CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 401-402
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

Open Problem

Voronovskaja-type Formula for the Bézier Variant of the Bernstein Operators

Ulrich Abel

For each function $f:[0,1]\to\mathbb{R}$, $n\in\mathbb{N}$ and $\alpha>0$, the Bernstein operators of Bézier type $B_{n,\alpha}$ are defined by

$$(B_{n,\alpha}f)(x)=\sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\big(J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)\big), \tag{1}$$

where

$$J_{n,k}(x)=\sum_{j=k}^{n} p_{n,j}(x) \qquad (k=0,\ldots,n), \qquad J_{n,n+1}(x)=0,$$

and $p_{n,j}$ is the Bernstein basis polynomial

$$p_{n,j}(x)=\binom{n}{j}x^j(1-x)^{n-j} \qquad (j=0,\ldots,n).$$

To my best knowledge the operators (1) were introduced by Chang [1] in 1983. Chang showed that, for $f\in C[0,1]$, $\lim_{n\to\infty}(B_{n,\alpha}f)(x)=f(x)$ uniformly on $[0,1]$. In 1985, Li and Gong [2] estimated the rate of convergence

$$\sup_{x\in[0,1]}|(B_{n,\alpha}f)(x)-f(x)|\le\begin{cases}(1+\frac{\alpha}{4})\omega(f;n^{-1/2}) & (\alpha\ge 1),\\ M\omega(f;n^{-\alpha/2}) & (0<\alpha<1),\end{cases}$$

where $\omega(f;\delta)$ denotes the modulus of continuity of f, and M is a constant depending only on α and f. In 1986 Liu [3] obtained an inverse theorem in the case $\alpha\ge 1$. Zeng and Piriou [5] studied the rate of convergence for functions of bounded variation. There are many further papers on the operators (1) and their variants.

It is obvious that $B_{n,\alpha}$ are positive linear operators which preserve constant functions. In the special case $\alpha = 1$, the operators $B_{n,\alpha}$ reduce to the Bernstein polynomials $B_{n,1} \equiv B_n$ given by

$$(B_n f)(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{n,k}(x).$$

The asymptotic behaviour of the sequence of Bernstein polynomials B_n as n tends to infinity is well known. The classical result by Voronovskaja [4]

$$\lim_{n\to\infty} n\big((B_n f)(x) - f(x)\big) = \frac{1}{2}\, x(1-x) f''(x) \tag{2}$$

is valid for all bounded functions f being twice differentiable in $x \in [0,1]$. As far as we know, nothing is known in this direction for the operators $B_{n,\alpha}$ when $\alpha \neq 1$. Numerical experiments suggest that the limit

$$\lim_{n\to\infty} \sqrt{n}\,\big((B_{n,\alpha} f)(x) - f(x)\big) \tag{3}$$

exists at least for sufficiently smooth functions f. The problem is to prove its existence, for certain functions f, and to find an explicit expression. In the particular case $\alpha = 1$ the limit (3) obviously exists, by Eq. (2), with value equal to 0, provided that f satisfies the conditions of Voronovskaja's theorem.

References

[1] G. Chang, Generalized Bernstein-Bézier polynomials, *J. Comput. Math.* **1** (1983), 322–327.

[2] P. Li and Y. H. Gong, The order of approximation by the generalized Bernstein-Bézier polynomial, *J. China Univ. Sci. Technol.* **15** (1985), 15–18.

[3] Z. Liu, Approximation of continuous functions by the generalized Bernstein-Bézier polynomials, *Approx. Theory Appl.* **2** (1986), 105–130.

[4] E. Voronovskaja, Détermination de la forme asymptotique de l'approximation des fonctions par les polynômes de M. Bernstein, *C. R. Acad. Sc. URSS, Ser. A* **4** (1932), 79–85.

[5] X. M. Zeng and A. Piriou, On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions, *J. Approx. Theory* **95** (1998), 369–387.

Ulrich Abel
Fachbereich MND
Technische Hochschule Mittelhessen
University of Applied Sciences
Wilhelm-Leuschner-Straße 13, 61169 Friedberg
GERMANY
E-mail: Ulrich.Abel@mnd.th-mittelhessen.de

CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 403-406
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

Open Problem

Interpolation of Wiener Amalgam Spaces and Associated Approximation Theory

Lubomir T. Dechevsky

The concept of *Wiener amalgam spaces (WAS)* was introduced by Feichtinger (see, e.g. [9] and the references therein), and a study of their properties was initiated by Feichtinger and an increasing number of other authors. The general concept of WAS unites in itself many previous particular constructions used in different contexts for different purposes without previous connection among these analogous constructions. One important instance are the spaces generated by the averaged moduli of smoothness [12], which, as discussed in [2, 4, 5], are important for numerical analysis particular cases of WAS. The results on the equivalence between the averaged moduli and K-functionals obtained in [1, 2, 4] show that it is very important to look for solution of the following general problem.

Problem A. *Extend the theory of K-functionals to the case when the spaces A, B in the K-functional $K(t,\cdot\,;A,B)$ depend on the step t.*

In [1] the following relevant conjecture was made.

Conjecture B. *The A-spaces induced by the averaged moduli (and, more generally, space scales obtained via the real interpolation functor applied on a K-functional between Wiener amalgam spaces with quasi-seminorms depending on the step of the K-functional) are, in general, not closed with respect to real or complex interpolation (which is a major difference with the classical interpolation spaces induced by the integral moduli, i.e., Besov spaces).*

Conjecture B has more recently been proved to be correct [10].

In contrast, the interpolation techniques developed in [1, 2, 4] work for both classical interpolation spaces and WAS. The interpolation techniques presented in [1, 2, 4] for several special WAS related to the averaged moduli of smoothness can be extended for general WAS depending on the step t of the K-functional.

In view of the above background of Problem A, we propose a sequence of several (sub)problems which cover various specific aspects of Problem A.

New results have been obtained about Marchaud-type inequalities, some of which involve spaces which are particular Wiener amalgams (see, e.g., [8] and the references therein). In relevance to this, we suggest the following.

Problem A.1. *Derive a new, more general kind of Marchaud-type inequalities in the context of Wiener amalgam spaces and even more general types of spaces and K-functionals, in which the spaces depend on the step of the K-functional.*

A close relationship was established between the theory of K-functionals and optimal smoothing techniques based on a small-penalty approach to Tikhonov regularization of ill-posed inverse problems (see [7, 6] and the references therein). In this case, of interest are only very small and very large values of the step t: $0 < t < \infty$, i. e., only values of t and $1/t$ close to 0. We note that, while intermediate values of t (i.e., t: $\varepsilon < t < 1/\varepsilon$ for any $\varepsilon > 0$) are "responsible" for the bulk of the norm of a real interpolation space, it is only the range of values of t and $1/t$ close to 0 that determines whether a function is, or is not, element of the space in the set-theoretic sense. In view of this observation, we propose the following concretization of Problem A.

