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Product Cubature Formulae with Finite-Differences Error Bound*

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Product cubature formulae are the usual tool for approximation of double integrals over rectangular domain. The aim of this note is to show how the error of a product cubature formula with equally spaced nodes can be estimated in terms of finite differences of the integrand. The result is based on a theorem from [4].

Keywords and Phrases: Cubature formulae, blending interpolation, finite differences.

1. Introduction

Let $m, n \geq 1$ be fixed integers. By B^2 we denote the rectangular region

$$B^2 := \big\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq n+1,\ 0 \leq y \leq m+1\big\}.$$

For a measurable subset M of $\mathbb{R}$ or of $\mathbb{R}^2$, $L(M)$ stands for the space of bounded and Lebesgue integrable functions on M, equipped with the uniform norm $\|\cdot\|$.

The set of real-valued bivariate polynomials $P(x,y)$ of degree at most n with respect to x and of degree at most m with respect to y we denote by $H_{n,m}$, i.e.,

$$H_{n,m} := \Big\{\sum_{i=0}^{n}\sum_{j=0}^{m} C_{ij}x^iy^j,\ C_{ij} \in \mathbb{R}\Big\}.$$

For any function f bounded on B^2, the best uniform approximation to f in B^2 by polynomials from $H_{n,m}$ is denoted by $E_{n,m}(f;B^2)$,

$$E_{n,m}(f;B^2) := \inf\{\| f - P \| : P \in H_{n,m}\}.$$

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In [4] we proved two-sided estimates for $E_{n,m}(f;B^2)$. There we have used the modulus

$$\omega_{n,m}(f;B^2) := \sup\{|\Delta^{n,m}_{\xi,\eta} f(x,y)| : (x,y), (x+n\xi, y+m\eta) \in B^2\},$$

where

$$\Delta^{n,m}_{\xi,\eta} f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{n+m+i+j}\binom{n}{i}\binom{m}{j} f(x+i\xi, y+j\eta).$$

The moduli $\omega_{n,0}(f;B^2)$ and $\omega_{0,m}(f;B^2)$ are defined in a similar way by taking supremum of $|\Delta^{n,0}_{\xi,\eta} f(x,y)|$ and $|\Delta^{0,m}_{\xi,\eta} f(x,y)|$, respectively, where

$$\begin{aligned}\Delta^{n,0}_{\xi,\eta} f(x,y) &= \sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i} f(x+i\xi, y),\\ \Delta^{0,m}_{\xi,\eta} f(x,y) &= \sum_{j=0}^{m}(-1)^{m+j}\binom{m}{j} f(x, y+j\eta).\end{aligned}$$

Further, we defined

$$\bar{\omega}_{n,m}(f;B^2) := \max\{\omega_{n,m}(f;B^2), \omega_{n,0}(f;B^2), \omega_{0,m}(f;B^2)\}.$$

With this notation, the result obtained in [4] reads as

Theorem 1. *For every bounded on B^2 function f, the inequalities*

$$2^{-n-m}\,\bar{\omega}_{n,m}(f;B^2) \le E_{n-1,m-1}(f;B^2) \le 56\,\bar{\omega}_{n,m}(f;B^2) \tag{1}$$

hold. Moreover, the lower bound in (1) *is exact.*

Notice that in view of properties of finite differences we have

$$f \in H_{n-1,m-1} \Longrightarrow \bar{\omega}_{n,m}(f;B^2) = 0,$$

hence (1) reproduces the fact that $E_{n-1,m-1}(f;B^2) = 0$ when $f \in H_{n-1,m-1}$.

The upper bound in (1) was established by showing that

$$\left\| f(x,y) - \sum_{i=1}^{n}\sum_{j=1}^{m} \ell_{n-1,i-1}(x-1)\ell_{m-1,j-1}(y-1) f(i,j)\right\| \le 56\,\bar{\omega}_{n,m}(f;B^2). \tag{2}$$

