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Asymptotic Expansions for a Durrmeyer Variant of Baskakov and Meyer-König and Zeller Operators and Quasi-Interpolants

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We consider a Durrmeyer variant of Baskakov and Meyer-König and Zeller operators. The main results include the local rate of convergence which is based on the eigenstructure of the operators. We present a complete asymptotic expansion for the Baskakov-Durrmeyer operators and their quasi-interpolants in terms of certain differential operators. Furthermore, we state analogues results for the Meyer-König and Zeller operators and introduce their quasi-interpolants.

Keywords and Phrases: Baskakov-Durrmeyer operators, Meyer-König and Zeller operators, quasi-interpolants, complete asymptotic expansion.

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1. Introduction

The Durrmeyer modification of the Baskakov-operators, introduced by Sahai and Prasad in [10] and considered independently by Heilmann in [6] and [7] in a more genereal setting, is defined as follows.

Definition 1. Let $(1+\cdot)^{-(n-1)}f(\cdot)\in L_\infty[0,\infty)$, $n\in\mathbb{N}$, then *the Baskakov-Durrmeyer operators* are given by

$$(B_{n+1}f)(\sigma) := \sum_{k=0}^{\infty} b_{n+1,k}(\sigma)\, n\int_0^\infty b_{n+1,k}(\tau)f(\tau)\,d\tau, \qquad \sigma\in[0,\infty),$$

where

$$b_{n+1,k}(\sigma) := \binom{n+k}{k}\sigma^k(1+\sigma)^{-(n+1+k)}.$$

The operators result from the classical Baskakov-operators, introduced in [4], by replacing the discrete point-evaluations $f\left(\frac{k}{n+1}\right)$ by the weighted integral $\int_0^\infty b_{n+1,k}(\tau)f(\tau)\,d\tau$ in order to approximate Lebesgue integrable functions on $[0,\infty)$. In the same way we define the "natural" Durrmeyer modification of the Meyer-König and Zeller operators by replacing the discrete values $f\left(\frac{k}{n}\right)$ in the classical case by a weighted integral. Then the "natural" Meyer-König and Zeller-Durrmeyer (MKZD) operators are defined as follows.

Definition 2. Let $(1-\cdot\,)^{n-1}f(\,\cdot\,) \in L_\infty[0,1)$, $n \in \mathbb{N}$, then *the "natural" Meyer-König and Zeller-Durrmeyer operators* are given by

$$(M_nf)(x) := \sum_{k=0}^{\infty} m_{n,k}(x)\, n \int_0^1 m_{n,k}(t)(1-t)^{-2} f(t)\,dt, \qquad x \in [0,1), \tag{1}$$

where

$$m_{n,k}(x) := \binom{n+k}{k} x^k (1-x)^{n+1}.$$

Remark 1. By using the same transformation as in [12] the operators B_{n+1} and M_n are related in the following way. Let $\sigma : [0,1) \to [0,\infty)$, $\sigma(x) := \frac{x}{1-x}$, $f(\,\cdot\,) = \tilde{f}(\sigma(\,\cdot\,))$. Then

$$(M_nf)(x) = (B_{n+1}\tilde{f})(\sigma(x)).$$

There exist several other Durrmeyer modifications of the Meyer-König and Zeller operators, where the basis functions are shifted in some way. The reader is referred to [2] or [9], where these modifications are listed. In this paper we consider only the "natural" Durrmeyer variation, because of its simple connection to the Baskakov-Durrmeyer operators. The representation (1) with weight-function $(1-t)^{-2}$ is due to Heilmann [9].

The following differential operators of order $2r$ will play an important role for our asymptotic expansion. Let $r \in \mathbb{N}$ and $\sigma \in [0,\infty)$, then we define

$$\widetilde{D}_B^{2r} := D(\sigma^r(1+\sigma)^r D) \quad \text{and} \quad \widetilde{D}_M^{2r} := U^r \frac{x^r}{(1-x)^{2r}}\, U^r, \tag{2}$$

with

$$U := \frac{1}{\sigma'(x)} D = (1-x)^2 D, \quad U^r = U^{r-1} \circ U, \qquad x \in [0,1).$$

If I denotes the identity operator, then we set $\widetilde{D}_B^0 := I$ and $\widetilde{D}_M^0 := I$.

