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Exact Condition to Generalize Monotonicity in L^1-convergence of Sine Series

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The present paper finds a more direct and clean version in treating with L^1-convergence.

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1. Introduction

Let $L_{2\pi}$ be the space of integrable functions of period 2π. Write

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx, \qquad f(x) := \sum_{n=1}^{\infty} a_n \cos nx.$$

As usual, let $S_n(g, x)$ be the n-th partial sum of the sine series g, i.e.

$$S_n(g, x) = \sum_{k=1}^{n} a_k \sin kx,$$

and $\| \cdot \|_{L^1} = \int_0^{2\pi} | \cdot | \, dx$.

In classical Fourier analysis, the integrability of trigonometric series is considered as an interesting but difficult topic. This plays an important role in identifying if a trigonometric series is a Fourier series or not, thus forms a foundation for most later investigations including convergence. For example, in studying L^1 convergence and approximation problems, people usually require an assumption that $f \in L_{2\pi}$. From this point of view, it really becomes a question what condition imposed on $\{a_n\}$ can guarantee the integrability for $f(x)$ and $g(x)$.

Quite long ago, Boas [2], Heywood [4] proved the following results:

Theorem BH. *Assume that* $a_n \searrow 0$. *Then for* $0 \leq \alpha < 2$,

$$g(x)/x^\alpha \in L_{2\pi} \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{\alpha-1} a_n < \infty. \tag{1}$$

For $0 < \alpha < 1$,

$$f(x)/x^\alpha \in L_{2\pi} \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{\alpha-1} a_n < \infty. \tag{2}$$

There have been some generalizations for the weighted integrability cases (i.e. for inequality (2) and (1) under condition $0 < \alpha < 2$) to various conditions from nonnegative decreasing condition imposed on the coefficient sequence $\{a_n\}$, see, for example, [1, 3, 8, 9, 10, 12]. Actually, the generalizations of (decreasing) monotonicity include various quasi-monotonicity conditions, the rest bounded variation condition, the group bounded variation condition, the non-onesided bounded variation condition and the mean value bounded variation condition etc., interested reader could find useful information in the reference [13], we give definitions of those conditions only related to the contents of this paper here.

Definition 1. A nonnegative sequence $\mathbf{A} = \{a_n\}_{n=1}^{\infty}$ is called a *Quasi-Monotone Sequence* if for some $\alpha \geq 0$, the sequence $\{a_n/n^\alpha\}_{n=1}^{\infty}$ is monotone (decreasing), in symbol, $\mathbf{A} \in \mathbf{QMS}$.

Definition 2. A nonnegative sequence $\mathbf{A} = \{a_n\}_{n=1}^{\infty}$ is called a *O-Regularly varying Quasi-Monotone Sequence* if for some O-regularly varying sequence $R(n)$, the sequence $\{a_n/R(n)\}_{n=1}^{\infty}$ is monotone (decreasing), in symbol, $\mathbf{A} \in \mathbf{RQMS}$. Here a O-regularly varying sequence $\{R(n)\}$ is an increasing sequence satisfying $\limsup\limits_{n\to\infty} \frac{R(2n)}{R(n)} < \infty$.

Definition 3 (Leindler [7]). If a nonnegative null sequence $\mathbf{A} = \{a_n\}_{n=1}^{\infty}$ satisfies

$$\sum_{k=n}^{\infty} |a_k - a_{k+1}| =: \sum_{k=n}^{\infty} |\Delta a_k| \leq M(\mathbf{A}) a_n$$

for all natural numbers n, where $M(\mathbf{A})$ is a positive constant depending only upon the sequence $\mathbf{A}$, we call it as a *Rest Bounded Variation Sequence*, in symbol, $\mathbf{A} \in \mathbf{RBVS}$.

Definition 4 (Le–Zhou [6]). If a nonnegative sequence $\mathbf{A} = \{a_n\}_{n=1}^{\infty}$ satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \leq M(\mathbf{A}) a_n$$

for all natural numbers n, we call it as a *Group Bounded Variation Sequence*, in symbol, $\mathbf{A} \in \mathbf{GBVS}$.

GBVS was also raised by Tikhonov [10] later as "General Monotone Sequences".

Definition 5 (S. Zhou–P. Zhou–Yu [13]). If a nonnegative sequence $\mathbf{A} = \{a_n\}_{n=1}^{\infty}$ satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \leq \frac{M(\mathbf{A})}{n} \sum_{k=[n/\lambda]}^{[\lambda n]} a_k$$

for all natural numbers n and some $\lambda \geq 2$, we call it as a *Mean Value Bounded Variation Sequence*, in symbol, $\mathbf{A} \in \mathbf{MVBVS}$.