Problem A.2. *Develop a new approach to the computation of K-functionals between spaces which depend on the step t of the K-functional (and for Wiener amalgam spaces, in particular), based on the following propositions:*

(a) Instead of trying to obtain embedding results (quantitative theory of interpolation spaces), try to obtain only set-theoretic inclusions (qualitative theory of interpolation spaces).

(b) The advantage of pursuing the more modest objective of qualitative theory is that the K-functionals have to be computed only for values of t and/or $1/t$ close to 0.

(c) As a main general new tool for computation of general classes of K-functionals

$$K(t,\cdot) = K^{(t)}(t,\cdot) = K(t,\cdot\,; A(t), B(t)) \tag{1}$$

between spaces $A(t)$, $B(t)$ which may depend on the step t of the K-functional, we propose to use standard perturbation expansion techniques, starting from the known "unperturbed" cases corresponding to $t = 0$ or $t = \infty$.

(In formula (1) we have used the same notation for the K-functional as in [1, 4, 5].)

Another modification of this idea is to use as "unperturbed" starting point of the perturbation expansion K-functionals which have already been successfully computed. For example, the solution of the following problem is of considerable heuristic interest.

Problem A.3. *Making use of the isometric quasilinearization of K-functionals between Hilbert spaces (see [3]), obtain (regular or singular) perturbation expansions of the K-functionals between Banach spaces which are "nearly"*

Hilbert. While in the formulation of Problem A.2 the expansion is in powers of t or $1/t$, in the present case the expansion is in powers of the small parameter measuring the proximity between the target (perturbed) Banach space and the original (unperturbed) Hilbert space (e.g., the parameter $1/p-1/2$ for p near 2).

The potential of the use of small perturbation techniques for computation of K-functionals extends beyond the range covered in Problems A.1–3, including, for example, problems related to interpolation of Banach algebrae of operators, spaces with (quasi-)norm dependent on the step of the K-functional which are not WAS, etc. We summarize these complementary cases, as follows.

Problem A.4. *Under the assumptions in items (a) and (b) in the formulation of Problem A.2, investigate also other prospective ways to use the "method of small parameter" for computing K-functionals.*

Taking in consideration the connections existing between real interpolation spaces and respective best-approximation rates (see, e.g., [11]), the solution of Problem A can be expected to provide an important extension of approximation-theoretic results available for Besov, Triebel-Lizorkin and other space scales to their Wiener amalgam analogues, as well as in the even more general context of (possibly non-Wiener amalgam) metrizable spaces depending on the step(s) of a respective approximation process.

References

[1] L. T. Dechevsky, τ-moduli and interpolation, *in* "Function Spaces and Applications" (M. Cwikel, J. Peetre, Y. Sakher, and H. Wallin, Eds.), Proc. US-Swedish Seminar on Function Spaces and Applications, Lund'86, Lecture Notes in Mathematics, Vol. 1302, pp. 177–190, Springer, Berlin, 1988.

[2] L. T. Dechevsky, "Some Applications of the Theory of Function Spaces in Numerical Analysis", Ph.D. Dissertation, Faculty of Mathematics and Mechanics, University of Sofia, Sofia, 1989 [in Bulgarian].

[3] L. T. Dechevsky, Explicit computation of the K-functional between semi-Hilbert spaces, *Int. J. Pure Appl. Math.* **33** (2006), no. 3, 287–332.

[4] L. T. Dechevsky, Properties of function spaces generated by the averaged moduli of smoothness, *Int. J. Pure Appl. Math.* **41** (2007), no. 9, 1305–1375.

[5] L. T. Dechevsky, Concluding remarks to paper "Properties of function spaces generated by the averaged moduli of smoothness", *Int. J. Pure Appl. Math.* **49** (2008), no. 1, 147–152.

[6] L. Dechevsky and B. MacGibbon, Cross validation in Besov spaces for multivariate density estimation and nonparametric regression with random design, *Int. J. Pure Appl. Math.* **33** (2006), no. 2, 225–256.

[7] L. T. Dechevsky, J. O. Ramsay, and S. I. Penev, Penalized wavelet estimation with Besov regularity constraints, *Math. Balkanica (N.S.)* **13** (1999), no. 3–4, 257–376.

[8] Z. Ditzian and A. V. Prymak, Sharp Marchaud and converse inequalities in Orlicz spaces, *Proc. Amer. Math. Soc.* **135** (2007), no. 4, 1115–1121.

[9] H. G. Feichtinger, S. S. Pandey, and T. Werther, Minimal norm interpolation in harmonic Hilbert spaces and Wiener amalgam spaces on locally compact Abelian groups, *J. Math. Kyoto Univ.* **47** (2007), no. 1, 65–78.

[10] N. Krugljak and E. Matvejev, Onesided approximation and real interpolation, *Collect. Math.* **48** (1997), no. 4–6, 619–634.

[11] P. P. Petrushev and V. A. Popov, "Rational Approximation of Real Functions", Encyclopedia of Mathematics and its Applications, Vol. 28, Cambridge University Press, Cambridge, 1987.

[12] Bl. Sendov and V. Popov, "Averaged Moduli of Smoothness", Bulg. Math. Monographs Vol. 4, Publ. House of Bulg. Acad. Sciences, Sofia, 1983 [in Bulgarian].

Lubomir T. Dechevsky

R&D Group for Mathematical Modelling
Numerical Simulation and Computer Visualization
Faculty of Technology
Narvik University College
P.O.Box 385, N-8505 Narvik
NORWAY
E-mail: ltd@hin.no

CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 407-409
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

Open Problem

Some Problems Posed by Borislav Bojanov

Nikola Naidenov*

Extremal problems is an area that was of great interest to Prof. Bojanov, and where he had obtained significant results. Here we formulate four open problems, due to Bojanov.

