^+} \frac{V_K(f(r))}{r-1}.$$

This verifies (24) and hence (20) in the case when K admits a continuous foliation.

In the above proof we assumed that ext$K$ admits a continuous foliation. Indeed, we have to emphasize the role of continuity in this one dimensional parametrization in the complex settings, which played a key role in the proof. Fortunately, according to the results of [15], there are enough many K with continuous foliations. In particular, all strictly convex bodies of $\mathbb{R}^d$ are such. Therefore, with an additional approximation argument, the proof can be extended from $\mathcal{C}$ to all convex bodies. Indeed, there are $K_1 \subset K \subset K_2$, $K_1, K_2 \in \mathcal{C}$, arbitrarily close (in the Hausdorff distance) to K, more precisely, we can construct strictly convex bodies K_1, K_2 satisfying that K_2 and K_1 are homothetic with a factor α arbitrarily close to 1. From this using also that both $b^*(K; x, y)$ and $D_y V_K(x)$ are monotonous in K, and are homogeneous with respect to dilation, the result can be extended even to general convex bodies of $\mathbb{R}^d$. For the details see [16]. $\square$

8. A Few Elementary Calculations and a Discussion of Conjecture 1

First of all, let us see a very fundamental issue, that of transformation of $\langle GP(x), y\rangle$ or $\langle DP(x), y\rangle$, and hence of $\mathcal{G}_K(x)$, under linear transformations. In fact, it is quite clear that rotations, as such, do not change anything, so in fact the whole issue is about affine transformations – so in the following we freely speak of affine invariance and linear invariance as synonyms.

To concretize things, let K be a convex body, $x \in \text{int}K$, and $y \in S_X$. Also, for simplicity let us restrict to $X = \mathbb{R}^d$ even if with some work everything we discuss can be carried over to normed spaces of arbitrary dimensions. Now assume that $T : K \leftrightarrow M$ with the linear, nonsingular mapping (invertible operator) T of X, so that also M is a convex body of X. We consider that the norm of X is unchanged in its two copies for the domain and range, and we analyze the rule how the derivatives of polynomials change. Clearly, Ty is a vector in X, and, as $y \neq 0$, also $Ty \neq 0$ by condition on T, but otherwise its norm may be arbitrary. Denote now $z := Ty/\|Ty\|$ the normalized version of Ty. What we are interested in, is the change, when a polynomial p is given on M so that it is related to another polynomial P, considered on K, as $P(x) := p(Tx)$. How will the gradient and the directional derivative be changed, between corresponding points x and Tx and in corresponding directions y and $z := Ty/\|Ty\|$?

By the chain rule, clearly, $\langle DP(x), y\rangle = \langle Dp(Tx) \cdot T, y\rangle = \langle Dp(Tx), Ty\rangle = \langle Dp(Tx), z\rangle \|Ty\|$.

Consider now the generalized Minkowski functional. It is clearly an affine invariant notion, so nothing will be changed.

How does the maximal chord changes? Taking $a, b \in \partial K$ with $b - a = \tau y$, where $\tau := \tau(K, y)$, the transformation carries them over to $A = Ta$, $B = Tb$, $A - B = T(a - b) = T\tau y = \tau\|Ty\|z$. So in fact it is easy to see then that $\tau(M, z) = \tau(K, y)\|Ty\|$.

Next we compute the change of $b^*(K; x, y)$, the best ellipse constant. Clearly if an ellipse of the form (10) is inscribed in K, then after applying T the new ellipse $R(t) := \cos tTa + b \sin tTy + T(x - a) = A \cos t + b\|Ty\| \sin tz + T(x - a)$ will be inscribed in M. Also, $r(0) = x$ and $R(0) = Tx$, and $r'(0)\|y$ and $R'(0)\|z$. Note that the description is one-to-one, as also the inverse mapping T^{-1} can be used to establish the converse relation between inscribed ellipses. Now the best ellipse constant is the supremum of the admissible parameter values: so we have $b^*(K; x, y) = \|Ty\|b^*(M; Tx, z)$, again a transformation rule in conformity with the change of derivatives.

Similar calculations – or referring to the equivalence theorem Theorem 9 – provide the same for $D_y^+ V_E(x)$ in (6), too.

These calculations confirm that in fact under affine transformations the directional derivatives change the same way as the quantities in their estimations in e.g. (12), (14).

So we see one reason, why it was not that easy to decide if Conjecture 2 was to hold or not. In fact, the conjecture – more precisely, its more precise version for all the directional derivatives – satisfies the natural criterion of being affine invariant, so it could have been expected logically to be valid. Furthermore, it was shown in [38], that if we restrict to *ridge polynomials*, then the conjecture actually holds true for them (at least on the simplex). Since our heuristics is very much after dimension one, moreover, in other problems, such as the multiply-quoted Chebyshev problem, such ridge polynomials provide the extremal cases, here it was more difficult to figure out, what polynomials, essentially multivariate in nature, could come up for a fine lower estimation of directional derivatives. That underlines the value of the work of Naidenov [47, 48] in this regard.