Also we use the symbol **MS** to indicate the set of all nonnegative (decreasing) Monotone Sequences.

The relationships of these sets can be described as follows:

$$\mathbf{MS} \subsetneqq \mathbf{QMS} \subsetneqq \mathbf{RQMS},$$
$$\mathbf{MS} \subsetneqq \mathbf{RBVS},$$
$$\mathbf{QMS} \not\subseteq \mathbf{RBVS}, \quad \mathbf{RBVS} \not\subseteq \mathbf{QMS} \quad \text{(see Leindler [8])},$$
$$\mathbf{RBVS} \cup \mathbf{RQMS} \subsetneqq \mathbf{GBVS} \quad \text{(see Le and Zhou [6])},$$
$$\mathbf{GBVS} \subsetneqq \mathbf{MVBVS} \quad \text{(see Zhou, Zhou, and Yu [13])},$$

altogether,

$$\mathbf{MS} \subsetneqq \mathbf{RBVS} \cup \mathbf{QMS} \subsetneqq \mathbf{RBVS} \cup \mathbf{RQMS} \subsetneqq \mathbf{GBVS} \subsetneqq \mathbf{MVBVS}.$$

MVBVS is one of the largest sets to essentially generalize (decreasing) monotone sequences and is used to generalize many classical results in Fourier analysis (see [13]).

There are some further generalizations of **MVBVS** (see Tikhonov [9], Korus [5]), however, since they are irrelevant to the contents of this paper, we will not mention them here.

Let us recall a recent result (Wang and Zhou [12]) on the weighted L^1-integrability.

Theorem WZ1. *Assume that* $\{a_n\} \in \mathbf{MVBVS}$. *Then for* $0 < \alpha < 2$,

$$g(x)/x^{\alpha} \in L_{2\pi} \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{\alpha-1} a_n < \infty.$$

For $0 < \alpha < 1$,

$$f(x)/x^{\alpha} \in L_{2\pi} \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{\alpha-1} a_n < \infty.$$

However, the most important real integrability case of (1) for $\alpha = 0$ is almost never touched in dozens years. Only very recently, Wang and Zhou [12] gave a try on this, but they indeed told a different story from usual generalizations of the weighted case:

Theorem WZ2. *Assume that* $\{a_n\} \in \mathbf{MVBVS}$*, then if* $g(x) \in L_{2\pi}$ *and* $\{a_n\}$ *are its Fourier coefficients, we have* $\sum_{n=1}^{\infty} n^{-1}a_n < \infty$*. On the other hand, there is a sequence* $\{a_n\} \in \mathbf{MVBVS}$ *such that* $\sum_{n=1}^{\infty} n^{-1}a_n < \infty$*, however the sine series* $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$ *does not belong to* $L_{2\pi}$*.*

If considering the indefinite integral of g, we actually can easily obtain

Proposition 1. *If* $g(x) \in L_{2\pi}$ *and* $\{a_n\}$ *are its Fourier coefficients, then* $\sum_{n=1}^{\infty} n^{-1}a_n$ *converges.*

The necessity part of Theorem WZ2 is a corollary of Proposition 1.

We may need another point of view now.

Given a sine series $\sum_{n=1}^{\infty} a_n \sin nx$, its sum function can be written as $g(x)$ at the point x where it converges. However, it is usually a very hard job to verify if the sum function or the sine series itself belongs to $L_{2\pi}$ or not. On the other hand, in studying L^1-convergence problems, people usually need a requirement that $g \in L_{2\pi}$, which also becomes a hard condition to check or a prior condition to set in most cases. For instance, the well-known classical results for L^1-convergence says (see Zygmund [14]): *let the real even (odd) function* $f \in L_{2\pi}$*, and its Fourier cosine (sine) coefficients* $\{a_n\}_{n=1}^{\infty} \in \mathbf{MS}$*, then* $\lim_{n\to\infty} \|f - S_n(f)\|_{L^1} = 0$ *if and only if* $\lim_{n\to\infty} a_n \log n = 0$. On the other hand, the classical result of uniform convergence of sine series (Chaudy-Jolliffe theorem, see [14]) says: *let* $\{a_n\}_{n=1}^{\infty} \in \mathbf{MS}$*, then the sine series* $\sum_{n=1}^{\infty} a_n \sin nx$ *uniformly converges if and only if* $\lim_{n\to\infty} na_n = 0$*.* The difference between the above two classical theorems is very clear: the first one needs a prior condition $f \in L_{2\pi}$, while the second one does not require that $f \in C_{2\pi}$ ($C_{2\pi}$ is the continuous function space of period 2π). Mathematicians surely prefer the latter to the former in mathematical sense. The reason that the prior condition in L^1 case cannot be avoided is explained mainly by the much more "computation complexity " in the integrable space than in the continuous space. Furthermore, by setting this prior condition $f \in L_{2\pi}$, the computation of an integral of the absolute value of a series (it is an infinite process) can be easily transferred into computation of an integral of the absolute value of a finite summation. People also note that the first important problem in the integrable space should be L^1-convergence, which may be achieved by the series itself (by coefficients) without mentioning the sum function.