Let π_n be the set of algebraic polynomials with real coefficients of degree not exceeding n and let $\pi_n^\circ := \{P \in \pi_n : \|P\|_{C[-1,1]} \le 1\}$. By Φ we denote the class of all strictly increasing convex functions on $[0,\infty)$. Then, see [3], the following extremal property of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$ is of interest:

Problem 1. *Prove or disprove the inequality*

$$\int_{-1}^{1} \varphi(|P^{(k)}(x)|)\, dx < \int_{-1}^{1} \varphi(|T_n^{(k)}(x)|)\, dx, \tag{1}$$

for every $\varphi \in \Phi$, $P \in \pi_n^\circ \setminus \{\pm T_n\}$ *and* $k \in \{2, \ldots, n-1\}$.

Note that for $k = n$ inequality (1) is a consequence of a well-known property of T_n. For $k = 1$ inequality (1) was proved in [4], while the important particular cases $\varphi(x) = \sqrt{1+x^2}$ and $\varphi(x) = x^p$, $p \ge 1$, were obtained in [2] and [3], correspondingly. The first case gives an affirmative answer to a problem posed by Erdős in 1939 (see [9]) and the second choice leads to the following Markov type inequality $\|P'\|_{L_p[-1,1]} \le \|T_n'\|_{L_p[-1,1]} \cdot \|P\|_{C[-1,1]}$.

Notice that Problem 1 was solved in [8] for the subclass $\mathcal{P}_n^\circ$ of π_n° which consists of polynomials having all their zeros in $[-1,1]$. The particular case $k = 2$ was confirmed in [1] in an intermediate class between $\mathcal{P}_n^\circ$ and π_n°.

In [10], Erdős extended his question asking about the "longest" polynomial in π_n°, but on a subinterval $[a,b]$ of $[-1,1]$. Prof. Bojanov ([5]) solved this problem when $[-\varepsilon_n, \varepsilon_n] \subset [a,b]$, with $\varepsilon_n = \cos\frac{\pi}{2n}$, under the additional restriction $P(-\varepsilon_n) = P(\varepsilon_n) = 0$. The extremal is again the Chebyshev polynomial of the first kind. He liked very much the following related problem.

*Supported by the National Science Fund of the Bulgarian Ministry of Education, Youth and Science under Grant DDVU-02/30.

Problem 2. *Let* $\eta_k = \cos\frac{(n-k)\pi}{n}$, $k = 0, \ldots, n$, *be the extremal points of* T_n *in* $[-1, 1]$. *Prove or disprove the inequality*

$$\int_{\eta_{k-1}}^{\eta_k} \varphi(|P'(x)|)\, dx < \int_{\eta_{k-1}}^{\eta_k} \varphi(|T_n'(x)|)\, dx,$$

for every $\varphi \in \Phi$, $P \in \pi_n^\circ \setminus \{\pm T_n\}$ *and* $k \in \{1, \ldots, n\}$.

Let us note that not for every subinterval of $[-1, 1]$ the extremal element is T_n. Indeed, if the interval $[a, b]$ tends to a point, then we arrive at the extremal problem $|P'(a)| \to \max$ for $P \in \pi_n^\circ$ whose solution, depending on a, can be T_n or some Zolotorev polynomial.

Prof. Bojanov was interested in a deeper understanding of the mechanism that makes the Chebyshev polynomial T_n extremal in so many optimization problems. This is not an end in itself, but would lead to a deeper understanding of many extremal properties of polynomials and other polynomial-like functions. For instance, the technique developed in connection with Problem 1 fits perfectly for establishing of Turán type inequalities (see [5]). The following problem can be considered as a good starting point for studying Karlin's hypothesis ([12]):

Problem 3 ([6]). *Prove or disprove that for every* $\varphi \in \Phi$ *and* $f \in W_\infty^n[-1, 1]$ *satisfying* $\|f\|_{C[-1,1]} \le 1$ *and* $\|f^{(n)}\|_{L_\infty[-1,1]} \le 2^{n-1}\, n!$,

$$\int_{-1}^{1} \varphi(|f^{(k)}(x)|)\, dx \le \int_{-1}^{1} \varphi(|T_n^{(k)}(x)|)\, dx, \qquad k \in \{1, \ldots, n-1\}.$$

Are the Chebyshev polynomials $\pm T_n(x)$ *the only extremal functions?*

(The corresponding problem for maximizing $\|f^{(k)}\|_{C[-1,1]}$ was solved in [11], while the particular cases $n = 2$ and $n = 3$ were obtained in [7].)

Several times in our personal communication Professor Bojanov mentioned the following integral extension of the Markov and Remez type inequalities.

Problem 4. *Prove or disprove that, for each* $\delta > 0$,

$$\int_{-1}^{1+\delta} \varphi(|P'(x)|)\, dx < \int_{-1}^{1+\delta} \varphi(|T_n'(x)|)\, dx,$$

for every $\varphi \in \Phi$ *and* $P \in \pi_n \setminus \{\pm T_n\}$ *such that* $\|P\|_{L_\infty(E)} \le 1$ *for some subset* $E \subset [-1, 1+\delta]$ *with measure* $\mu(E) = 2$.

References

[1] L. S. Avvakumova, An extension of V. A. Markov's inequality for second derivative to a wider polynomial class, *East J. Approx.* **6** (2000), no. 4, 493–519.

[2] B. Bojanov, Proof of a conjecture of Erdős about the longest polynomial, *Proc. Amer. Math. Soc.* **84** (1982), no. 1, 99–103.

[3] B. Bojanov, An extension of the Markov inequality, *J. Approx. Theory* **35** (1982), no. 2, 181–190.

[4] B. Bojanov, An extremal problem for polynomials, *in* "Constructive Function Theory'81", pp. 229–233, BAN, Sofia, 1983.

[5] B. Bojanov, Polynomial inequalities, *in* "Open Problems in Approximation Theory" (B. Bojanov, Ed.), pp. 25–42, SCT Publishing, Singapore, 1994.

[6] B. Bojanov, Markov-type inequalities for polynomials and splines, *in* "Approximation Theory X: Abstract and Classical Analysis" (C. Chui, L. L. Schumaker, and J. Stöckler, Eds.), pp. 31–90, Vanderbilt University Press, Nashville, TN, 2002.

[7] B. Bojanov and N. Naidenov, Examples of Landau-Kolmogorov inequalities in integral norms on a finite interval, *J. Approx. Theory* **117** (2002), 55–73.

[8] B. D. Bojanov and Q. I. Rahman, On certain extremal problems for polynomials, *J. Math. Anal. Appl.* **189** (1995), 781–800.

[9] P. Erdős, An extremum-problem concerning trigonometric polynomials, *Acta Sci. Math. (Szeged)* **9** (1939), 113–115.

