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Linear Relative n-Widths of Sets of Smooth Functions*

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The main topic of this paper is approximation by means of linear shape-preserving operators and estimations of the error of such type of approximation. The results will be presented in the form of estimations of relative (shape-preserving) linear widths.

Keywords and Phrases: Shape-preserving approximation, linear n-widths.

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1. Introduction

In various applications it is necessary to approximate functions preserving properties like monotonicity, convexity, concavity, etc. The last 25 years have marked extensive research in the theory of shape-preserving approximation by means of polynomials and splines. The most significant results were gathered in [1], [2].

Note that a function f possesses some shape properties in the interval $[0,1]$ usually means that the element f belongs to some cone V in $C[0,1]$.

Let X be a normed linear space, X_n be a n-dimensional subspace of X, and V be a cone in X. In the theory of shape-preserving approximation the following classical problems are of interest:

1. the problem of existence, uniqueness, and characterization of the best shape-preserving approximation $g^* \in X_n \cap V$ to $f \in V$, where

$$\|f - g^*\|_X = \inf_{g \in X_n \cap V} \|f - g\|_X;$$

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2. estimation of the deviation of $A \cap V$ from $X_n \cap V$, i.e.

$$E(A \cap V; X_n \cap V) = \sup_{f \in A \cap V} \inf_{g \in X_n \cap V} \|f - g\|_X;$$

3. estimation of the (nonlinear) relative n-widths

$$d_n(A \cap V, V)_X = \inf_{X_n} \sup_{f \in A \cap V} \inf_{g \in X_n \cap V} \|f - g\|_X,$$

   the leftmost infimum being taken over all n-dimensional subspaces X_n of X, such that $X_n \cap V \neq \emptyset$;

4. estimation of the linear relative n-widths.

In this paper we obtain some results related to the latter task.

Note that the definition of (nonlinear) relative n-width was first introduced in 1984 by Konovalov [3]. Though he considered a problem not connected with preserving shapes, the concept of relative n-width arises in the theory of shape-preserving approximation naturally. Of course, it is impossible to obtain $d_n(A \cap V, V)_X$ and determine optimal subspaces X_n (if they exist) for all A, V, X. Nevertheless, some estimations of (nonlinear) relative shape-preserving n-widths have been obtained in papers [4, 5, 6].

Definition 1. Let $L : X \to X$ be a linear operator and V, W be cones in X, $V, W \neq \emptyset$. We say that the operator L has *the shape-preserving property* relative to the cones V, W, if $L(V) \subset W$.

Let X be a linear normed space, and let V, W be some cones in X, $W \subset V$.

Definition 2. *Linear relative n-width* of a set $A \cap V \subset X$ in X relative to the cones (V, W) is defined by

$$\delta_n(A \cap V, V, W)_X := \inf_{L_n(V) \subset W} \sup_{f \in A \cap V} \|f - L_n f\|_X,$$

where the infimum is taken over all linear continuous operators $L_n : X \to X$ of finite rank n and $L_n(V) \subset W$.

If $\delta_n(A, V, W)_X = \sup_{f \in A} \|(I - L_n)f\|_X$, where L_n is a linear continuous operator of rank at most n, such that $L_n(V) \subset W$, then L_n is said to be *an optimal linear operator* for $\delta_n(A, V, W)_X$.

Determination of linear relative n-widths is of interest in the theory of shape-preserving approximation as, knowing the value of the relative linear n-width $\delta_n(A \cap V, V, W)_X$, we can estimate how good or bad (in terms of optimality) this or that finite-dimensional method with shape-preserving property $L_n(V) \subset W$ is.

The estimations of linear relative n-widths of some sets of algebraic polynomials in $X = C[0, 1]$ relative to the cone of all non-negative continuous functions defined on $[0, 1]$ was considered in [7].