Based upon these reasons, why do we not try to find a more direct and clean version of L^1-convergence? Let us recall another version of the classical theorem, equivalent to Theorem BH for sine series in case $\alpha = 0$ under monotone (decreasing) condition:

Theorem BH*. *Write $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$ at those points x where it converges. Assume that $\{a_n\} \in$ **MS**. Then*

$$\lim_{n\to\infty} \|g - S_n(g)\|_{L^1} = 0$$

if and only if

$$\sum_{n=1}^{\infty} n^{-1} a_n < \infty.$$

Here, we emphasize that, in any case, $\lim\limits_{n\to\infty} \|g - S_n(g)\|_{L^1} = 0$ is interpreted as the Cauchy's convergence principle $\|S_n(g) - S_m(g)\|_{L^1} \to 0,\ n, m \to \infty$ holds, and vice versa.

2. Results

Starting from here, we begin our research.

By C, C_1, C_2 etc. we denote constants which may differ in the different occurrences.

First, we construct a nonnegative sequence which shows that although it is quasi-monotone and the inequality $\sum\limits_{n=1}^{\infty} n^{-1} a_n < \infty$ holds, the corresponding sine series does not converge in L^1-norm.

Theorem 1. *There exists a positive sequence $\{a_n\} \in$ **QMS** such that*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty, \tag{3}$$

however for the sine series

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx,$$

we have

$$\|g - S_n(g)\|_{L^1} \not\to 0, \quad n \to \infty.$$

Proof. We give a sketch of construction here. Set $n_1 = 1$, $n_2 = 2$, and

$$m_j = 2^{j^2}, \quad n_j = j^2 2^{j^2}, \qquad j = 3, 4, \dots$$

Let a_1, a_2, a_3 be any given positive numbers, and for $j \geq 2$, let

$$a_n = \begin{cases} \frac{n}{m_{j+1}\log^2 n}, & 2n_j \leq n < m_{j+1}, \\ \frac{n}{m_{j+1}\log^2 n_{j+1}}, & m_{j+1} \leq n < n_{j+1}, \\ \frac{1}{2\log n_{j+1}}, & n_{j+1} \leq n < 2n_{j+1}. \end{cases}$$

In such a way we define a sine series

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx$$

at x where it converges. This series can serve for the purpose of Theorem 1. □

Next, we formulate a new condition which can guarantee the condition (3) to be necessary and sufficient for the corresponding sine series to converge in L^1-norm.

Theorem 2. *Let $\{a_n\}$ be a nonnegative bounded sequence. Write $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$ at the points x where it converges. Given an integer $N > 0$, we set*

$$R_n = \sum_{k=n}^{\infty} \left|\Delta \frac{a_k}{\log^N k}\right|,$$

and assume that

$$R_n \leq M(\mathbf{A}) \frac{a_n}{\log^N n} \quad \text{for} \quad n = 1, 2, \ldots \tag{4}$$

Then $g \in L_{2\pi}$, $\{a_n\}$ are its Fourier coefficients and

$$\lim_{n\to\infty} \|g - S_n(g)\|_{L^1} = 0$$

if and only if

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty.$$

Proof. The necessity part is a direct corollary of Proposition 1.