The differential operators $\widetilde{D}_B^{2r}$ correspond to the Baskakov-Durrmeyer operators. We get the differential operators $\widetilde{D}_M^{2r}$, corresponding to the MKZD operators, by using the same transformation as in Remark 1.

2. Eigenfunctions and Eigenvalues

In [8] it was shown, that the eigenfunctions of the Baskakov-Durrmeyer and the MKZD operators can be represented by a Rodriguez-type formula.

Lemma 1. *Let $m \in \mathbb{N}_0$, $m \le n-1$,*

$$\widetilde{g_m}(\sigma) := D^m(\sigma^m(1+\sigma)^m) \quad \text{and} \quad g_m(x) := U^m\Big(\frac{x^m}{(1-x)^{2m}}\Big).$$

Then

$$(B_{n+1}\widetilde{g_m})(\sigma) = \lambda_{n,m}\,\widetilde{g_m}(\sigma) \quad \text{and} \quad (M_n g_m)(x) = \lambda_{n,m}\,g_m(x),$$

where

$$\lambda_{n,m} := \frac{(n-m-1)!\,(n+m)!}{(n-1)!\,n!}.$$

Our approximation operators have the same eigenvalues and the eigenfunctions are connected by the transformation of Remark 1. In the Baskakov case the eigenfunctions are polynomials of order m, while in the MKZD case the eigenfunctions are of the type $\frac{x^j}{(1-x)^j}$, $j = 0, \ldots, m$. As polynomials are easier to handle, we will first prove our results for the Baskakov-Durrmeyer operators and then use the transformation of Remark 1.

Now we study the eigenfunctions of the differential operators (2). We will see that our differential operators have the same eigenfunctions as the Baskakov-Durrmeyer and the MKZD operators, respectively.

Lemma 2. *For $m, r \in \mathbb{N}_0$ we have*

$$\widetilde{D}_B^{2r}\,\widetilde{g_m}(\sigma) = \gamma_{r,m}\,\widetilde{g_m}(\sigma) \quad \text{and} \quad \widetilde{D}_M^{2r} g_m(x) = \gamma_{r,m}\,g_m(x),$$

with

$$\gamma_{r,m} := \begin{cases} \frac{(m+r)!}{(m-r)!}, & \text{for } r \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For $r > m$ we have $\widetilde{D}^{2r}\widetilde{g_m}(\sigma) = 0$. So we have to prove the proposition only for $r \le m$. We derive using the Leibniz-formula

$$\begin{aligned}
&\sigma^r(1+\sigma)^r D^{r+m}\,(\sigma^m(1+\sigma)^m) \\
&\qquad = \sigma^r(1+\sigma)^r \sum_{\nu=r}^{m} \binom{r+m}{\nu} \frac{m!}{(m-\nu)!}\,\sigma^{m-\nu}\,\frac{m!}{(\nu-r)!}\,(1+\sigma)^{\nu-r} \\
&\qquad = \sum_{\nu=0}^{m-r} \binom{r+m}{\nu+r} \frac{m!\,m!}{(m-r-\nu)!\,\nu!}\sigma^{m-\nu}(1+\sigma)^{\nu+r} \\
&\qquad = \frac{(m+r)!}{(m-r)!} \sum_{\nu=0}^{m-r} \binom{m-r}{\nu} \frac{m!}{(m-\nu)!}\,\sigma^{m-\nu}\,\frac{m!}{(\nu+r)!}\,(1+\sigma)^{\nu+r} \\
&\qquad = \frac{(m+r)!}{(m-r)!}\,\frac{d^{m-r}}{d\sigma^{m-r}}\,\sigma^m(1+\sigma)^m.
\end{aligned}$$

By differentiating both sides r times we get our Lemma. □

The following Lemma gives a relation between the eigenvalues of the Baskakov-Durrmeyer and the MKZD operators, respectively, and the differential operators (2). We can write the eigenvalues of the approximation operators as a weighted sum of the eigenvalues of our differential operators, where the coefficients are of order n^{-k}. We will use this relation in the next section to derive an asymptotic expansion for the approximation operators $B_{n+1}f$ and M_nf.