Based on the ideas of Korovkin [8], Videnskii [9] and Vassiliev [10], the work [11] presents some estimations of the order of approximation of the r-th derivative of a function by means of linear operators under different assumptions related to shape preserving properties. In this paper estimations of the error of approximation by means of linear shape-preserving rank n operators will be presented in the form of estimations of linear relative n-widths.

2. The Cone

A function $f : [0,1] \to \mathbb{R}$ is said to be *p-monotone* on $[0,1]$, $p \geq 1$, if and only if for all choices of $p+1$ distinct points $t_0, \ldots, t_p$ in $[0,1]$ the inequality

$$[t_0, \ldots, t_p]f \geq 0$$

holds, where $[t_0, \ldots, t_p]f = \sum_{j=0}^{p} f(t_j)/w'(t_j)$ denotes the p-th divided difference of f at $0 \leq t_0 < t_1 < \cdots < t_p \leq 1$, and $w(t) = \prod_{j=0}^{p}(t - t_j)$.

Note that 2-monotone functions are just convex functions. The class of all p-monotone functions on $[0,1]$ is denoted by $\Delta^p[0,1]$. If $f \in C^p[0,1]$, then $f \in \Delta^p[0,1]$ if and only if $f^{(p)}(t) \geq 0$, $t \in [0,1]$. We set for completeness $\Delta^0[0,1] := \{f \in C[0,1] : \ f(t) \geq 0, \ t \in [0,1]\}$.

Let $\sigma = (\sigma_0, \ldots, \sigma_k) \in \mathbb{R}^{k+1}$, $\sigma_i \in \{-1, 0, 1\}$, and $\sigma_k \neq 0$.

Following ideas of [12], we denote

$$\Delta^{0,k}(\sigma) := \{f \in C[0,1] : \ \sigma_p f \in \Delta^p[0,1], \ 0 \leq p \leq k\}. \tag{1}$$

Without loss of generality we will assume that $\sigma_0 = 1$. By $\Delta^{0,0}(\sigma)$ we denote the cone of all non-negative continuous functions, defined on [0,1]:

$$\Delta^{0,0}(\sigma) := \{f \in C[0,1] : \ f \geq 0 \ \text{ in } \ [0,1]\}.$$

3. Examples of Linear Shape-Preserving Operators

Let $L_{k-1}f(\,\cdot\,; y_0, y_1, \ldots, y_{k-1}) \in \operatorname{span}\{e_0, \ldots, e_{k-1}\}$, $e_i(t) = t^i$, denote the Lagrange interpolating polynomial, which coincides with the function f at the points $0 \leq y_0 < y_1 < \ldots < y_{k-1} < 1$:

$$L_{k-1}f(y_i; y_0, y_1, \ldots, y_{k-1}) = f(y_i), \qquad i = 0, \ldots, k-1.$$

Set $y_{-1} = -\infty$, $y_k = \infty$.

Lemma 1. *Let $f \in \Delta^{0,k}(\sigma)$.*

(a) If $\sigma_0\sigma_k > 0$, then for all $x \in \cup_{i=0}^{[(k-1)/2]}[y_{k-1-(2i+1)}, y_{k-1-2i}]$

$$\sigma_0 L_{k-1}f(x; y_0, \ldots, y_{k-1}) \geq 0. \tag{2}$$

(b) If $\sigma_0\sigma_k < 0$, then for all $x \in \cup_{i=-1}^{[(k-2)/2]}[y_{k-1-(2i+2)}, y_{k-1-(2i+1)}]$ the inequality (2) *holds.*

Proof. Suppose that $x \in (y_{l-1}, y_l)$, $l = 0, \dots, k$. It follows from $f \in \Delta^{0,k}(\sigma)$ that $\sigma_k \Delta_{k-1} f(x; y_0, \dots, y_{k-1}) \geq 0$, where

$$\Delta_{k-1} f(x; y_0, \dots, y_{k-1}) = (-1)^l \begin{vmatrix} e_0(x) & e_0(y_0) & \dots & e_0(y_{k-1}) \\ \cdots & \cdots & \cdots & \cdots \\ e_{k-1}(x) & e_{k-1}(y_0) & \dots & e_{k-1}(y_{k-1}) \\ f(x) & f(y_0) & \dots & f(y_{k-1}) \end{vmatrix}.$$