[10] P. Erdős, Open problems, *in* "Open Problems in Approximation Theory" (B. Bojanov, Ed.), pp. 238–242, SCT Publishing, Singapore, 1994.

[11] B. O. Eriksson, Some best constants in the Landau inequality on a finite interval, *J. Approx. Theory* **94** (1998), no. 3, 420–454.

[12] S. Karlin, Oscillatory perfect splines and related extremal problems, *in* "Studies in Spline Functions and Approximation Theory" (S. Karlin et all, Eds.), pp. 371–460, Academic Press, New York, 1976.

Nikola Naidenov

Department of Mathematics and Informatics
University of Sofia
Boulevard James Bourchier 5
1164 Sofia
BULGARIA
E-mail: nikola@fmi.uni-sofia.bg

CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 410-413
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

Open Problem

V. A. Markov's Inequality for the Coefficients of Bounded Multivariate Polynomials

Heinz-Joachim Rack

Consider multivariate polynomials P_m^r in $r \geq 1$ variables of total degree $\leq m$, i.e. $P_m^r \in \Phi_m^r = \operatorname{span}\{\underline{x}^{\underline{k}} : |\underline{k}| \leq m\}$ with

$$P_m^r(\underline{x}) = \sum_{|\underline{k}| \leq m} A_{\underline{k}} \underline{x}^{\underline{k}},$$

where $\underline{k} = (k_1, k_2, \ldots, k_r)$ is a multiindex with integer components $k_i \geq 0$, $|\underline{k}| = k_1 + k_2 + \cdots + k_r$ is a (total) degree, $A_{\underline{k}} \in \mathbb{R}$ is a coefficient, $\underline{x} = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$ is a variable, and $\underline{x}^{\underline{k}} = x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$ is a monomial.

Let B_m^r denote the unit ball in Φ_m^r with respect to the unit cube $I^r = [-1,1]^r = \{\underline{x} : |x_i| \leq 1, \text{ for } i = 1, 2, \ldots, r\} \subset \mathbb{R}^r$ and with uniform norm $\|P_m^r\| = \sup_{\underline{x} \in I^r} |P_m^r(\underline{x})|$, i.e.,

$$B_m^r = \{P_m^r \in \Phi_m^r : \|P_m^r\| \leq 1\}.$$

For $r = 1$ the m-th Chebyshev polynomial of the first kind with respect to $I^1 = [-1,1]$, T_m, belongs to B_m^1 since $T_m(x_1) = \cos(m \arccos x_1)$ if $x_1 \in I^1$. T_m with $T_m(x_1) = \sum_{k_1=0}^{m} t_{m,k_1} x_1^{k_1}$ satisfies the three-term recurrence relation

$$T_m(x_1) = 2x_1 T_{m-1}(x_1) - T_{m-2}(x_1), \qquad m \geq 2,$$

with $T_0(x_1) = 1$ and $T_1(x_1) = x_1$, and is hence an even resp. odd polynomial, according to the parity of m, so that $t_{m,k_1} = 0$, if $m - k_1$ is odd, whereas, if $m - k_1$ is even, the coefficients t_{m,k_1} are nonzero integers given in descending order by:

$$t_{m,m-2j} = (-1)^m \frac{m 2^{m-2j-1}}{m-j} \binom{m-j}{j}, \qquad 0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

V. A. Markov in his celebrated paper of 1892 [3, pp. 80–81], [4, p. 248] (see also [1, p. 248], [5, p. 423], [6, p. 56], [9, pp. 672–673], [11, p. 147], [12,

p. 167]) has provided exact estimates for all the $m+1$ coefficients of $P_m^1 \in B_m^1$, in terms of the nonzero coefficients of T_m and T_{m-1}:

V. A. Markov's Univariate Coefficient Inequality. *If $P_m^1 \in B_m^1$ with $P_m^1(x_1) = \sum\limits_{k_1=0}^{m} A_{k_1} x_1^{k_1}$, then*

$$|A_{k_1}| \leq |t_{m,k_1}| = \frac{m 2^{k_1-1}\,(\frac{m+k_1-2}{2})!}{k_1!\,(\frac{m-k_1}{2})!}, \qquad \text{if } m-k_1 \text{ even}, \tag{1}$$

$$|A_{k_1}| \leq |t_{m-1,k_1}| = \frac{(m-1) 2^{k_1-1}\,(\frac{m+k_1-3}{2})!}{k_1!\,(\frac{m-k_1-1}{2})!}, \qquad \text{if } m-k_1 \text{ odd}, \tag{2}$$

with equality if $P_m^1 = \pm T_m$ resp. $P_m^1 = \pm T_{m-1}$.

It is well known that determining the optimal upper bound for a certain coefficient of a bounded polynomial is equivalent to solving a dual problem of best approximation to the corresponding monomial, see [12, Satz 1.2] or [13, Exercise 2.13.1].

We pose the following:

Problem. *Find an analogue to V. A. Markov's univariate coefficient inequality in the multivariate setting $P_m^r \in B_m^r$ with $r > 1$.*

This problem has already been touched upon in [2, Remark 1] and [7, p. 131]. Alternative extensions of V. A. Markov's univariate two-staged coefficient inequality (as well as of the related G. Szegő's coefficient inequality) to multivariate polynomials, on the unit cube and on the unit ball, are discussed in [8].

Concerning the posed problem, the following is known, compare with Theorem I in [8]:

Sharp Estimates for the Coefficients $\boldsymbol{A_{\underline{k}}}$ of $\boldsymbol{P_m^r \in B_m^r}$. *If $P_m^r \in B_m^r$ $(r \geq 2, m \geq 3)$ with $P_m^r(\underline{x}) = \sum\limits_{|\underline{k}| \leq m} A_{\underline{k}} \underline{x}^{\underline{k}}$, and $\check{r}$ denotes the number of nonvanishing components of $\underline{k}$, then,*

$$|A_{\underline{k}}| \leq 2^{m-\check{r}}, \qquad \text{if } |\underline{k}| = m, \tag{3}$$

with equality if $P_m^r(\underline{x}) = \prod\limits_{i=1}^{r} T_{k_i}(x_i) \in B_m^r$ (see [10]),

$$|A_{\underline{k}}| \leq 2^{m-\check{r}-1}, \qquad \text{if } |\underline{k}| = m-1, \tag{4}$$

with equality if $P_m^r(\underline{x}) = \prod\limits_{i=1}^{r} T_{k_i}(x_i) \in B_{m-1}^r$ (see [7]),

$$|A_{\underline{k}}| \leq \begin{cases} m, & \text{if } |\underline{k}| = 1 \text{ and } m \text{ odd}, \\ m-1, & \text{if } |\underline{k}| = 1 \text{ and } m \text{ even}, \end{cases} \tag{5}$$

with equality if $P_m^r(\underline{x}) = T_m(x_i)$ *resp.* $P_m^r(\underline{x}) = T_{m-1}(x_i)$, $i = 1, 2, \ldots, r$, *see [2],*

$$|A_{\underline{k}}| \le 1, \qquad \textit{if } |\underline{k}| = 0, \tag{6}$$

with equality if $P_m^r(\underline{x}) = T_0(x_i)$, $i = 1, 2, \ldots, r$ *(trivial, since* $P_m^r \in B_m^r$*).*

