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Error Bounds for Scattered Data Interpolation in $\mathbb{R}^3$ by Minimum Norm Networks

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We consider the problem of interpolating scattered data in $\mathbb{R}^3$ assuming that the data are sampled from a smooth bivariate function $F = F(x, y)$. For a fixed triangulation T associated with the projections of the data onto the plane Oxy we consider Nielson's minimum norm interpolation network S defined in [6] and prove an estimate of the form $\|F-S\|_{L_2(T)} \leq C(T)\,\|F^{IV}\|_{L_2(T)}$. The dependence of the term $C(T)$ on the triangulation T is analysed.

1. Introduction

Scattered data interpolation is a fundamental problem in approximation theory and finds applications in both theory and practice in areas like geology, meteorology, cartography, medicine, computer graphics, geometric modeling etc. There exist different methods for solving the problem, excellent surveys can be found, e.g. in [2, 3, 4, 5].

The problem can be stated as follows: Given a set of points $(x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, n$, find a bivariate function F possessing continuous partial derivatives up to a given order and such that $F(x_i, y_i) = z_i$. One of the possible approaches to solving the problem is due to Nielson [6]. The method consists of the following three steps:

Step 1. Triangulation. Construct a triangulation T of the projection points $V_i = (x_i, y_i)$, $i = 1, \ldots, n$, in the plane Oxy.

Step 2. Minimum norm network. The interpolant F and its first order partial derivatives are defined on the edges of T so as to satisfy an extremal property. The minimum norm network is a cubic curve network, i.e. on every edge of T it is a cubic polynomial. Hereafter we denote the minimum norm network by S.

Step 3. Blending. The obtained network is extended to F by an appropriate *blending method*.

In [1] Andersson et al. pay special attention to the second step of the above method – the construction of the minimum norm network. By a different approach, the authors give a new proof of Nielson's result. They construct a system of simple linear curve networks called *basic curve networks* and then represent the second derivative of the minimum norm network as a linear combination of these basic curve networks.

In this paper we estimate the L_2-norm of the error of the interpolation by the minimum norm network S. We show that when the data are sampled from a smooth bivariate function F then

$$\|F - S\|_{L_2(T)} \leq C(T)\, \|F^{IV}\|_{L_2(T)}, \tag{1}$$

where the norm is taken over the edges of T. Furthermore we analyse the constant $C(T)$ and show that it depends on the geometry of T. Precisely $C(T) = 2\,\boldsymbol{\epsilon}^5/(\sqrt{3}\,\Lambda_1)$ where $\boldsymbol{\epsilon}$ is the maximal edge length of T and Λ_1 is the minimal eigenvalue of a matrix related to T.

The paper is organised as follows: In Section 2 we introduce the notation and present some related results from [1]. In Section 3 we derive error bound (1) and analyse the term $C(T)$.

2. Preliminaries

Let $n \geq 3$ be an integer and $P_i := (x_i, y_i, z_i)$, $i = 1, \ldots, n$, be different points in $\mathbb{R}^3$. We call this set of points *data*. The data are *scattered* if the projections $V_i := (x_i, y_i)$ onto the plane Oxy are different and non-colinear.

A collection of non-overlapping, non-degenerate triangles in $\mathbb{R}^2$ is a *triangulation* of the points V_i, $i = 1, \ldots, n$, if the set of the vertices of the triangles coincides with the set of the points V_i, $i = 1, \ldots, n$. Let T be a given triangulation of the points V_i, $i = 1, \ldots, n$. The union of all triangles in T is a polygonal domain which we denote by D. In general D is a collection of polygons with holes. The set of the edges of the triangles in T is denoted by E. If there is an edge between V_i and V_j in E, it will be referred to by e_{ij} or simply by e if no ambiguity arises. Similarly, $\mathbf{e_{ij}}$ will denote the vector corresponding to e_{ij} starting at V_i.

Definition 1. A *curve network* is a collection of real-valued univariate functions $\{f_e\}_{e \in E}$ defined on the edges in E.