It follows from

$$\begin{aligned} &\Delta_{k-1} f(x; y_0, \dots, y_{k-1}) \\ &\qquad = (-1)^{k-1+l}[L_{k-1} f(x; y_0, \dots, y_{k-1}) - f(x)] \det \big(e_i(y_j)\big)_{i=0,\, j=0}^{k-1,\, k-1}, \end{aligned} \tag{3}$$

that $\sigma_k(-1)^{k-1+l} L_{k-1} f(x; y_0, \dots, y_{k-1}) \geq \sigma_k(-1)^{k-1+l} f(x)$. Since $\sigma_0 f \geq 0$, the inequality (2) holds for appropriate x. □

It is obvious from (3) that

$$L_{k-1} e_i(\,\cdot\,; y_0, \dots, y_{k-1}) = e_i, \qquad i = 0, \dots, k-1. \tag{4}$$

Let $k, n \in \mathbb{N}$, $2 \leq k < n$. We set $x_i = \frac{i-1}{n-1}$, $i = 1, \dots, n$, and denote

$$Z_i := \{j : \ 0 \leq j < n \text{ and } i - k + 2 \leq j \leq i\}.$$

We define the linear operator $\Lambda_{k,n}^{[\sigma]} : C[0,1] \to C[0,1]$ by

$$\Lambda_{k,n}^{[\sigma]} f(x) = L_{k-1} f(x; x_{j(i)}, \dots, x_{j(i)+k}), \quad x \in [x_i, x_{i+1}],\ i = 0, \dots, n-1, \tag{5}$$

where $j(i) \in Z_i$ satisfies

$$L_{k-1} f(x; x_{j(i)}, \dots, x_{j(i)+k}) \geq 0 \quad \text{for all } f \in \Delta^{0,k}(\sigma) \text{ and } x \in [x_{i-1}, x_i].$$

It is obvious that $\Lambda_{k,n}^{[\sigma]}$ is of finite rank n.

Lemma 2. *Let $\Lambda_{k,n}^{[\sigma]} : C[0,1] \to C[0,1]$ be defined by* (5)*. Then*

$$\|\Lambda_{k,n}^{[\sigma]} e_k - e_k\|_{C[0,1]} \leq c_1(k)\, n^{-k}, \tag{6}$$

where $c_1(k)$ does not depend on n.

Proof. It follows from the definition of the operator $\Lambda_{k,n}^{[\sigma]}$ and (3) that

$$\begin{aligned}
&\|\Lambda_{k,n}^{[\sigma]} e_k - e_k\|_{C[0,1]} \\
&\qquad \le \max_{0\le i<n} \max_{j\in Z_i} \sup_{x\in[x_i,x_{i+1}]} |e_k(x) - L_{k-1}e_k(x;x_j,\dots,x_{j+k-1})| \\
&\qquad \le \max_{0\le i<n} \max_{j\in Z_i} \sup_{x\in[x_i,x_{i+1}]} \frac{|\Delta_{k-1}f(x;x_j,\dots,x_{j+k-1})|}{\det\big(e_i(x_s)\big)_{i=0,\, s=j}^{k-1,\, j+k-1}} \\
&\qquad = \max_{0\le i<n} \max_{j\in Z_i} \sup_{x\in[x_i,x_{i+1}]} \Big|\prod_{j\le s\le j+k-1}(x-x_s)\Big| =: c_1(k)\, n^{-k}.
\end{aligned}$$

□

Theorem 1. *Let* $\Lambda_{k,n}^{[\sigma]} : C[0,1] \to C[0,1]$ *be defined by* (5). *Then:*