Lemma 3. *For $m \le n-1$ there holds*

$$\lambda_{n,m} = \sum_{k=0}^{m} \frac{(n-1-k)!}{k!\,(n-1)!}\,\gamma_{k,m} = 1 + \sum_{k=1}^{m} \frac{(n-1-k)!}{k!\,(n-1)!}\,\gamma_{k,m}.$$

Proof. Using the Pochhammer symbol $(x)_k := x(x+1)\cdots(x+k-1)$ we have:

$$\begin{aligned}
\sum_{k=0}^{m} \frac{(n-1-k)!}{k!\,(n-1)!}\,\gamma_{k,m} &= \sum_{k=0}^{m} \frac{(n-1-k)!}{k!\,(n-1)!}\,\frac{(m+k)!}{(m-k)!} \\
&= \sum_{k=0}^{m} \frac{(n-1-k)!}{k!\,(n-1)!}\,\frac{(m+k)!}{m!}\,\frac{m!}{(m-k)!} \\
&= \sum_{k=0}^{m} \frac{(m+1)_k\,(-m)_k}{(-(n-1))_k\,k!} \\
&\overset{(\star)}{=} \frac{(-(n-1)-(m+1))_m}{(-(n-1))_m} \\
&= \frac{(-(n+m))_m}{(-(n-1))_m} \\
&= \frac{(-1)^m (n+m)!\,(n-1-m)!}{(-1)^m (n+m-m)!\,(n-1)!} \\
&= \frac{(n+m)!}{n!}\,\frac{(n-m-1)!}{(n-1)!} = \lambda_{n,m}.
\end{aligned}$$

In $(\star)$ we applied the Chu-Vandermonde convolution formula, see e.g. [3, Corollary 2.2.3]. □

As the eigenfunctions of the Baskakov-Durrmeyer operators are polynomials, we can expand this result to arbitrary polynomials and therefore we get an asymptotic expansion for polynomials in this case. Denote by $\mathbb{P}_q$ the set of all polynomials of degree at most q.

Corollary 1. *For $p \in \mathbb{P}_q$, $q \le n-1$, there holds*

$$B_{n+1}p = p + \sum_{k=1}^{q} \frac{(n-1-k)!}{k!\,(n-1)!}\,\widetilde{D}^{2k}p. \tag{3}$$

Proof. By Lemma 1, Lemma 3 and Lemma 2 we have

$$B_{n+1}\widetilde{g}_m = \widetilde{g}_m + \sum_{k=1}^{q} \frac{(n-1-k)!}{k!\,(n-1)!}\,\widetilde{D}_B^{2k}\widetilde{g}_m.$$

Since $p \in \mathbb{P}_q$ can be represented as a sum of $\widetilde{g_m}$, $0 \le m \le q$, (3) holds. □

This result will be expanded for arbitrary, sufficiently smooth functions in the next section.

3. Asymptotic Expansion of Baskakov-Durrmeyer and MKZD Operators

In this section we will present our main result, asymptotic expansions of the Baskakov-Durrmeyer and MKZD operators. Our basic tool is a theorem of Sikkema proved in [11] for arbitrary positive linear operators. First we need to define an appropriate class of functions.

For an interval I we denote by $H^{(q)}(\xi)$, $\xi \in I$, the set of all functions $f : I \to \mathbb{R}$ possessing the following properties:

- f is q times differentiable at ξ;
- f is bounded on every finite interval $I' \subset I$;
- $f(x) = \mathcal{O}(x^q)$ as $x \to \infty$.

Theorem 1. *For $q \in \mathbb{N}$, let $\{L_n\}$, $n \in \mathbb{N}$, be a sequence of linear positive operators $L_n : H^{(2q)}(x) \to C[a,b]$, such that*

$$\big(L_n(t-x)^r\big)(x) = \mathcal{O}\big(n^{\lfloor -(r+1)/2 \rfloor}\big) \qquad (n \to \infty), \quad r = 0,1,\ldots,2q+2.$$

Then we have

$$(L_n f)(x) = \sum_{\nu=0}^{2q} \frac{f^{(\nu)}(x)}{\nu!}\,(L_n(t-x)^\nu)(x) + o(n^{-q}) \qquad (n \to \infty).$$

(Here $\lfloor u \rfloor$ denotes the integral part of u.)