With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i.e. for

$e = e_{ij} \in E$,

$$f_e(t) := F\big((1 - \tfrac{t}{\|e\|})x_i + \tfrac{t}{\|e\|}x_j, (1 - \tfrac{t}{\|e\|})y_i + \tfrac{t}{\|e\|}y_j\big),$$
$$\text{where } 0 \le t \le \|e\|, \text{ and } \|e\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \tag{2}$$

Furthermore, according to the context F will denote either a real-valued bivariate function or a curve network defined by (2). We introduce the following class of functions defined on D:

$$\mathcal{F} := \{F(x,y) : F(x_i, y_i) = z_i,\ i = 1, \dots, n,\ \partial F/\partial x, \partial F/\partial y \in C(D),$$
$$f_e' \in AC_{[0,\|e\|]},\ f_e'' \in L^2_{[0,\|e\|]},\ e \in E\},$$

and the corresponding class of so-called *smooth interpolation curve networks*

$$\mathcal{C}(E) := \{F_{|E} = \{f_e\}_{e \in E} : F(x,y) \in \mathcal{F},\ e \in E\}.$$

Let $F \in \mathcal{C}(E)$ and $G := \{g_e\}_{e \in E} \in \mathcal{C}(E)$. We define inner product for F and G and norm for F as

$$\langle F, G \rangle := \int_E FG = \sum_{e \in E} \int_0^{\|e\|} f_e(t) g_e(t)\, dt,$$
$$\|F\|_{L_2(T)} := \|F\|_2 = \Big(\sum_{e \in E} \int_0^{\|e\|} |f_e(t)|^2 dt\Big)^{1/2}.$$

For $i = 1, \dots, n$, let m_i denote the degree of the vertex V_i, i.e. the number of the edges in E incident to V_i. Furthermore, let $\{e_{ii_1}, \dots, e_{ii_{m_i}}\}$ be the edges incident to V_i listed in clockwise order around V_i. The first edge e_{ii_1} is chosen so that the coefficient $\lambda_{1,i}^{(s)}$ defined below is non-zero – this is always possible. A *basic curve network* B_{is} is defined on E for any pair of indices i, s such that $i = 1, \dots, n$ and $s = 1, \dots, m_i - 2$. The support of the basic curve network B_{is} consists of the three consecutive edges e_{ii_s}, $e_{ii_{s+1}}$, $e_{ii_{s+2}}$ where B_{is} is linear. More precisely, B_{is} is defined by

$$B_{is}(t) := \begin{cases} \lambda_{r,i}^{(s)}\big(1 - \frac{t}{\|e_{ii_{s+r-1}}\|}\big) & \text{on } e_{ii_{s+r-1}},\ r = 1,2,3,\ 0 \le t \le \|e_{ii_{s+r-1}}\|, \\ 0 & \text{on the other edges of } E. \end{cases}$$

The coefficients $\lambda_{r,i}^{(s)}$, $r = 1, 2, 3$, form a zero linear combination of the three unit vectors along the edges $e_{ii_{s+r-1}}$ starting at V_i. It is shown in [7] that they can be determined in the following way. Let $\alpha_{ii_{s+r-1}}$ denote the angle between the vectors $\mathbf{e_{ii_{s+r-1}}}$ and $\mathbf{e_{ii_{s+r}}}$, $r = 1, 2$. Then

$$\lambda_{1,i}^{(s)} := \sin \alpha_{ii_{s+1}}, \qquad \lambda_{2,i}^{(s)} := -\sin(\alpha_{ii_s} + \alpha_{ii_{s+1}}), \qquad \lambda_{3,i}^{(s)} := \sin \alpha_{ii_s}.$$

Hence $|\lambda_{r,i}^{(s)}| \le 1$, $r = 1, 2, 3$.

Note that basic curve networks are associated with points that have at least three edges incident to them. Thus, if a point is incident to two edges only (this might happen on the boundary of D) then no basic curve network is associated with that point. We denote by N_B the set of pairs of indices i, s for which a basic curve network is defined, i.e.,

$$N_B := \{is \,:\, m_i \geq 3,\ i = 1, \dots, n,\ s = 1, \dots, m_i - 2\}.$$

With each basic curve network B_{is} for $is \in N_B$ we associate a number d_{is} defined by

$$d_{is} = \frac{\lambda_{1,i}^{(s)}}{\|e_{ii_s}\|}(z_{i_s} - z_i) + \frac{\lambda_{2,i}^{(s)}}{\|e_{ii_{s+1}}\|}(z_{i_{s+1}} - z_i) + \frac{\lambda_{3,i}^{(s)}}{\|e_{ii_{s+2}}\|}(z_{i_{s+2}} - z_i),$$

which reflects the position of the data in the supporting set of B_{is}. The numbers d_{is}, $is \in N_B$, possess interesting properties and can be viewed as a generalization of the second-order divided differences in the univariate case.