Example 1. For $r = 2$ and $m = 3$ the foregoing sharp estimates yield a complete analogue to (1) and (2) with the aid of (products of) coefficients of univariate Chebyshev polynomials:

$$\begin{aligned}
&|A_{(3,0)}| \le 4 = t_{3,3}t_{0,0}, && |A_{(2,1)}| \le 2 = t_{2,2}t_{1,1},\\
&|A_{(1,2)}| \le 2 = t_{1,1}t_{2,2}, && |A_{(0,3)}| \le 4 = t_{0,0}t_{3,3},\\
&|A_{(2,0)}| \le 2 = t_{2,2}t_{0,0}, && |A_{(1,1)}| \le 1 = t_{1,1}t_{1,1},\\
&|A_{(0,2)}| \le 2 = t_{0,0}t_{2,2}, && |A_{(1,0)}| \le 3 = |t_{3,1}|,\\
&|A_{(0,1)}| \le 3 = |t_{3,1}|, && |A_{(0,0)}| \le 1 = t_{0,0}.
\end{aligned}$$

Example 2. However, for $r = 2$ and $m = 4$, neither products nor any rational functions of coefficients of univariate Chebyshev polynomials are enough to sharply majorize all coefficients of $P_4^2 \in B_4^2$. A solution to the posed problem for the special case $r = 2$ and $m = 4$ reads as follows:

$$\begin{aligned}
&|A_{(4,0)}| \le 8 = t_{4,4}t_{0,0}, && |A_{(3,1)}| \le 4 = t_{3,3}t_{1,1},\\
&|A_{(2,2)}| \le 4 = t_{2,2}t_{2,2}, && |A_{(1,3)}| \le 4 = t_{1,1}t_{3,3},\\
&|A_{(0,4)}| \le 8 = t_{0,0}t_{4,4}, && |A_{(3,0)}| \le 4 = t_{3,3}t_{0,0},\\
&|A_{(2,1)}| \le 2 = t_{2,2}t_{1,1}, && |A_{(1,2)}| \le 2 = t_{1,1}t_{2,2},\\
&|A_{(0,3)}| \le 4 = t_{0,0}t_{3,3}, && |A_{(1,0)}| \le 3 = |t_{3,1}|,\\
&|A_{(0,1)}| \le 3 = |t_{3,1}|, && |A_{(0,0)}| \le 1 = t_{0,0}.
\end{aligned}$$

These estimates follow from inequalities (3) to (6).

$$|A_{(1,1)}| \le 2(1 + \sqrt{2}) = 4.82842\ldots$$

with equality if $P_4^2(x_1, x_2) = 2(1 + \sqrt{2})x_1x_2 + (\frac{-3}{2} - \sqrt{2})(x_1^3x_2 + x_1x_2^3)$. This estimate is deduced in [7, Example], see also [8, Example 3].

$$|A_{(2,0)}| \le 8 = |t_{4,2}|, \qquad |A_{(0,2)}| \le 8 = |t_{4,2}|.$$

These estimates are obtained as follows: $P_4^2 \in B_4^2$ implies that $|P_4^2(0, x_2)| \le 1$ resp. $|P_4^2(x_1, 0)| \le 1$ holds, and hence (1) may be applied to the univariate polynomials $P_4^2(0, x_2)$ and $P_4^2(x_1, 0)$, yielding $|A_{(2,0)}| \le 8$ and $|A_{(0,2)}| \le 8$.

Thus, for $r = 2$ the problem remains open for $m \ge 5$, and in general, for $r \ge 3$ and $m \ge 4$.

References

[1] P. Borwein and T. Erdélyi, "Polynomials and Polynomial Inequalities", Graduate Texts in Mathematics Vol. 161, Springer-Verlag, New York, 1995.

[2] M. I. Ganzburg, A Markov-type inequality for multivariate polynomials on a convex body, *J. Comput. Anal. Appl.* **4** (2002), 265–268.

[3] V. A. Markov, "On Functions Deviating Least from Zero in a Given Interval", Izdat. Akad. Nauk, St. Petersburg, 1892 [in Russian]; `URL:www.math.technion.ac.il/hat/fpapers/vmar.pdf`.

[4] W. A. Markoff, Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen, *Math. Ann.* **77** (1916), 213–258 [German translation of [3]].

[5] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, "Topics in Polynomials: Extremal Problems, Inequalities, Zeros", World Scientific, Singapore, 1994.

[6] I. P. Natanson, "Constructive Function Theory, Volume I: Uniform Approximation", F. Ungar Publ., New York, 1964 [Russian original 1949].

[7] H.–J. Rack, A generalization of an inequality of V. Markov to multivariate polynomials, II, *J. Approx. Theory* **40** (1984), 129–133.

[8] H.–J. Rack, On V. A. Markov's and G. Szegő's inequality for the coefficients of polynomials in one and several variables, *East J. Approx.* **14** (2008), 319–352.

[9] Q. I. Rahman and G. Schmeisser, "Analytic Theory of Polynomials", London Mathematical Society Monographs New Series Vol. 26, Oxford University Press, Oxford, 2002.

[10] M. Reimer, On multivariate polynomials of least deviation from zero on the unit cube, *J. Approx. Theory* **23** (1978), 65–69.

[11] Th. J. Rivlin, "Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory", 2nd ed., J. Wiley & Sons, New York, 1990.

[12] A. Schönhage, "Approximationstheorie", Walter de Gruyter, Berlin, 1971.

[13] A. F. Timan, "Theory of Approximation of Functions of a Real Variable", Pergamon Press, Oxford, 1963 [Russian original 1960].