Therefore we have to prove that the assumptions of Theorem 1 are fulfilled for our operators and we have to find an appropriate representation of our operators applied to the moments $(\sigma - a)^s$ in terms of the differential operators (2).

Lemma 4. *Let $a, \sigma \in [0,\infty)$ and $2r < s$. Then*

$$\big(\widetilde{D}_B^{2r}(\sigma - a)^s\big)(a) = 0. \tag{4}$$

Proof. We have

$$\widetilde{D}_B^{2r}(\sigma-a)^s = \frac{d^r}{d\sigma^r}\,\sigma^r(1+\sigma)^r\,\frac{d^r}{d\sigma^r}(\sigma-a)^s = (\sigma-a)^{s-2r}p(\sigma),$$

where p is a polynomial (which also depends on s) of degree $2r$. Since $s-2r>0$ (4) holds. □

Lemma 5. *Let $\sigma\in[0,\infty)$ be fixed and $s\in\mathbb{N}_0$. Then*

$$\big(B_{n+1}(\tau-\sigma)^s\big)(\sigma) = \mathcal{O}\big(n^{-\lfloor (s+1)/2\rfloor}\big) \qquad (n\to\infty). \tag{5}$$

Proof. Since B_{n+1} reproduces constant functions, (5) holds for $s=0$. Now let $s>0$. By Corollary 1 and Lemma 4 we have

$$\big(B_{n+1}(\tau-\sigma)^s\big)(\sigma) = \sum_{k=\lfloor (s+1)/2\rfloor}^{s}\frac{(n-1-k)!}{k!\,(n-1)!}\,\big(\widetilde{D}_B^{2k}(t-\sigma)^s\big)(\sigma).$$

Since $\frac{(n-1-k)!}{(n-1)!}=\mathcal{O}(n^{-k})$ as $n\to\infty$, the statement follows. □

Now we are able to prove the main result, which gives a complete asymptotic expansion for the Baskakov-Durrmeyer operators.

Theorem 2. *Let $\sigma\in[0,\infty)$ be fixed, $q\in\mathbb{N}$ and $f\in H^{(2q)}(\sigma)$. Then*

$$(B_{n+1}f)(\sigma) = f(\sigma) + \sum_{k=1}^{q}\frac{(n-1-k)!}{k!\,(n-1)!}\,\big(\widetilde{D}_B^{2k}f\big)(\sigma) + o(n^{-q}) \quad (n\to\infty). \tag{6}$$

Proof. Let

$$T_{2q,\sigma}(\tau) := \sum_{\nu=0}^{2q}\frac{f^{(\nu)}(\sigma)}{\nu!}\,(\tau-\sigma)^\nu$$

be the Taylor polynomial of degree $2q$ of the function f at point σ. By Lemma 4 and Theorem 1 we have

$$(B_{n+1}f)(\sigma) = (B_{n+1}T_{2q,\sigma})(\sigma) + o(n^{-q}), \qquad (n\to\infty).$$

By Corollary 1 we have

$$(B_{n+1}T_{2q,\sigma})(\sigma) = T_{2q,\sigma}(\sigma) + \sum_{k=1}^{2q}\frac{(n-1-k)!}{k!\,(n-1)!}\,\widetilde{D}_B^{2k}T_{2q,\sigma}(\sigma).$$

For $0 \le k \le q$ we get

$$\begin{aligned}\big(\widetilde{D}_B^{2k} T_{2q,\sigma}\big)(\sigma) &= \sum_{l=0}^{2q} \frac{f^{(l)}(\sigma)}{l!} \big(\widetilde{D}_B^{2k} (\tau-\sigma)^l\big)(\sigma) \\ &= \sum_{l=0}^{2q} \frac{f^{(l)}(\sigma)}{l!} \big(D^k \tau^k (1+\tau)^k D^k (\tau-\sigma)^l\big)(\sigma) \\ &= \sum_{l=k}^{2q} \frac{f^{(l)}(\sigma)}{(l-k)!} \sum_{\nu=0}^{k} \binom{k}{\nu} \big[\big(\tau^k (1+\tau)^k\big)^{(k-\nu)} \big((\tau-\sigma)^{l-k}\big)^{(\nu)}\big](\sigma) \\ &= \sum_{l=k}^{2k} f^{(l)}(\sigma) \binom{k}{l-k} \big[\big(\tau^k (1+\tau)^k\big)^{(2k-l)}\big](\sigma) \\ &= \sum_{l=0}^{k} \binom{k}{l} \big(\sigma^k (1+\sigma)^k\big)^{(k-l)} f^{(l+k)}(\sigma) \\ &= D^k \sigma^k (1+\sigma)^k D^k f(\sigma) \\ &= (\widetilde{D}^{2k} f)(\sigma).\end{aligned}$$