Here,

$$\ell_{n,i}(x) = \prod_{\substack{j=0\\ j\ne i}}^{n} \frac{x-j}{i-j}, \quad i = 0,\dots,n$$

and

$$\ell_{m,j}(y) = \prod_{\substack{i=0\\ i\ne j}}^{m} \frac{y-i}{j-i}, \quad j = 0,\dots,m$$

are the Lagrange basis polynomials for interpolation at $x_i = i,\ i = 0, 1, \ldots, n$ and at $y_j = j,\ j = 0, 1, \ldots, m$, respectively. The key ingredient of the proof of (2) is an integral representation of a univariate function $f(x)$, found by Sendov [1] (see also [2], and exploited to prove the uniform (with respect to n) boundedness of the Whitney constants W_n (for the story of Whitney constants see [1] and the references therein). For the reader convenience, we quote below this integral representation. If $x \in [\mu, \mu+1)$ for some $\mu \in \{0, \ldots, n\}$, we may write $x = \mu + \sigma,\ 0 \le \sigma < 1$. For such a x, Sendov defined the operator

$$\varphi_n(x) = \varphi(\mu + \sigma) = \frac{(-1)^{n-\mu}}{\binom{n}{\mu}} \int_0^1 \Delta_t^n f(x - \mu t)dt.$$

Then the aforementioned integral representation of $f(x)$ is

$$\begin{aligned} f(x) &= f(\mu+\sigma) \\ &= \sum_{i=0}^{n} \ell'_{n,i}(x) \int_0^i f(u)du + \phi_n(f; \mu+\sigma) \\ &\quad + \int_0^\sigma \sum_{i=0}^{n} \phi_n(f; \mu+u) \ell'_{n,i}(\mu+\sigma-u)du, \qquad \mu = 0, \ldots, n. \end{aligned}$$

2. An Error Bound of a Product Cubature Formula

Product cubature formulae are the usual tool for approximation of a double integral on a rectangular region (see, e.g., [8]). For practice, it is important to know estimates for the error of the cubature formulae in terms of certain (easily accessible) characteristics of the integrand.

Here we would like to point out that (2) furnishes a cubature formula with a useful error bound. Indeed, by integrating (2) over B^2 we obtain

$$\Big| \iint\limits_{B^2} f(x,y)\, dxdy - Q[f] \Big| \le 56(n+1)(m+1)\, \bar{\omega}_{n,m}(f; B^2)\,, \tag{3}$$

where

$$Q[f] = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{i,j} f(i,j) \tag{4}$$

is a product cubature formula with coefficients $\{C_{i,j}\}_{i=1,j=1}^{n\ \ m}$ given by

$$C_{ij} = c_i\, d_j, \qquad c_i = \int\limits_0^{n+1} \ell_{n-1,i}(x-1)dx, \qquad d_j = \int\limits_0^{m+1} \ell_{m-1,j}(y-1)dy\,.$$

In view of the definition of the modulus $\bar{\omega}_{n,m}(f;B^2)$, we have

$$\bar{\omega}_{n,m}(f;B^2) \le \max\Big\{\Big\|\frac{\partial^{m+n} f}{\partial x^n \partial y^m}\Big\|, \Big\|\frac{\partial^n f}{\partial x^n}\Big\|, \Big\|\frac{\partial^m f}{\partial y^m}\Big\|\Big\}$$

provided the derivatives $\frac{\partial^{m+n} f}{\partial x^n \partial y^m}$, $\frac{\partial^n f}{\partial x^n}$ and $\frac{\partial^m f}{\partial y^m}$ exist and are continuous on B^2. Hence, for such integrands f the inequality (3) becomes

$$\Big|\iint\limits_{B^2} f(x,y)\,dxdy - Q[f]\Big| \le c\,\max\Big\{\Big\|\frac{\partial^{m+n} f}{\partial x^n \partial y^m}\Big\|, \Big\|\frac{\partial^n f}{\partial x^n}\Big\|, \Big\|\frac{\partial^m f}{\partial y^m}\Big\|\Big\},$$

with $c = 56(n+1)(m+1)$. Thus, the error of the product cubature formula $Q[f]$ is estimated by the uniform norms on B^2 of only three partial derivatives of the integrand.

Similar estimates for the error of generalized product cubature formulae (using both function evaluations and line integrals) in the Sobolev classes of functions have been obtained recently in [5], [6], by means of the Peano kernel theory. Notice that our error estimate (3) of the product cubature formula (4) applies to the wider class of functions bounded and integrable on B^2. Unfortunately, this approach seems not applicable to other product cubature formulae.

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