For $F \in \mathcal{C}(E)$ we denote the networks of the first and the second derivatives of F by $F' := \{f'_e\}_{e\in E}$ and $F'' := \{f''_e\}_{e\in E}$, respectively. The following statements are proved in [1].

Lemma 1. $F \in \mathcal{C}(E) \Leftrightarrow \lambda_{1,i}^{(s)} f'_{ii_s}(0) + \lambda_{2,i}^{(s)} f'_{ii_{s+1}}(0) + \lambda_{3,i}^{(s)} f'_{ii_{s+2}}(0) = 0,\ is \in N_B.$

Theorem 1. *For the second derivative of the minimum norm network* $S := \{s_e\}_{e\in E}$ *we have* $S'' = \sum_{is\in N_B} \alpha_{is} B_{is}$. *The coefficients* α_{is} *are obtained as the unique solution of the following linear system of equations*

$$\sum_{kl\in N_B} \alpha_{kl}\langle B_{kl}, B_{is}\rangle = d_{is}, \qquad is \in N_B. \tag{3}$$

The matrix of the system is symmetric and positive definite.

The system (3) is denoted by $\mathbf{A}\alpha = \mathbf{d}$, where $\mathbf{A} := \langle B_{kl}, B_{is}\rangle_{kl,is}$ and $\mathbf{d} := (d_{is})$, $is, kl \in N_B$.

3. Error Bounds

In this section we evaluate the norms $\|F - S\|_2$, $\|F' - S'\|_2$, $\|F'' - S''\|_2$ and analyse the term $C(T)$ in (1).

Let $N := \dim \mathbf{A}$ and $\|x\|_2 := \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$, $\|\mathbf{A}\|_2 := \sup_{x\neq 0} \frac{\|\mathbf{A}x\|_2}{\|x\|_2}$ be the standard 2-norm for vectors and matrices respectively. We prove the following theorem.

Theorem 2. *Let $F \in \mathcal{C}(E)$ and $\boldsymbol{\epsilon} := \max_{e\in E} \|e\|$. Then*

$$\|F - S\|_2 \le \boldsymbol{\epsilon}\, \|F' - S'\|_2 \le \boldsymbol{\epsilon}^2 \|F'' - S''\|_2. \tag{4}$$

Furthermore, if in addition $F^{IV} := \{f_e^{IV}\}_{e\in E}$ is such that $f_e^{IV} \in L_2[0, \|e\|]$, $e \in E$, then

$$\|F'' - S''\|_2 \le \frac{2}{\sqrt{3}}\, \boldsymbol{\epsilon}^3\, \|\mathbf{A}^{-1}\|_2\, \|F^{IV}\|_2. \tag{5}$$

Proof. First, we prove (4). Since F and S are interpolating networks we have $(f_e - s_e)(0) = (f_e - s_e)(\|e\|) = 0$. Then for every $e \in E$ there exists ξ_e, $0 < \xi_e < \|e\|$ such that $(f_e' - s_e')(\xi_e) = 0$. Hence

$$f_e'(t) - s_e'(t) = \int_{\xi_e}^{t} (f_e''(\tau) - s_e''(\tau))\, d\tau. \tag{6}$$