(a) $\Lambda_{k,n}^{[\sigma]}(\Delta^{0,k}(\sigma)) \subset \Delta^{0,0}(\sigma)$*;*

(b) $\Lambda_{k,n}^{[\sigma]} e_i = e_i$, $i = 0,\dots,k-1$*;*

(c) $\lim_{n\to\infty} \big\|\Lambda_{k,n}^{[\sigma]} e_k - e_k\big\|_{C[0,1]} = 0$*;*

(d) for every $f \in C[0,1]$, $\lim_{n\to\infty} \big\|\Lambda_{k,n}^{[\sigma]} f - f\big\|_{C[0,1]} = 0$.

Proof. Part (a) follows from Lemma 1, (b) follows from (4), and (c) follows from (6). Finally, proposition (d) follows from (a)–(c) and [12]. □

4. Estimations of Linear Relative Shape-Preserving n-Widths

Denote by Π_k the subspace of $C[0,1]$, spanned by $\{e_0, e_1, \dots, e_k\}$, where $e_i(t) = t^i/i!$. Set $P_k := \{p \in \Pi_k : \|D^k p\|_{C[0,1]} \le 1\}$, where D^k denotes the differential operator of order k, $D^k = d^k/dx^k$.

Theorem 2. *Let* $\Delta^{0,k}(\sigma)$, $k \ge 2$, *be the cone defined by* (1). *Then*

(a) $\delta_n\big(P_m \cap \Delta^{0,k}(\sigma), \Delta^{0,k}(\sigma), \Delta^{0,0}(\sigma)\big)_{C[0,1]} = 0$, $m = 0,\dots,k-1$*;*

(b) $c_2(k)\, n^{-k} \le \delta_n\big(P_k \cap \Delta^{0,k}(\sigma), \Delta^{0,k}(\sigma), \Delta^{0,0}(\sigma)\big)_{C[0,1]} \le c_1(k)\, n^{-k}$, *where* $c_1(k)$, $c_2(k)$ *do not depend on* n.

Proof. The first claim (a) of Theorem 2 follows from Theorem 1 (b). We proceed with the proof of the claim (b).

Denote $R_k := \{a = (a_0, \dots, a_k) \in \mathbb{R}^{k+1} : |a_k| \le 1\}$. We have

$$\begin{aligned}
&\delta_n\big(P_k \cap \Delta^{0,k}(\sigma), \Delta^{0,k}(\sigma), \Delta^{0,0}(\sigma)\big)_{C[0,1]} \\
&\quad = \inf_{L_n(\Delta^{0,k}(\sigma)) \subset \Delta^{0,0}(\sigma)} \ \sup_{p \in P_k \cap \Delta^{0,k}(\sigma)} \|p - L_n p\|_{C[0,1]} \\
&\quad = \inf_{L_n(\Delta^{0,k}(\sigma)) \subset \Delta^{0,0}(\sigma)} \ \sup_{p \in P_k \cap \Delta^{0,k}(\sigma)} \ \sup_{x \in [0,1]} |p(x) - L_n p(x)| \\
&\quad = \inf_{L_n(\Delta^{0,k}(\sigma)) \subset \Delta^{0,0}(\sigma)} \ \sup_{x \in [0,1]} \ \sup_{a \in R_k} \sum_{r=0}^{k} |a_r| |e_r(x) - L_n e_r(x)| \\
&\quad = \inf_{L_n \in \mathcal{L}_n(\sigma)} \ \sup_{x \in [0,1]} |e_k(x) - L_n e_k(x)|,
\end{aligned} \tag{7}$$

where $\mathcal{L}_n(\sigma)$ stands for the set of all linear continuous operators L_n of finite rank n, $L_n(\Delta^{0,k}(\sigma)) \subset \Delta^{0,0}(\sigma)$, with

$$L_n e_j = e_j, \qquad j = 0, \dots, k-1. \tag{8}$$

The upper inequality in (b) follows from (7) and Lemma 2.