We give a sketch of proof of the sufficiency. Given arbitrary sufficiently large m, n, say $m > n$, by Abel's transformation we obtain

$$\begin{aligned} S_m(x) - S_n(x) = \sum_{k=n+1}^{m} a_k \sin kx &= \sum_{k=n+1}^{m} \frac{a_k}{\log^N k} \log^N k \, \sin kx \\ &= -\frac{a_{n+1}}{\log^N (n+1)} \sum_{j=1}^{n} \log^N j \, \sin jx + \frac{a_m}{\log^N m} \sum_{j=1}^{m} \log^N j \, \sin jx \\ &\quad + \sum_{k=n+1}^{m-1} \Delta \frac{a_k}{\log^N k} \sum_{j=1}^{k} \log^N j \, \sin jx \\ &=: I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

By standard methods, we have

$$\int_0^\pi \Big|\sum_{j=1}^{k} \log^N j \, \sin jx\Big| dx \le \sum_{j=1}^{k-1} |\Delta \log^N j| \int_0^\pi \Big|\sum_{l=1}^{j} \sin lx\Big| dx + \log^N k \int_0^\pi \Big|\sum_{l=1}^{k} \sin lx\Big| dx$$
$$\le C \log^{N+1} k,$$

and consequently

$$\int_0^\pi |I_1(x)| dx \le C \frac{a_{n+1}}{\log^N (n+1)} \log^{N+1} n \le C a_{n+1} \log n,$$

and

$$\int_0^\pi |I_2(x)| dx \le C a_m \log m.$$

For the third term we have

$$\int_0^\pi |I_3(x)| dx \le \sum_{k=n+1}^{m-1} \Big|\Delta \frac{a_k}{\log^N k}\Big| \int_0^\pi \Big|\sum_{j=1}^{k} \log^N j \, \sin jx\Big| dx$$
$$\le \sum_{k=n+1}^{m-1} \Big|\Delta \frac{a_k}{\log^N k}\Big| \log^{N+1} k =: J.$$

Further investigation shows that, by condition (4),

$$J = \sum_{k=n+1}^{m-1} (R_k - R_{k+1}) \log^{N+1} k$$
$$= \Big|R_{n+1} \log^{N+1}(n+1) - R_m \log^{N+1} m + \sum_{j=n+1}^{m-1} R_{j+1} (\log^{N+1}(j+1) - \log^{N+1} j)\Big|$$
$$\le C_N \Big(a_{n+1} \log(n+1) + a_m \log(m+1) + \sum_{j=n+1}^{m-1} \frac{a_{j+1}}{j+1}\Big),$$

where we have applied Abel's transformation again. On using (4) we see that, for $[\sqrt{n}] \le j \le n$,

$$\frac{a_n}{\log^N n} = -\sum_{k=j}^{n-1} \Delta \frac{a_k}{\log^N j} + \frac{a_j}{\log^N j} \le \sum_{k=j}^{\infty} \Big|\Delta \frac{a_k}{\log^N j}\Big| + \frac{a_j}{\log^N j} \le (M(\mathbf{A})+1) \frac{a_j}{\log^N j}.$$

Therefore, $a_n \le C a_j$ for all $[\sqrt{n}] \le j \le n$. Hence,

$$a_{n+1} \log n \le C a_n \sum_{j=[\sqrt{n}]}^{n} \frac{1}{j} \le C \sum_{j=[\sqrt{n}]}^{n} \frac{a_j}{j}, \qquad a_m \log m \le C \sum_{j=[\sqrt{m}]}^{m} \frac{a_j}{j}.$$

Combining all the above estimates with Cauchy's criterion, we finally achieve the required result. $\square$

If a nonnegative bounded sequence $\{a_n\}$ satisfies condition (4), we can call it as a *Logarithm Rest Bounded Variation Sequence*, indicated by $\{a_n\} \in \mathbf{LRBVS}_N$.

Similarly, a nonnegative sequence $\{a_n\}$ is called as a *Logarithm Quasi-Monotone Sequence*, if $\{a_n/\log^N n\}$ is decreasing for some integer N, then it is indicated by $\{a_n\} \in \mathbf{LQMS}_N$.

The following theorem shows relationships among the related sets including that **RBVS** is a real subset of $\mathbf{LRBVS}_N$.