Using (6) we obtain consecutively

$$\begin{aligned}
\|F' - S'\|_2^2 &= \sum_{e\in E} \int_0^{\|e\|} |f_e'(t) - s_e'(t)|^2\, dt \\
&= \sum_{e\in E} \int_0^{\|e\|} \Big| \int_{\xi_e}^{t} (f_e''(\tau) - s_e''(\tau))\, d\tau \Big|^2\, dt \\
&\le \sum_{e\in E} \int_0^{\|e\|} \Big((t - \xi_e) \int_{\xi_e}^{t} |f_e''(\tau) - s_e''(\tau)|^2\, d\tau \Big) dt \\
&\quad \text{(by Cauchy-Schwarz inequality)} \\
&\le \sum_{e\in E} \|e\|^2 \int_0^{\|e\|} |f_e''(\tau) - s_e''(\tau)|^2\, d\tau \\
&\le \boldsymbol{\epsilon}^2 \sum_{e\in E} \int_0^{\|e\|} |f_e''(\tau) - s_e''(\tau)|^2\, d\tau = \boldsymbol{\epsilon}^2 \|F'' - S''\|_2^2\ .
\end{aligned}$$

Therefore $\|F' - S'\|_2 \le \boldsymbol{\epsilon}\, \|F'' - S''\|_2$.

In a similar way and using that $(f_e - s_e)(t) = \int_0^t (f_e'(\tau) - s_e'(\tau)) d\tau$ we obtain

$$\begin{aligned}
\|F - S\|_2^2 &= \sum_{e\in E} \int_0^{\|e\|} |f_e(t) - s_e(t)|^2\, dt \\
&= \sum_{e\in E} \int_0^{\|e\|} \Big| \int_0^{t} (f_e'(\tau) - s_e'(\tau))\, d\tau \Big|^2\, dt \\
&\le \sum_{e\in E} \int_0^{\|e\|} \Big(t \int_0^{t} |f_e'(\tau) - s_e'(\tau)|^2\, d\tau \Big) dt \\
&\le \sum_{e\in E} \|e\|^2 \int_0^{\|e\|} |f_e'(\tau) - s_e'(\tau)|^2\, d\tau \le \boldsymbol{\epsilon}^2 \|F' - S'\|_2^2\ .
\end{aligned}$$

Therefore $\|F - S\|_2 \le \boldsymbol{\epsilon}\, \|F' - S'\|_2$ and (4) is established.

Further on for the sake of simplicity we shall also use the notation $\lambda_r := \lambda_{r,i}^{(s)}$, $e_r := e_{ii_{s+r-1}}$ and $f_r := f_{ii_{s+r-1}}$. Now we define a set of curve networks $\{G_{is}\}_{is\in N_B}$ by

$$\begin{cases} \operatorname{supp} G_{is} \equiv \operatorname{supp} B_{is}, & is \in N_B, \\ G_{is}(t)\big|_{e_r} = \lambda_r \|e_r\|^2 \dfrac{t}{\|e_r\|}\Big(1 - \dfrac{t}{\|e_r\|}\Big)\Big(\dfrac{t}{\|e_r\|} - 2\Big), & r = 1, 2, 3. \end{cases}$$

Since

$$\begin{aligned} G'_{is}(t)\big|_{e_r} &= \lambda_r \|e_r\| \Big(-\frac{3t^2}{\|e_r\|^2} + \frac{6t}{\|e_r\|} - 2\Big), \\ G''_{is}(t)\big|_{e_r} &= 6\lambda_r \Big(1 - \frac{t}{\|e_r\|}\Big), \end{aligned}$$

it is easy to see that

$$\begin{aligned} \max_{e_r} |G_{is}| &= |G_{is}((1 - 1/\sqrt{3})\|e_r\|)| \\ &= |\lambda_r|\, \|e_r\|^2 (1 - 1/\sqrt{3})(1/\sqrt{3})(1 + 1/\sqrt{3}) = \frac{2\sqrt{3}}{9}\, |\lambda_r|\, \|e_r\|^2. \end{aligned} \tag{7}$$

Let $F \in \mathcal{C}(E)$ be such that $f_e^{IV} \in L_2[0, \|e\|]$, $e \in E$. Using that

$$G_{is}\big|_{e_r}(0) = G_{is}\big|_{e_r}(\|e_r\|) = 0, \quad G'_{is}\big|_{e_r}(0) = -2\lambda_r \|e_r\|, \quad G'_{is}\big|_{e_r}(\|e_r\|) = \lambda_r \|e_r\|$$

we obtain

$$\begin{aligned} \int_E G_{is} F^{IV} &= \int_E G_{is}\, dF''' = \sum_{r=1}^{3} G_{is} F'''\big|_0^{\|e_r\|} - \int_E F'''\, dG_{is} \\ &= -\int_E G'_{is}\, dF'' = -\sum_{r=1}^{3} G'_{is} F''\big|_0^{\|e_r\|} + \int_E F'' dG'_{is} \\ &= -\sum_{r=1}^{3} \lambda_r \|e_r\| f''_r(\|e_r\|) - 2\sum_{r=1}^{3} \lambda_r \|e_r\| f''_r(0) + \int_E F'' dG'_{is}\ . \end{aligned} \tag{8}$$