Consider a linear operator $L_n \in \mathcal{L}_n(\sigma)$. Let $\{v_1, \dots, v_n\}$ be a system of functions generating the linear space $\{L_n f : f \in C[0,1]\}$, i.e.

$$\operatorname{span}\{v_1, \dots, v_n\} = \{L_n f : f \in C[0,1]\}.$$

Consider the matrix $A = \big(v_j(z_i)\big)_{j=1,\, i=1}^{n,\quad n}$, where $z_i = \frac{i-1}{n-1}$, $i = 1, \dots, n$. Observe that the rank of the matrix A is not equal to 0. Indeed, if $\operatorname{rank} A = 0$, then $L_n f(z_i) = \sum_{j=1}^{n} a_j(f) v_j(z_i) = 0$, $i = 0, \dots, n$, for all $f \in C[0,1]$, which is impossible in view of (8).

Next, we take a vector $\delta = (\delta_0, \dots, \delta_n) \in \mathbb{R}^{n+1}$, such that

$$\sum_{i=0}^{n} |\delta_i| = 1, \quad \sum_{i=0}^{n} \delta_i v_j(z_i) = 0, \qquad j = 1, \dots, n.$$

Let a function $h \in C[0,1]$ be such that $h(z_i) = \operatorname{sign} \delta_i$, $i = 0, \dots, n$. Define a function $g \in C[0,1]$ by $g(x) = L_k h(x; z_j, \dots, z_{j+k})$ on $[z_i, z_{i+1}]$, $i = 0, \dots, n-1$, with j being taken arbitrary from the set Z_i. It is easy to verify that the function g possesses the following properties: $g(z_i) = \operatorname{sign} \delta_i$, $i = 0, \dots, n$; $D^k g$ is continuous and finite on every interval (z_i, z_{i+1}), $i = 0, \dots, n-1$.

Since $D^k e_k = 1$, the value of $D^k g$ is equal to the leading coefficient of the polynomial $L_k h(\,\cdot\,; z_j, \dots, z_{j+k})$. Then for $x \in (z_i, z_{i+1})$ there is $j \in Z_i$, such that

$$D^k g(x) = \frac{\begin{vmatrix} e_0(z_j) & e_0(z_{j+1}) & \dots & e_0(z_{j+k}) \\ \hdotsfor{4} \\ e_{k-1}(z_j) & e_{k-1}(z_{j+1}) & \dots & e_{k-1}(z_{j+k}) \\ \operatorname{sign} \delta_j & \operatorname{sign} \delta_{j+1} & \dots & \operatorname{sign} \delta_{j+k} \end{vmatrix}}{\det \big(e_i(z_s)\big)_{i=0,\ s=j}^{k,\quad j+k}}.$$

Then for all $x \in [0,1] \setminus \{z_0, \dots, z_n\}$

$$
\begin{aligned}
|D^k g(x)| &\le \max_{0\le i<n-1} \max_{j\in Z_i} \frac{\sum_{p=j}^{j+k} \det\big(e_i(z_s)\big)_{i=0,\ s=j,\ s\ne p}^{k,\quad j+k}}{\det\big(e_i(z_s)\big)_{i=0,\ s=j}^{k,\quad j+k}} \\
&= \max_{0\le i<n-1} \max_{j\in Z_i} \frac{\sum_{p=j}^{j+k} \prod_{j\le l<m\le j+k,\ l,m\ne p} (z_m - z_l)}{\prod_{j\le l<m\le j+k} (z_m - z_l)} \\
&\le (k+1) \max_{0\le i<n-1} \max_{j\in Z_i} \frac{\prod_{j+1\le l<m\le j+k} (z_m - z_l)}{\prod_{j\le l<m\le j+k} (z_m - z_l)} \\
&= (k+1) \max_{0\le i<n-1} \max_{j\in Z_i} \frac{1}{\prod_{j+1\le m\le j+k} (z_m - z_j)} \\
&=: c_2(k)\, n^{-k}\,.
\end{aligned}
\tag{9}
$$