Heinz-Joachim Rack
Steubenstrasse 26 a
58097 Hagen
GERMANY
E-mail: `heinz-joachim.rack@drrack.com`

CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 414-416
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

Open Problem

The Longest Polynomial in the Unit Disk

Heinz-Joachim Rack

In 1939 Erdös [4] conjectured that the m-th Chebyshev polynomial T_m of the first kind (with $T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x)$ for $m \geq 2$, and $T_0(x) = 1$, $T_1(x) = T_{-1}(x) = x$) is the extremizer for the arc-length functional on $I = [-1, 1] \subset \mathbb{R}$ among all real m-th degree polynomials $P_m \in \Phi_m = \operatorname{span}\{x^k : 0 \leq k \leq m\}$ whose graph on I is assumed to lie entirely in the unit square, i.e. among all $P_m \in B_m = \{P_m \in \Phi_m : |P_m(x)| \leq 1 \text{ for } x \in I\}$. This conjecture was proved by B. Bojanov in 1982, see [1], [2], [3, p. 31], [5, p. 600], [8, p. 149].

In 1970 Turán [5, p. 546], [8, p. 145], [10] proposed to investigate extremal problems not only for polynomials from B_m, but also for polynomials whose graph lies entirely in the unit disc, i.e., for

$$P_m \in G_m = \{P_m \in \Phi_m : |P_m(x)| \leq \sqrt{1 - x^2} \text{ for } x \in I\},$$

see also [9, pp. 110–111]; more generally, one may consider polynomials bounded on I by some non-negative "curved majorant". The function $\varphi(x) = \sqrt{1 - x^2}$ is called the "circular majorant" and we will denote the unit disk centered at the origin by C_2. We pose the problem to carry over the extremal arc-length property of $\pm T_m$ from the unit square to the unit disc, and ask:

Problem 1. *Is it true that among all $P_m \in G_m$ the polynomial $M_m = (T_m - T_{m-2})/2 \in G_m$ is (up to the sign) the longest in C_2, i.e.,*

$$L(P_m) = \int_{-1}^{1} \sqrt{1 + (P_m'(x))^2}\, dx \leq \int_{-1}^{1} \sqrt{1 + (M_m'(x))^2}\, dx = L(M_m) \qquad (1)$$

for all $P_m \in G_m$?

Note that M_m alternately touches $\pm\varphi(x)$ at the $m + 1$ points

$$x_{m,0} = -1, \qquad x_{m,i} = -\cos\frac{(2i - 1)\pi}{2m - 2}, \quad 1 \leq i \leq m - 1, \qquad x_{m,m} = 1.$$

For $m = 1$ we get $M_1(x) = 0$, and trivially the horizontal diameter is a longest straight line segment in C_2, with $L(M_1) = 2$.

For $m = 2$ we get $M_2(x) = -1+x^2$. An arbitrary second-degree polynomial (with positive leading coefficient) from class G_2 is necessarily of the form $P_{2,\gamma}(x) = -\gamma + \gamma x^2$ where $0 < \gamma \leq 1$. It follows from calculus that

$$L(P_{2,\gamma}) = \int_{-1}^{1} \sqrt{1 + (P'_{2,\gamma}(x))^2}\, dx = \sqrt{1+4\gamma^2} + \frac{1}{2\gamma} \operatorname{arcsinh} 2\gamma,$$

and this parameterized value will be maximized if we choose $\gamma = 1$, yielding $L(M_2) = 2.95788\ldots$. Thus (1) holds for $m = 2$, too. Is (1) true for $m \geq 3$?

Problem 2. *More generally, is it true that among all $P_m \in \Phi_m$ satisfying*

$$|P_m(x)| \leq \sqrt{1 + (\alpha^2 - 1)x^2} \quad \textit{for} \quad x \in I$$

("Videnskii majorant", see also [7]), the polynomial $P_m = M_{m,\alpha}$ is the longest on I, i.e. has maximal arc-length on I, for every $\alpha \in [0, 1]$ and every $m \geq 2$? Here,

$$M_{m,\alpha} = \frac{(\alpha+1)T_m + (\alpha-1)T_{m-2}}{2}.$$

The case $\alpha = 0$ takes us back to C_2 and the case $\alpha = 1$ comes down to the already solved P. Erdös / B. Bojanov constellation in the unit square.

The problem of determining the longest polynomial within C_2 is meaningful even if the condition $P_m \in G_m$ is violated: Consider the diameters of C_2. Their arc-length is 2 and they are longer than any straight line segment (secant) within C_2 which does not pass through the midpoint $(0, 0)$ of C_2. In generalizing this basic extremal property from plane Euclidean geometry, we interpret the diameters and the straight line segments (except the vertical ones) as traces of first-degree polynomials passing through C_2 and correspondingly we ask, in a first step, for the longest traces of second-degree polynomials traversing C_2. Due to the symmetries of C_2 we confine ourselves to parabolas $P_2(x) = a_0+a_2x^2$ with $a_2 > 0$.

It might be suspected that the longest parabolic segment within C_2 will be an improper one which degenerates into the vertical diameter traversed twice, and hence would have arc-length $= 4$. But this is not the case since we have found that the longest parabola P_2^* belonging to C_2 has arc-length $4.00267\ldots$ there. It is given by $P_2^*(x) = -1 + 94.09128\ldots x^2$, satisfies $|P_2^*(x)| \leq \varphi(x)$ for $x \in [-A^*(2), A^*(2)]$, where $A^*(2) = 0.14540\ldots$, and alternates at the endpoints and at the midpoint of that subinterval of I, see [6]. It is also shown in [6] that P_2^* transforms into T_2 if we continuously transform the unit disc into the unit square by considering $\partial C_q = \pm(1 - |x|^q)^{\frac{1}{q}}$ and letting $q \geq 1$ tend to infinity.

Problem 3. *We pose the additional problem to determine the longest polynomial segment within C_2, i.e., the longest trace of an m-th degree polynomial ($m \geq 3$) traversing C_2, if it exists.*

Hint for $m = 3$: Consider $P_3(x) = Dx + Cx^3$ on the interval $[-A(3), A(3)]$, where $A = A(3) \in (0, 1)$ is given. The coefficient C is the largest solution of the quartic equation

$$C^4 + \frac{4A^2 - 3}{A^3\sqrt{1 - A^2}} C^3 + \frac{15 - 16A^4}{4A^6(A^2 - 1)} C^2 + \frac{-1 - 4A^2}{A^9\sqrt{1 - A^2}} C + \frac{1}{A^{10}(A^2 - 1)} = 0,$$

and the coefficient D is given by $D = \frac{\sqrt{1-A^2}}{A} - CA^2$. Then evaluate the arc-length of P_3 on $[-A(3), A(3)]$, and continue by letting A vary in $(0, 1)$.