Since

$$\frac{(n-1-k)!}{k!\,(n-1)!}\, \widetilde{D}_B^{2k} T_{2q,\sigma}(\sigma) = o(n^{-q}) \qquad (n \to \infty) \text{ for } q+1 \le k \le 2q,$$

we obtain (6). □

For the MKZD operators we prove a similar result.

Theorem 3. *Let $x \in [0,1]$, $q \in \mathbb{N}$ and $f \in H^{2q}(x)$. Then*

$$(M_n f)(x) = f(x) + \sum_{k=1}^{q} \frac{(n-1-k)!}{k!\,(n-1)!} \big(\widetilde{D}_M^{2k} f\big)(x) + o(n^{-q}).$$

In the case $q = 1$ we have a Voronovskaja-type result:

$$\lim_{n\to\infty} n[(M_n f)(x) - f(x)] = \widetilde{D}_M^2 f(x) = (1-x)^2 f'(x) + x(1-x)^2 f''(x).$$

This result coincides with that of Abel, Gupta and Ivan in [2], where they developed an asymptotic expansion for the MKZD operators of the following form.

Let $x \in [0,1]$, $q \in \mathbb{N}$ and $f \in H^{2q}(x)$, then

$$(M_n f)(x) = f(x) + \sum_{k=1}^{q} c_k(f,x) \frac{n!}{(n+k)!} + o(n^{-q}),$$

where the coefficients c_k, $k = 1, 2, \dots$, are given by

$$c_k(f,x) = \sum_{s=0}^{2k} \frac{f^{(s)}(x)}{s!} (1-x)^s a_{s,k}(x),$$

with

$$a_{s,k}(x) = (k-1)! \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{k-j} x^j \sum_{r=0}^{s} (-1)^r \binom{s}{r} \binom{r+j-1}{j} \binom{r+j-1}{k-1} r.$$

Our expansion seems to be much more simple and elegant.

4. Quasi-Interpolants

As the approximation order of the Baskakov-Durrmeyer and the MKZD operators is $\mathcal{O}\left(\frac{1}{n}\right)$, we want to construct an operator of higher approximation order. A manner to do this is by quasi-interpolation. In [1] Abel introduced quasi-interpolants for the Baskakov-Durrmeyer operators as follows.

Definition 3. For $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $(1+\cdot)^{-(n-1)} f(\cdot) \in L_\infty[0,\infty)$, *the quasi-interpolants of Baskakov-Durrmeyer operators* are defined by

$$B_{n+1}^{(r)} f := \sum_{k=0}^{r} \frac{n!}{(n+k)!} \frac{(-1)^k}{k!} \widetilde{D}_B^{2k} (B_{n+1} f). \tag{7}$$

In [5] Berdysheva investigated these quasi-interpolants. She showed that they are uniformly bounded and gave some direct results.

An important property of the quasi-interpolants (7) is that we can represent them as linear combinations of the Baskakov-Durrmeyer operators.

Theorem 4. *Let* $n \in \mathbb{N}$, *then*

$$B_{n+1}^{(r)} f = \sum_{l=0}^{r} (-1)^{r-l} \frac{(n+l-1)!}{l!\,(r-l)!\,(n+l-r-1)!} B_{n+1+l} f. \tag{8}$$

Linear combinations of the Baskakov-Durrmeyer operators were studied by Heilmann in [6] and [7]. She proved that the linear combinations (8) preserve polynomials of degree r.

With the transformation in Remark 1 we are able to define quasi-interpolants of the MKZD operators.