It follows from $L_n g \in \operatorname{span}\{v_0, \dots, v_n\}$ that $\sum_{i=0}^n \delta_i L_n g(z_i) = 0$. Then

$$
\begin{aligned}
1 = \sum_{i=0}^n |\delta_i| &= \sum_{i=0}^n \delta_i g(z_i) = \sum_{i=0}^n \delta_i (g(z_i) - L_n g(z_i)) \\
&\le \sum_{i=0}^n |\delta_i|\, |L_n g(z_i) - g(z_i)| \le \|L_n g - g\|_{C[0,1]}.
\end{aligned}
\tag{10}
$$

From (8) it follows that for $z \in [0,1]$

$$
|L_n g(z) - g(z)| = |L_n(g - e_0 g(z))(z)|. \tag{11}
$$

Let us denote $g_z = g - e_0 g(z)$. According to a result from [12], there exist $\varphi_{z,j} \in \operatorname{span}\{e_0, \dots, e_k\}$, $j = 1,2$, such that $\varphi_{z,j} + (-1)^j g_z \in \Delta^{0,k}(\sigma)$, $j = 1,2$, and

$$
\begin{aligned}
&\varphi_{z,j} \in \Delta^{0,k}(\sigma) \setminus \Delta^{0,0}(\sigma);\\
&\varphi_{z,j}(z) = 0 < \varphi_{z,j}(x) \text{ for all } x \in [0,1] \setminus \{z\};\\
&D^k \varphi_{z,j} = \sigma_k \|D^k g\|, \text{ where } \|D^k g\| := \sup_{x\in[0,1]\setminus\{z_0,\dots,z_n\}} |D^k g(x)|.
\end{aligned}
$$

Then $L_n(\varphi_{z,j} + (-1)^j g_z) \in \Delta^{0,0}(\sigma)$ for $j = 1,2$, and consequently,

$$
|L_n(g - e_0 g(z))(z)| \le \max\{|L_n \varphi_{z,1}(z)|, |L_n \varphi_{z,2}(z)|\}. \tag{12}
$$

From $\varphi_{z,j}(z) = 0$, $j = 1,2$, and (8) we deduce that

$$
\begin{aligned}
|L_n \varphi_{z,j}(z)| &= |L_n \varphi_{z,j}(z) - \varphi_{z,j}(z)| \\
&\le \|L_n \varphi_{z,j} - \varphi_{z,j}\|_{C[0,1]} \le \|D^k g\| \cdot \|L_n e_k - e_k\|_{C[0,1]}.
\end{aligned}
\tag{13}
$$

From (9), (10), (11), (12) and (13) we get

$$
\|L_n e_k - e_k\|_{C[0,1]} \ge \frac{1}{\|D^k g\|}.
$$

We conclude by (7) that

$$\inf_{L_n(\Delta^{0,k}(\sigma))\subset\Delta^{0,0}(\sigma)} \sup_{p\in P_k} \|p - L_n p\|_{C[0,1]} \geq \frac{1}{\|D^k g\|}.$$

Now the left-hand side inequality in Theorem 2(b) follows from (9). □

5. Conclusion

Let X be a normed linear space, and V, W be some cones in X, $W \subset V$. It is obvious that

$$\delta_n(A \cap V, V, W)_X \geq \delta_n(A \cap V)_X, \tag{14}$$

where $\delta_n(A \cap V)_X$ denotes the linear n-width of $A \cap V$ in X. Note that if $V = X$, then the relative n-width of A in X is equal to the linear n-width of A in X for all A.