As is given in Theorem 2, the density function $\lambda(x)$ of the equilibrium measure satisfies $\lambda(x)/d! = \operatorname{vol} \mathcal{G}(x)$ if K is a symmetric convex body. Few concrete cases were known above that, but Baran [6] computed this density for the standard triangle Δ. He found $\lambda(x) = 2\pi/\sqrt{x_1 x_2 (1 - x_1 x_2)}$. In [50], on the other hand, we described the set, what results from the known estimates for the gradient: we of course must have $\mathcal{G}_\Delta(x) \subset \{y : D_y V_\Delta(x) \leq 1\}^*$, and the estimating quantities $D_y V_\Delta(x)$ were there already computed, so it opened up the way for a direct calculation of this example. The result was that we in fact have $\operatorname{vol}\{y : D_y V_\Delta(x) \leq 1\}^* = \lambda(x)/2$ for all $x \in \operatorname{int}\Delta$.

This was interesting for the exact values of the available estimates for the standard triangle fell short of Conjecture 2. So we found that either the set

$\mathcal{G}_\Delta(x)$ cannot be further restricted, i.e. $\mathcal{G}_\Delta(x) = \{y : D_y V_\Delta(x) \le 1\}^*$, and then Conjecture 1 holds for the triangle, but Conjecture 2 fails, or there is room for further improvement, maybe even Conjecture 2 can be reached, but then $\operatorname{vol}\mathcal{G}_\Delta(x) < \lambda(x)/2$, and Conjecture 1 must fail. That led us to the somewhat strange conclusion of Proposition 4, which, in turn, greatly motivated the further search for a disproof of Conjecture 2.

Regarding inequality (7), the work [16] also shed more light on the situation. Indeed, we have also showed in [16, Corollary 4.5] the following formula.

Theorem 10 (Burns, Levenberg, Ma'u, Révész, 2007). *For any convex body $K \subset \mathbb{R}^d$, and with the density $\lambda(x)$ of the equilibrium measure $\lambda_K(x)$, the formula $\lambda(x)/d! = \operatorname{vol}\{y : D_y V_K(x) \le 1\}^*$ holds true.*

This shows the same situation, for the general convex body, too, as seen before for the triangle.

So once again, we have a dichotomy: either Conjecture 1 holds, and then the currently available estimates, through the equivalent methods of Baran or Sarantopoulos, must yield the best possible restriction on the set $\mathcal{G}_K(x)$ – and then in particular Conjecture 3 below is essentially correct – or there is room for sharpening of the available bounds on $\mathcal{G}_K(x)$, and then also Conjecture 1 must fail. It thus became clear, that all the currently standing conjectures are essentially equivalent.

Corollary 3. *Either Conjectures 1 and 3 both fail, or Conjecture 1 holds true and then there can be no essential improvement to the bounds $\delta_D(K;x,y) \le \delta_B(K;x,y) = \delta_S(K;x,y)$ in the sense, that the resulting set $\{y : \delta_B(K;x,y)(= \delta_S(K;x,y)) \le 1\}^*$ equals to $\mathcal{G}_K(x,y)$.*

Note that, at least in principle, it is possible that in some direction y we have $\delta_D(K;x,y) < \delta_B(K;x,y) = \delta_S(K;x,y)$, but the resulting niveau sets, and thus their polar sets, are the same for all three metrics. This is because the redundant increase of one estimate, or some estimates, of these directional estimates can be "corrected' by taking the intersection of all halfplanes determined by the inequalities $\langle v, y\rangle \le \delta(K;x,y)$. For an example of this sort see [50, §5].

9. What's Next – Problems and Possible Approaches

Now the natural question is if these equivalent methods lead to the best possible result, i.e. if $\delta_D(x,y) = \delta_S(x,y) = \delta_B(x,y)$? These were formulated as Hypothesis B and Hypothesis C in [50].

Conjecture 3. *We have $\delta_D(x,y) = \delta_S(x,y) = \delta_B(x,y)$. That is, the estimates obtained by the pluripotential theoretic approach of Baran as well as by the inscribed ellipse method of Sarantopoulos, are sharp.*

What is missing, is the lower estimation. We look for polynomials with large gradient vectors. To find the largest, we should consider all polynomials of a given degree n on $\mathbb{R}^d$, i.e. of d variables, and compute $DP(x)$ and $GP(x)$.