Definition 4. For $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $(1-\cdot)^{(n-1)} f(\cdot) \in L_\infty[0,1)$ *the quasi-interpolants of the MKZD operators* are defined by

$$M_n^{(r)} f := \sum_{k=0}^{r} \frac{n!}{(n+k)!} \frac{(-1)^k}{k!} \widetilde{D}_M^{2k} (M_n f). \tag{9}$$

The quasi-interpolants $M_n^{(r)}$ do not preserve polynomials of degree r, but they preserve functions of the type $\frac{x^j}{(1-x)^j}$, $j = 0, \dots, r$. Therefore we hope to get a higher approximation order.

With the representation as linear combinations of the Baskakov-Durrmeyer operators (8) we get the following identity for the eigenvalues of the quasi-interpolants (7).

Lemma 6. *Let $m, r \in \mathbb{N}_0$ and $m \le n-1$, then*

$$\big(B_{n+1}^{(r)}\widetilde{g_m}\big)(\sigma) = \lambda_{n,m}^{(r)}\,\widetilde{g_m}(\sigma) \quad \text{and} \quad \big(M_n^{(r)} g_m\big)(x) = \lambda_{n,m}^{(r)}\, g_m(x),$$

where

$$\lambda_{n,m}^{(r)} := \sum_{l=0}^{r} (-1)^{r-l} \frac{(n+l-1)!}{l!\,(r-l)!\,(n+l-r-1)!}\,\lambda_{n+l,m}.$$

The analogue of Lemma 3 is the following identity for the eigenvalues $\lambda_{n,m}^{(r)}$ of $B_{n+1}^{(r)}$ and $M_n^{(r)}$.

Lemma 7. *For $m, r \in \mathbb{N}_0$, $m \le n-1$ there holds*

$$\lambda_{n,m}^{(r)} = \begin{cases} 1, & \text{for } m \le r, \\ 1 + \sum\limits_{k=r+1}^{m} (-1)^r \gamma_{k,m} \frac{(n-1-k)!}{(n-1)!\,k!} \binom{k-1}{r}, & \text{for } m > r. \end{cases}$$

Proof. By Lemma 6 we have

$$\lambda_{n,m}^{(r)} = \sum_{l=0}^{r} (-1)^{r-l} \frac{(n+l-1)!}{l!\,(r-l)!\,(n+l-r-1)!}\,\lambda_{n+l,m}.$$

Now from Lemma 3 it follows that

$$\begin{aligned}
\lambda_{n,m}^{(r)} &= \sum_{l=0}^{r} (-1)^{r-l} \frac{(n+l-1)!}{l!\,(r-l)!\,(n+l-r-1)!} \sum_{k=0}^{m} \frac{(n+l-1-k)!}{k!\,(n+l-1)!}\,\gamma_{k,m} \\
&= \sum_{k=0}^{m} \frac{\gamma_{k,m}}{k!} \sum_{l=0}^{r} (-1)^{r-l} \frac{(n+l-1-k)!}{(n+l-r-1)!} \frac{1}{l!\,(r-l)!} \\
&= \sum_{k=0}^{m} \frac{\gamma_{k,m}}{k!} (-1)^r \frac{1}{r!} \underbrace{\sum_{l=0}^{r} \frac{(-r)_l}{l!} \frac{(n+l-1-k)!}{(n+l-r-1)!}}_{=:S}.
\end{aligned}$$

For the quantity S we obtain:

$$\begin{aligned} S &= \frac{(n-1-k)!}{(n-r-1)!} \sum_{l=0}^{r} \frac{(-r)_l}{l!} \frac{(n-k)_l}{(n-r)_l} \\ &\overset{(\star)}{=} \frac{(n-1-k)!}{(n-r-1)!} \frac{(k-r)_r}{(n-r)_r} \\ &= \frac{(n-1-k)!}{(n-r-1)!} \frac{(k-r)_r (n-r-1)!}{(n-1)!} \\ &= \frac{(n-1-k)!}{(n-1)!} (k-r)_r, \end{aligned}$$

In $(\star)$ we apply again the Chu-Vandermonde convolution formula, see e.g. [3, Corollary 2.2.3], and therefore

$$\lambda_{n,m}^{(r)} = \sum_{k=0}^{m} \frac{\gamma_{k,m}}{k!} \frac{(n-1-k)!}{(n-1)!} \frac{(k-r)_r}{r!}.$$

Since $(-1)^r(-r)_r = r!$ and $(k-r)_r = 0$ for $1 \le k \le r$ we get

$$\lambda_n^{(r)} = 1 + \sum_{k=r+1}^{m} (-1)^r \gamma_{k,m} \frac{(n-1-k)!}{(n-1)!\,k!} \binom{k-1}{r}.$$

□

Hence we obtain an asymptotic expansion for $B_{n+1}^{(r)}$ for polynomials.