As it has been shown in the preceding section,

$$\delta_n(P_m, \Delta^{0,k}(\sigma), \Delta^{0,0}(\sigma))_{C[0,1]} = \begin{cases} 0, & m = 0, \dots, k-1, \\ cn^{-k}, & m = k, \\ \infty, & m = k+1, \dots, n-1. \end{cases}$$

Thus, if a linear operator with finite rank n has the shape-preserving property relative to the cone $\Delta^{0,k}(\sigma)$, then the degree of approximation of continuous functions by this operator cannot be better than n^{-k}.

It is known that

$$\delta_n(P_m \cap \Delta^{0,k}(\sigma))_{C[0,1]} = \delta_n(P_m)_{C[0,1]} = 0, \qquad m = 0, 1, \dots, n-1.$$

Thus, if $A = P_m$, $m = k, \dots, n-1$, $V = \Delta^{0,k}(\sigma)$, $W = \Delta^{0,0}(\sigma)$, $X = C[0,1]$, we have strong inequality in (14).

If we compare the value of linear n-width $\delta_n(P_m)_{C[0,1]}$ with the value of the relative linear n-width $\delta_n(P_m, \Delta^{0,k}(\sigma), \Delta^{0,0}(\sigma))_{C[0,1]}$, we can see that the shape-preserving property relative to the cone $\Delta^{0,k}(\sigma)$ is negative in a sense that the error of approximation by such operators does not decrease with the increase of the smoothness of the approximated functions.

It is worth noting that there is a connection between Korovkin theory and the theory of shape-preserving approximation. It turns out that if we have Korovkin-type theorem for a sequence of shape-preserving linear operators [12], then the degree of approximation of continuous functions by linear shape-preserving finite-dimensional operator is low [11].

References

[1] D. Leviatan, Shape-preserving approximation by polynomials, *J. Comput. Appl. Math.* **121** (2000), 73–94.

[2] S. G. Gal, "Shape-Preserving Approximation by Real and Complex Polynomials", Springer, Berlin, 2008.

[3] V. Konovalov, Estimates of diameters of Kolmogorov type for classes of differentiable periodic functions, *Mat. Zametki* **35** (1984), no. 3, 369-380 [in Russian].

[4] V. Konovalov and D. Leviatan, Shape preserving widths of Sobolev-type classes of k-monotone functions on a finite interval, *Israel J. Math.* **133** (2003), 239–268.

[5] J. Gilewicz, V. N. Konovalov, and D. Leviatan, Widths and shape-preserving widths of Sobolev-type classes of s-monotone functions, *J. Approx. Theory* **140** (2006), no. 2, 101–126.

[6] V. Konovalov and D. Leviatan, Shape-preserving widths of weighted Sobolev-type classes of positive, monotone, and convex functions on a finite interval, *Constr. Approx* **19** (2008), 23–58.

[7] S. P. Sidorov, Estimation of relative linear width of unit ball for the class of positive operators, *Sib. Zh. Ind. Mat.* **10** (2007), no. 4, 122-128 [in Russian].

[8] P. P. Korovkin, On the order of approximation of functions by linear positive operators, *Dokl. Akad. Nauk SSSR* **114** (1957), no. 6, 1158–1161 [in Russian].

[9] V. S. Videnskii, On the exact inequality for linear positive operators of finite rank, *Dokl. Akad. Nauk Tadzhik. SSR* **12** (1980), no. 24, 715–717 [in Russian].

[10] R. K. Vasiliev and F. Guendouz, On the order of approximation of continuous functions by positive linear operators of finite rank, *J. Approx. Theory* **69** (1992), no. 2, 133–140.

[11] S. P. Sidorov, Approximation of the r-th differential operator by means of linear shape preserving operators of finite rank, *J. Approx. Theory* **124** (2003), no. 2, 232–241.

[12] F. J. Muñoz-Delgado, V. Ramírez-González, and D. Cárdenas-Morales, Qualitative Korovkin-type results on conservative approximation, *J. Approx. Theory* **94** (1998), 144–159.

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