For the last term in (8) we have

$$\begin{aligned} \int_E F''\, dG'_{is} &= \int_E G''_{is}\, dF' = \sum_{r=1}^{3} G''_{is} F'\big|_0^{\|e_r\|} - \int_E F' dG''_{is} \\ &= -6\sum_{r=1}^{3} \lambda_r f'_r(0) - \int_E F' G'''_{is}\ . \end{aligned} \tag{9}$$

Since $F \in \mathcal{C}(E)$, Lemma 1 implies $\lambda_1 f_1'(0) + \lambda_2 f_2'(0) + \lambda_3 f_3'(0) = 0$. Further on we have

$$-\int_E F' G_{is}''' = 6 \sum_{r=1}^{3} \frac{\lambda_r}{\|e_r\|} \int_0^{\|e_r\|} F'\, dt = 6 \sum_{r=1}^{3} \frac{\lambda_r}{\|e_r\|} \left(z_{i_{s+r-1}} - z_i \right) = 6 d_{is}. \quad (10)$$

From (8), (9) and (10) we obtain

$$\int_E G_{is} F^{IV} = -\sum_{r=1}^{3} \lambda_r \|e_r\|\, f_r''(\|e_r\|) - 2 \sum_{r=1}^{3} \lambda_r \|e_r\|\, f_r''(0) + 6 d_{is}. \quad (11)$$

The equality (11) holds true for any curve network $F \in \mathcal{C}(E)$ such that $f_e^{IV} \in L_2[0, \|e\|]$, $e \in E$. In the case $F \equiv S$ we have $S^{IV} \equiv 0$ and hence

$$0 = -\sum_{r=1}^{3} \lambda_r \|e_r\|\, s_r''(\|e_r\|) - 2 \sum_{r=1}^{3} \lambda_r \|e_r\|\, s_r''(0) + 6 d_{is}, \qquad is \in N_B. \quad (12)$$

By subtracting (12) from (11) we obtain

$$\begin{aligned} \int_E G_{is} F^{IV} = &-\sum_{r=1}^{3} \lambda_r \|e_r\| \left[f_r''(\|e_r\|) - s_r''(\|e_r\|) \right] \\ &- 2 \sum_{r=1}^{3} \lambda_r \|e_r\| \left[f_r''(0) - s_r''(0) \right], \qquad is \in N_B. \end{aligned} \quad (13)$$

Note that the linear systems (12) and (13) have the same matrix. They differ only in their left-hand sides which are $-6d_{is}$ for the system (12) and $\int_E G_{is} F^{IV}$ for the system (13), $is \in N_B$.

According to Theorem 1, $S'' = \sum_{is \in N_B} \alpha_{is} B_{is}$, where $\alpha = \{\alpha_{is}\}_{is \in N_B}$ is the unique solution of the system (3), i. e. $\mathbf{A}\alpha = \mathbf{d}$. The matrix $\mathbf{A}$ is invertible (as a positive definite matrix), and we have

$$\|\alpha\|_2 \le \|\mathbf{A}^{-1}\|_2\, \|\mathbf{d}\|_2. \quad (14)$$

Now we shall obtain an upper bound for $\|S''\|_2$ in terms of $\|\alpha\|_2$. For a fixed $is \in N_B$ we have

$$\begin{aligned} s_{ii_1}''(0) &= \alpha_{ii_1} \lambda_{1,i}^{(1)} \\ s_{ii_2}''(0) &= \alpha_{ii_1} \lambda_{2,i}^{(1)} + \alpha_{ii_2} \lambda_{1,i}^{(2)} \\ s_{ii_3}''(0) &= \alpha_{ii_1} \lambda_{3,i}^{(1)} + \alpha_{ii_2} \lambda_{2,i}^{(2)} + \alpha_{ii_3} \lambda_{1,i}^{(3)} \\ &\cdots\cdots\cdots\cdots \\ s_{ii_s}''(0) &= \alpha_{ii_{s-2}} \lambda_{3,i}^{(s-2)} + \alpha_{ii_{s-1}} \lambda_{2,i}^{(s-1)} + \alpha_{ii_s} \lambda_{1,i}^{(s)} \\ &\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}$$