A possibly important observation is that on the vector space $\mathcal{P}_n(X)$, the functional $\|DP(x)\|$, or even $|\langle DP(x), y\rangle|$ are convex functionals. So it comes to mind to extremalize by the method which can be called (in reference to the Krein-Milmnan theorem) the *Krein-Milman approach*, an early champion of which was Voronovskaya [69]. In approximation theory probably Konheim and Rivlin [32] made this approach well-known, and it is extensively used e.g. by Shapiro [61], too. Working with this idea, first we describe the *extreme points* of the convex set formed by the polynomials of norm ≤ 1 – the *unit ball* of our polynomial space – and then to find the maximum of these convex functionals it suffices to maximize them on these extreme points.

This type of approach has seen some successes in other questions, see e.g. [2, 20, 21, 22, 53, 41, 42, 49], and in particular [43], where the method is explicitly in focus. However, one has to see that these successes were possible only in rather low dimensional polynomial spaces, e.g. in spaces of trinomials, or spaces of homogeneous polynomials (which restriction of course reduces the dimension). Also it is much easier to handle symmetric sets, particularly the disk or the square, than to treat the standard triangle $\Delta \subset \mathbb{R}^2$.

Still, to settle the case of the triangle is, as in several instances in the above described researches, important. As mentioned above, Δ is a prototype of a nonsymmetric convex body, and it is the extreme case, too, because in a completely precise mathematical sense it is the least symmetric among all convex bodies of $\mathbb{R}^2$, see [23]. Also note that proving sharpness for the triangle already implies sharpness for all planar convex bodies, see the closing remark of [16]. Therefore it is both natural and not too special to deal with Δ in these considerations.

We of course cannot handle the very large dimensional – even less the infinite dimensional – cases, but if the space of polynomials $\mathcal{P}_n(\mathbb{R}^d)$ is reasonably decent, then we may get somewhere with these investigations. However, we also encounter certain difficulties.

(i) The normalization by the degree is fine, but $\sqrt{\|P\|^2 - P(x)^2}$ is ugly. It seems to spoil the convexity of the functional to be maximized.

(ii) Normalizing with respect to the maximum norm on Δ determines a rather complicated type of norm, when expressed in function of the coefficients as coordinates. E.g. the paper [42] describes the space of *homogeneous* degree 2 polynomials with this norm (i.e., provides a geometric description of the unit ball) – and it is already quite complicated. Even if the dimension of the extreme points is smaller, than the total dimension of the polynomial space in question, their structure is complicated. We encounter several manifolds of extreme points of different type and dimension.

(iii) The number of parameters blow up with the degree. $\mathcal{P}_n^d$ is a $\binom{n+d-1}{d}$-dimensional vector space in the coefficients as variables. Degree 1 is linear – nothing can be seen as DP is a constant. Degree 2 is, even for $d = 2$, a space of dimension 6. Then degree 3 is already of dimension 10, and seems essentially impossible to deal with.

So it seems that the dimension 2 case is kind of decisive – if we can squeeze out something from the description of the 6 dimensional space of bivariate quadratic polynomials, equipped with the maximum norm of function values on Δ, then we won – if not, then very likely the approach will not lead to a definite answer concerning the optimality of the available estimates. It is quite possible, that our estimates in the Bernstein problem are asymptotically sharp, so they cannot be improved, but degree 2 or 3 polynomials do not come that close to the upper estimations. If this is the case, then there is no hope to follow the geometry of the unit ball of higher dimensional polynomial spaces so far that we detect asymptotic behavior. Only if already at low dimension and degree this sharpness occurs, we can hope for grasping it by this laborious method.

Nevertheless, the work has begun. Milev and Naidenov [36, 37] already described "two thirds" of the set of extreme points of the unit ball of $\mathcal{P}_2(\mathbb{R}^2)$ with the maximum norm on Δ. Calculation of the best inscribed ellipse for the triangle led to rather interesting further insight, conjectures and finally theorems – maybe, this effort will also pay off one day.

There is another approach, which deserves some mention, too. Observe that in the proof of the basic Sarantopoulos estimate – that of the inscribed ellipse lemma in (11) – sharpness of the steps, which we have done, were always clear, apart form the estimate of $\|T\|$ – that is, $\|p|_{\mathcal{E}}\|$ – by $\|p\|$. We could not do any better, but did we it right? Analyzing this question leads to the following extension problem: given an ellipse $\mathcal{E}$, and a polynomial restricted on it, do we always have an extension to the triangle, which has about the same norm (so not larger) than the absolute maximum was on $\mathcal{E}$? For more about this extension problem see the Open Problem section, where the question is presented in full detail.

In conclusion, let us settle with the general impression, or at least our hope, that the exact form of Bernstein's inequality on convex bodies is perhaps about to be clarified.

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