References

[1] B. Bojanov, Proof of a conjecture of Erdös about the longest polynomial, *Proc. Am. Math. Soc.* **84** (1982), 99–103.

[2] B. Bojanov, Markov-type inequalities for polynomials and splines, *in* “Approximation Theory X: Abstract and Classical Analysis” (C. K. Chui, L. L. Schumaker, and J. Stöckler, Eds.), pp. 31–90, Vanderbilt University Press, Nashville, 2002.

[3] P. Borwein and T. Erdélyi, “Polynomials and Polynomial Inequalities”, Graduate Texts in Mathematics Vol. 161, Springer-Verlag, New York, 1995.

[4] P. Erdös, An extremum-problem concerning trigonometric polynomials, *Acta. Litt. Sci. Univ. Szeged, Sect. Sci. Math.* **9** (1939), 113–115.

[5] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, “Topics in Polynomials: Extremal Problems, Inequalities, Zeros”, World Scientific, Singapore, 1994.

[6] H.–J. Rack, Der quadratische Durchmesser eines Kreises, *Praxis der Mathematik* (Aulis Verlag, Cologne/Köln) **26** (1984), 109–111.

[7] H.–J. Rack, On the length and height of Chebyshev polynomials in one and two variables, *East J. Approx.* **16** (2010), 35–91.

[8] Th. J. Rivlin, “Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory”, 2nd ed., J. Wiley & Sons, New York, 1990.

[9] V. T. Sós, Turbulent years: Erdös in his correspondence with Turán from 1934 to 1940, *in* “Paul Erdös and His Mathematics I” (G. Halász et al., Eds.), pp. 85–146, Bolyai Society Mathematical Studies vol. 11, Springer, Berlin, 2002.

[10] P. Turán, Unsolved Problem on p. 360, *in* “Constructive Function Theory” (B. Penkov and D. Vaĉov, Eds.), Proc. Conf. Varna 1970, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1972.

Heinz-Joachim Rack

Steubenstrasse 26 a
58097 Hagen
GERMANY
E-mail: heinz-joachim.rack@drrack.com

CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 417-420
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

Open Problem

Norm of Extension from a Circle to a Triangle

Szilárd Gy. Révész

This problem I posed at an approximation theory conference first in Bommerholz, Germany in 2005. In 2007 it was also incorporated into the collection of open problems [1], but this is the first time it also gets printed.

In recent years we have seen a number of quite good estimates on derivatives of multivariate polynomials P under condition of controlling the maximum norm of P on say a convex, or a symmetric convex body of $\mathbb{R}^N$. For details we refer to [4] and to our survey [9] in this very volume. The problem, if the otherwise converging estimates are really sharp, seem to be the next question to answer. The following simple-looking question is related to lower estimations, that is, sharpness questions of the Bernstein problem.

Problem. *Let $\Delta \subset \mathbb{R}^2$ be any triangle, with its inscribed circle denoted by $\mathcal{C}$. Determine (at least asymptotically, when $n \to \infty$)*

$$M_n(\Delta) := \sup_{\substack{P \in \mathcal{P}_n \\ \|P|_{\mathcal{C}}\|=1}} \left\{ \inf \|Q(x,y)\|_{C(\Delta)} : Q|_{\mathcal{C}} = P|_{\mathcal{C}},\ Q \in \mathcal{P}_n \right\}.$$

Equivalently, determine (at least asymptotically)

$$M_n(\Delta) = \sup_{\substack{T \in \mathcal{T}_n \\ \|T\|_{\mathbb{T}}=1}} \left\{ \inf \|Q(x,y)\|_{C(\Delta)} : Q(\cos t, \sin t) = T(t) \right\}.$$

Clearly, knowing the minimax type quantity $M_n(\Delta)$, we can then determine, by suitable affine transformations, the same quantities for any pair of triangles and inscribed ellipses $\mathcal{E}$: we just have to consider the affine transformation which takes $\mathcal{E}$ to a circle.

The strongest possible hypothesis would be $M_n(\Delta) = 1+o(1)$, when $n \to \infty$, for all triangles. However, not even the question if $M_n(\Delta) \sim M_n(\Delta_0)$ $(n \to \infty)$, if Δ_0 is say the standard triangle, seems to be simple. It may well be, in particular when these quantities do not converge to 1, that they are indeed different for different triangles. A warning sign may be the following. Naidenov found [6] – also using computer search – several counterexamples to a conjecture of mine with Sarantopoulos. The conjecture was to say that gradients of polynomials may be subject to an estimate with the so-called generalized Minkowski functional in place of $G_K(x, y)$ below. Now what happened is that the counterexamples showed varying degree of failure, with constants from something like one percent in case of Δ_0 to "rather large" (say 10–20%) deficiencies when Δ is a rather elongated, flat triangle. As may be seen from what follows, this phenomenon may suggest a problem with the above extension constants.

As already noted, my interest in the question comes from the multivariate Bernstein problem, that is, estimates from above the directional derivative of a polynomial P, say of norm 1 on a convex body on $K \subset \mathbb{R}^d$, at a point $x \in K^o$ and in a direction y. The known estimates have the form

$$|D_y P(x)| \leq \deg P \sqrt{\|P\|^2_{C(K)} - P(x)^2}\, G_K(x, y),$$

where this $G_K(x, y)$ are constants only depending on the geometry, i.e. the body K and the points x, y, but independent from P. That is, the estimation separates the effects of geometry and analysis, giving the degree and the so-called "Bernstein-Szegő factor" (the square root term) as the result of the "analysis inputs", plus another factor, which is a purely geometry-related quantity.

In fact, we have basically two types of quantities for $G_K(x, y)$, one being the semiderivative $(V_K)'_y(x) := \lim_{t \to 0+} V_K(x + ity)/t$ of the Siciak-Zaharjuta extremal function, and the other the reciprocal of the (tangentially) best inscribed ellipse constant $E_K(x, y)$. For details see [9]. Now these quantities are rarely known precisely – a nice exception being when K is a simplex, see [5] – but one of the astonishing recent findings was that *they are equal* in case of any convex body K, interior point x and directional vector y [4]. This of course strengthened the expectation that these estimates then may as well be "the right ones", that is, sharp. In fact, in the form of the Siciak-Zaharjuta extremal function semiderivative this was already conjectured by Baran [2].