Corollary 2. *For $p \in \mathbb{P}_q$ and $q \le n-1$ there holds*

$$B_{n+1}^{(r)} p = \begin{cases} p, & \text{for } q \le r, \\ p + \sum\limits_{k=r+1}^{q} \frac{(n-1-k)!}{k!\,(n-1)!} \widetilde{D}_B^{2k} p, & \text{for } q > r. \end{cases}$$

We omit the proof. Compare with Corollary 1.

Next we want to expand this result for arbitrary, sufficiently smooth functions.

Theorem 5. *Let $\sigma \in [0,\infty)$, $q, r \in \mathbb{N}$ and $f \in H^{(2q+2r)}(\sigma)$, then*

$$B_{n+1}^{(r)} f = f + \sum_{k=r+1}^{q} \frac{(n-1-k)!}{k!\,(n-1)!} \widetilde{D}_B^{2k} f + o(n^{-q}) \qquad (n \to \infty). \tag{10}$$

Proof. Let

$$T_{2(q+r),\sigma}(\tau) := \sum_{\nu=0}^{2(q+r)} \frac{f^{(\nu)}(\sigma)}{\nu!} (\tau - \sigma)^\nu$$

be the Taylor polynomial of order $2(q+r)$ of the function f at point σ. By Theorem 4 we can write the quasi-interpolants as linear combinations of the Baskakov-Durrmeyer operators:

$$\big(B_{n+1}^{(r)} f\big)(\sigma) = \sum_{l=0}^{r} (-1)^{r-l} \frac{(n+l-1)!}{l!\,(r-l)!\,(n+l-r-1)!} \,(B_{n+1+l} f)(\sigma).$$

Since

$$(B_{n+1+l} f)(\sigma) = (B_{n+1+l} T_{2(q+r),\sigma})(\sigma) + o(n^{-(q+r)}) \qquad (n \to \infty),$$

by Theorem 1 and $\frac{(n+l-1)!}{l!\,(r-l)!\,(n+l-r-1)!} = \mathcal{O}(n^r)$, we have

$$\big(B_{n+1}^{(r)} f\big)(\sigma) = \big(B_{n+1}^{(r)} T_{2(q+r),\sigma}\big)(\sigma) + o(n^{-q}) \qquad (n \to \infty).$$

By Corollary 2 we get

$$(B_{n+1}^{(r)} T_{2(q+r),\sigma})(\sigma) = T_{2(q+r),\sigma}(\sigma) + \sum_{k=r+1}^{2(q+r)} \frac{(n-1-k)!}{k!\,(n-1)!} \,\big(\widetilde{D}_B^{2k} T_{2(q+r),\sigma}\big)(\sigma).$$

Taking into account that

$$\widetilde{D}_B^{2k} T_{2(q+r),\sigma}(\sigma) = \big(\widetilde{D}_B^{2k} f\big)(\sigma) \qquad \text{for} \quad 0 \le k \le q+r,$$

comparing with the proof of Theorem 2 and using

$$\frac{(n-1-k)!}{k!\,(n-1)!} \,\widetilde{D}_B^{2k} T_{2q,\sigma}(\sigma) = o(n^{-q}) \qquad (n \to \infty) \ \text{ for } \ q \le k \le 2(q+r),$$

we obtain (10). □

Applying the transformation of Remark 1 yields an asymptotic expansion of $M_n^{(r)}$ in terms of the differential operator $\widetilde{D}_M^{2l}$.

Theorem 6. *Let $x \in [0,1]$, $q, r \in \mathbb{N}$ and $f \in H^{2q+2r}(x)$, then*

$$\big(M_n^{(r)} f\big)(x) = f(x) + \sum_{k=r+1}^{q} \frac{(n-1-k)!}{k!\,(n-1)!} \big(\widetilde{D}_M^{2k} f\big)(x) + o(n^{-q}) \qquad (n \to \infty).$$

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