Theorem 3. *For any integer $N \geq 2$, we have*

$$\mathbf{RBVS} \cup \mathbf{LQMS} \subsetneqq \mathbf{LRBVS}_1 \subsetneqq \mathbf{LRBVS}_2 \subsetneqq \cdots \subsetneqq \mathbf{LRBVS}_N.$$

Proof. The relationships

$$\mathbf{RBVS} \subset \mathbf{LRBVS}_1, \qquad \mathbf{LQMS} \subsetneqq \mathbf{LRBVS}_1$$

are clear. By the following example, we can show that $\mathbf{RBVS} \neq \mathbf{LRBVS}_1$. Set $n_1 = 1$, and $n_j = 2^{2^{j^2}}$ for $j \geq 2$. Choose $a_1, a_2, \ldots, a_{n_2-1}$ as arbitrary positive numbers, and define

$$a_n = \frac{\log n}{\log^3 n_{j+1}}, \qquad n_j \leq n < n_{j+1},\ j = 2, 3, \ldots$$

The other inclusions $\mathbf{LRBVS}_1 \subsetneqq \mathbf{LRBVS}_2 \subsetneqq \cdots \subsetneqq \mathbf{LRBVS}_N$ for $N \geq 2$ can be treated similarly. $\square$

Corollary 1. *Let $\{a_n\} \in \mathbf{LQMS}_N$ for some integer $N > 0$, and $g(x)$ be defined as in Theorem 2. Then $g \in L_{2\pi}$, $\{a_n\}$ are its Fourier coefficients and $\|g - S_n(g)\|_{L^1} = 0$ if and only if*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty.$$

Corollary 2. *Let $\{a_n\} \in \mathbf{RBVS}$ and $g(x)$ be defined as in Theorem 2. Then $g \in L_{2\pi}$ and $\|g - S_n(g)\|_{L^1} = 0$ if and only if*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty.$$

Remark 1. In all the above results, L^1-convergence conclusions cannot be directed by using ordinary L^1-convergence theorems, because there is no a prior requirement that $g \in L_{2\pi}$.

In the classical case when $\{a_n\} \in \mathbf{MS}$, it is known that the conditions

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$$

and

$$\sum_{n=1}^{\infty} |\Delta a_n| \log n < \infty$$

are equivalent. It can be shown that in the case $\{a_n\} \in \mathbf{LRBVS}_N$ they are equivalent, too.

Theorem 4. *Let $\{a_n\} \in \mathbf{LRBVS}_N$, then the conditions*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$$

and

$$\sum_{n=1}^{\infty} |\Delta a_n| \log n < \infty$$

are equivalent.

Finally, we prove that the newly raised condition cannot be weakened further in some sense.

Theorem 5. *Let $R(n)$ is an increasing sequence satisfying $\lim_{n\to\infty} R(n) = \infty$ and $\limsup_{n\to\infty} \frac{R(2n)}{R(n)} < \infty$. If*

$$\sum_{k=1}^{n} \frac{R(k)}{k} \le CR(n), \qquad n = 1, 2, \dots, \tag{5}$$

then there exists a positive sequence $\{a_n\}$ such that $\{a_n/R(n)\}$ is decreasing and satisfies further

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty,$$

however for the sine series

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

we have $\|g - S_n(g)\|_{L^1} \not\to 0$ as $n \to \infty$.

Proof. The proof follows the basic idea of Theorem 1. Set $n_1 = 1$, $n_2 = 2$, and select n_j inductively. If, for $j \geq 2$, n_j is selected, we choose

$$m_{j+1} = 2^{n_j^2}.$$

From (5) we see that

$$CR(n) \geq \sum_{k=[\sqrt{n}]}^{n} \frac{R(k)}{k} \geq C_1 R([\sqrt{n}]) \log n,$$

whence

$$\frac{R(n)}{\log n} \geq C_2 R([\sqrt{n}]) \to \infty, \quad n \to \infty.$$

Thus let

$$n_{k+1} = \min\Big\{ n \geq m_{k+1} : \frac{R(n)}{\log n} \geq R(m_{k+1}) \Big\}.$$

By definition, $n_{k+1} > m_{k+1}$ and

$$\frac{R(n_{k+1} - 1)}{\log(n_{k+1} - 1)} \leq R(m_{k+1}).$$

Now define

$$a_n = \begin{cases} \frac{R(n)}{R(m_{j+1}) \log^2 n}, & 2n_j \leq n < m_{j+1}, \\ \frac{R(n)}{R(m_{j+1}) \log^2 n_{j+1}}, & m_{j+1} \leq n < n_{j+1}, \\ \frac{1}{\log n_{j+1}}, & n_{j+1} \leq n < 2n_{j+1}. \end{cases}$$

This example can serve for the purpose of Theorem 5. □

Remark 2. Except those "too slowly increasing" sequences like $\{\log^N n\}$, the other "not too slowly increasing" sequences like $\{n^\alpha\}$, $\alpha > 0$, etc. evidently satisfy condition (5). Therefore, in a sense, Theorem 5 shows that LRBV condition cannot weakened further in L^1-convergence for sine series.

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