So these Bernstein-type estimates are conjecturally best possible, at least when the degrees are not restricted, but we consider all polynomials of all degrees. To arrive at this, one approach would be to show that the estimates in the course of proofs are sharp. So let us have a closer look at the method of the inscribed ellipses, which yields $G_K(x, y) = 1/E_K(x, y)$. Here we consider an *inscribed ellipse* $\mathcal{E} \subset K$, and estimate the derivative by considering $T := P|_{\mathcal{E}}$, which then has a derivative along the curve. This is then used to estimate $|D_y P(x)|$. For getting the best estimate, we choose the inscribed ellipse $\mathcal{E}$ (in a certain well-specified sense) maximal.

So now we are to see that once restricting to $\mathcal{E}$ or $\mathcal{C}$, we do not loose anything. In the course of proof we always estimate sharply, except when the yield of the trigonometrical Bernstein inequality, which is of the form $n\sqrt{\|T\|^2_{C(\mathcal{E})} - T^2(t_0)}$, where $T = P|_{\mathcal{E}}$ is estimated by $n\sqrt{\|P\|^2_{C(K)} - P^2(x)}$. That is, in the Bernstein-

Szegő factor we substituted $\|P\|_{C(K)}$ for $\|T\|_{C(\mathcal{E})}$. Now this is put in the focus by the above extension problem, at least when K is a triangle. But, although the question seems to be rather particular, as for the choice of $K = \Delta$, note that it is already shown that sharpness of the above Bernstein type inequalities for this particular case already entail sharpness for all convex bodies of dimension 2, see the closing remark of [4]. One may then pose the analogous question to Δ being a simplex and the inscribed ellipse $\mathcal{E}$, or circle $\mathcal{C}$ being maximal in the appropriate sense.

Of course, it may well happen that for some polynomials P or T the extension increases the norm, while for others it does not. So if $M(\Delta)$ is large, it still may happen that in the case when the trigonometrical Bernstein inequality is sharp – when $T(t) = \cos(n(t - t_0))$ – then the extension has small norm. That also means that the question in its general form requires more, than is necessary for the affirmative answer in question of sharpness of the currently known Bernstein type inequalities.

Let us note one more related thing, which, however well-known to some, seems to cause surprise to others. That observation is that if we now denote by D the disk, encircled by the circle $\mathcal{C}$, then defining $M(D)$ as the corresponding extension quantity to D, we always have $M(D) = 1$. So extending a polynomial *into* $\mathcal{C}$ does not increase its norm at all. This comes from the fact that we always have some *harmonic* polynomial extensions, which then satisfy the maximum principle and thus $\max_{\mathcal{C}} |Q| = \max_D |Q|$. This fact is hard to look up in the literature, so D. Burns at al. describes an elegant proof – which they attribute to D. Khavinson – on [3, page 101].

The argument runs as follows. Fix $\mathcal{C}$ to be the unit circle together with a polynomial $P \in \mathcal{P}_n = \mathcal{P}_n(\mathbb{R}^2)$ to be extended, and consider the mapping $T : p \to \Delta(pq)$, Δ being the Laplace operator, and $q(x, y) := (1 - x^2 - y^2)$. This mapping is now clearly a linear mapping from $\mathcal{P}_m \to \mathcal{P}_m$, for any $m \in \mathbb{N}$, and it is *injective*; for if $T(p) = 0$, then pq satisfies the Laplace equation, i.e. harmonic, but as it vanishes on the boundary $\mathcal{C}$ (for there $q(x, y) \equiv 0$), by the maximum principle the harmonic function pq vanishes everywhere and is thus also $p \equiv 0$. But as $\mathcal{P}_m$ is a finite dimensional vector space, $\ker T = 0$ means that T is also *surjective*. We take now $m = n-2$, and $R := \Delta P \in \mathcal{P}_{n-2}$. Because T is surjective, there is $r \in \mathcal{P}_{n-2}$ such that $Tr = R$, that is, $\Delta(qr) = \Delta P$. Clearly $Q := P - qr$ is then the right polynomial to pick, for $\Delta Q \equiv 0$ and $Q|_{\mathcal{C}} = P|_{\mathcal{C}}$.

For another discussion of extensions, and harmonic extensions in particular, see also [7], where the rather similar question of finding sharp norm estimates for extensions from $\mathcal{C}$ to a concentric circle $\mathcal{C}_r$ of radius r is solved. (This work also settles the above existence question of a harmonic extension, even if in a more involved way.) I would like to thank this reference to Professor V. V. Arestov.

I would say that the minimax problem of determining $M_n(\Delta)$ is certainly of some degree of difficulty and of independent interest, too.

References

[1] Problèmes Ouverts, *Groupe de travail "Theorie Analytique Des Polynômes Et Analyse Harmonique"* organisé par Aline Bonami, Szilárd Révész, Bahman Saffari, 2006-2007.
`http://www.renyi.hu/~revesz/OpenProblems.pdf`

[2] M. Baran, Complex equilibrium measure and Bernstein type theorems for compact sets in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* **123** (1995), no. 2, 485–494.

[3] D. Burns, N. Levenberg, and S. Ma'u, Pluripotential theory for convex bodies in $\mathbb{R}^N$, *Math. Z.* **250** (2005), no. 1, 91–111.

[4] D. Burns, N. Levenberg, S. Ma'u, and Sz. Révész, Monge-Ampère measures for convex bodies and Bernstein-Markov type inequalities, *Trans. Amer. Math. Soc.* **362** (2010), 6325–6340.

[5] L. Milev and Sz. Révész, Bernstein's inequality for multivariate polynomials on the standard simplex, *J. Inequal. Appl.* (2005), no. 2, 145–163.

[6] N. Naidenov, Note on a conjecture about Bernstein-type inequalities for multivariate polynomials, *Analysis Math.* **33** (2007), no. 1, 55–62.

[7] A. V. Parfenenkov, The best extension of algebraic polynomials from the unit circle, *Proc. Steklov Inst. Math.* (2009), suppl. 2, 194–204.

[8] Sz. Révész, A comparative analysis of Bernstein type estimates for the derivative of multivariate polynomials, *Ann. Polon. Math.* **88** (2006), no. 3, 229–245.

[9] Sz. Révész, Conjectures and results on the multivariate Bernstein inequality on convex bodies, *in* "Constructive Theory of Functions, Sozopol 2010" (G. Nikolov and R. Uluchev, Eds.), pp. 318–353, Prof. Marin Drinov Academic Publishing House, Sofia, 2012.

Szilárd Gy. Révész

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
1364 Budapest, P.O.B. 127
HUNGARY
E-mail: `revesz@renyi.hu`