Since $|\lambda_{r,i}^{(s)}| \leq 1$ it follows that

$$\begin{aligned}
&|s''_{ii_1}(0)| \leq |\alpha_{ii_1}| \\
&|s''_{ii_2}(0)| \leq |\alpha_{ii_1}| + |\alpha_{ii_2}| \\
&|s''_{ii_3}(0)| \leq |\alpha_{ii_1}| + |\alpha_{ii_2}| + |\alpha_{ii_3}| \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&|s''_{ii_s}(0)| \leq |\alpha_{ii_{s-2}}| + |\alpha_{ii_{s-1}}| + |\alpha_{ii_s}| \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots .
\end{aligned}$$

In the above estimate for $s''_{ii_s}(0)$ at most three α's participate and every α is present in exactly three inequalities. We have from Jensen inequality

$$|s_{ii_s}(0)|^2 \leq 3\big(|\alpha_{ii_{s-2}}|^2 + |\alpha_{ii_{s-1}}|^2 + |\alpha_{ii_s}|^2\big).$$

Therefore

$$\sum_{e\in E} \big[|s''_e(0)|^2 + |s''_e(\|e\|)|^2\big] \leq 9 \sum_{is\in N_B} |\alpha_{is}|^2$$

and so

$$\Big(\sum_{e\in E} \big[|s''_e(0)|^2 + |s''_e(\|e\|)|^2\big]\Big)^{1/2} \leq 3\Big(\sum_{is\in N_B} |\alpha_{is}|^2\Big)^{1/2} = 3\|\alpha\|_2. \tag{15}$$

On the other hand, $s_e(t)$ is a cubic polynomial, whence

$$s''_e(t) = \Big(1 - \frac{t}{\|e\|}\Big) s''_e(0) + \frac{t}{\|e\|}\, s''_e(\|e\|)$$

and it follows that

$$|s''_e(t)|^2 \leq \big[|s''_e(0)| + |s''_e(\|e\|)|\big]^2 \leq 2\big[|s''_e(0)|^2 + |s''_e(\|e\|)|^2\big]. \tag{16}$$

From (16) and (15) we obtain

$$\begin{aligned}
\|S''\|_2 &= \Big(\sum_{e\in E} \int_0^{\|e\|} |s''_e(t)|^2\, dt\Big)^{1/2} \leq \Big(\sum_{e\in E} 2\|e\|\, \big[|s''_e(0)|^2 + |s''_e(e)|^2\big]\Big)^{1/2} \\
&\leq \sqrt{2}\, \boldsymbol{\epsilon}^{1/2} \Big(\sum_{e\in E} \big[|s''_e(0)|^2 + |s''_e(e)|^2\big]\Big)^{1/2} \\
&\leq 3\sqrt{2}\, \boldsymbol{\epsilon}^{1/2} \|\alpha\|_2. \qquad (17)
\end{aligned}$$

Now from (17) and (14) we obtain

$$\|S''\|_2 \leq 3\sqrt{2}\, \boldsymbol{\epsilon}^{1/2}\, \|\mathbf{A}^{-1}\|_2\, \|\mathbf{d}\|_2. \tag{18}$$

Since the linear systems (12) and (13) have the same matrix, we deduce from (18) that

$$\|F'' - S''\|_2 \leq \frac{1}{\sqrt{2}}\, \boldsymbol{\epsilon}^{1/2} \|\mathbf{A}^{-1}\|_2 \Big(\sum_{is\in N_B} \Big|\int_E G_{is} F^{IV}\Big|^2\Big)^{1/2}. \tag{19}$$

Now we shall estimate the right hand side of the inequality (19). Let $g_r := G_{is}\big|_{e_{ii_{s+r-1}}}$, $r = 1, 2, 3$. We obtain consecutively

$$\Big|\int_{\operatorname{supp} B_{is}} G_{is} F^{IV}\Big|^2 = \Big|\sum_{r=1}^{3}\int_{e_r} g_r f_r^{IV}\Big|^2 \le 3\sum_{r=1}^{3}\Big|\int_{e_r} g_r f_r^{IV}\Big|^2. \tag{20}$$

By applying Cauchy-Schwarz inequality to the terms in (20) we obtain

$$\Big|\int_{e_r} g_r f_r^{IV}\Big|^2 \le \int_{e_r} |g_r|^2 \cdot \int_{e_r} |f_r^{IV}|^2, \qquad r = 1, 2, 3. \tag{21}$$

From (21), (7) and $|\lambda_r| \le 1$, $r = 1, 2, 3$, it follows that

$$\Big|\int_{e_r} g_r f_r^{IV}\Big|^2 \le \|e_r\|\Big(\frac{2\sqrt{3}}{9}\,\|e_r\|^2\Big)^2 \int_{e_r} |f_r^{IV}|^2 \le \frac{4}{27}\,\boldsymbol{\epsilon}^5 \int_{e_r} |f_r^{IV}|^2. \tag{22}$$

From (22) and (20) we have

$$\Big(\sum_{is\in N_B}\Big|\int_{\operatorname{supp} B_{is}} G_{is}\, F^{IV}\Big|^2\Big)^{1/2}$$
$$\le \Big(3\cdot\frac{4}{27}\,\boldsymbol{\epsilon}^5\cdot 3\cdot 2\sum_{is\in N_B}\int |f_{is}^{IV}|^2\Big)^{1/2} = \frac{2\sqrt{2}}{\sqrt{3}}\,\boldsymbol{\epsilon}^{5/2}\,\|F^{IV}\|_2. \tag{23}$$

From (23) and (19) we obtain

$$\|F'' - S''\|_2 \le \frac{2}{\sqrt{3}}\,\boldsymbol{\epsilon}^3\,\|\mathbf{A}^{-1}\|_2\,\|F^{\mathrm{IV}}\|_2.$$

Thus we established (5) and Theorem 2 is proved. □

Corollary 1. *If $F \in \mathcal{C}(E)$ is such that $f_e^{IV} \in L_2[0, \|e\|]$, $e \in E$ then*

$$\|F - S\|_2 \le \frac{2}{\sqrt{3}}\,\boldsymbol{\epsilon}^5\,\|\mathbf{A}^{-1}\|_2\,\|F^{IV}\|_2.$$

At the end of this section let us say a few words about the evaluation of $\|\mathbf{A}^{-1}\|_2$. By $\mathbf{A}^T$ we denote the transpose matrix of $\mathbf{A}$, then it is known that

$$\|\mathbf{A}\|_2 := \sup_{x\neq 0}\frac{\|\mathbf{A}x\|_2}{\|x\|_2} = \max_{1\le i\le N}\{\sqrt{\Lambda_i} \,:\, \Lambda_i \text{ is an eigenvalue of } \mathbf{A}^T\mathbf{A}\}.$$

In our case $\mathbf{A}$ is symmetric and positive definite, hence its eigenvalues $\Lambda_1,\ldots,\Lambda_N$, are real and positive. Assuming that $0 < \Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_N$, then $\|\mathbf{A}\|_2 = \Lambda_N$, and since the eigenvalues of the inverse matrix $\mathbf{A}^{-1}$ are $1/\Lambda_N, \ldots, 1/\Lambda_1$ then $\|\mathbf{A}^{-1}\|_2 = 1/\Lambda_1$.

Corollary 2. *Let Λ_1 be the minimal eigenvalue of matrix* $\mathbf{A}$. *Then*

$$\|F - S\|_2 \leq \frac{2\,\boldsymbol{\epsilon}^5}{\sqrt{3}\,\Lambda_1}\,\|F^{IV}\|_2. \tag{24}$$

The estimate (24) can be used to evaluate the error of interpolation by minimum norm networks a priori for a specific triangulation by applying some numerical method for finding the minimal eigenvalue of $\mathbf{A}$.

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