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Natural Weights for Uniform Approximation by the Szász-Mirakjan Operator*

Borislav R. Draganov and Kamen G. Ivanov

We establish a sharp characterization of the error of the Szász-Mirakjan operator in uniform norm with power-type weights in terms of a K-functional. The weight exponents are optimal. We also state a sharp characterization of the K-functional by means of the classical unweighted fixed-step modulus of smoothness.

Keywords and Phrases: Szász-Mirakjan operator, modulus of smoothness, K-functional, rate of convergence, weighted uniform approximation.

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1. Introduction

1.1. Notations and Main Result

The Szász-Mirakjan operator for a function f defined on $[0,\infty)$ is given by

$$S_n f(x) = \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{n,k}(x), \quad s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \qquad n \ge 1, \quad x \ge 0.$$

Here n is not necessarily an integer.

In order to describe the approximation properties of S_n we need a number of function spaces. Let $C[0,\infty)$ denote the space of all continuous functions on $[0,\infty)$ and $L_\infty[0,\infty)$ denote the Lebesgue measurable and essentially bounded on $[0,\infty)$ functions with the essential supremum norm $\|\cdot\|$. For continuous functions $\|\cdot\|$ coincides with the uniform norm on $[0,\infty)$. Also, we denote the first derivative operator by $D = \frac{d}{dx}$, thus $Dg(x) = g'(x)$ and $D^2 g(x) = g''(x)$. We further set $\varphi(x) = x$. Now, for a weight function w on $(0,\infty)$ we put

$$C(w)(0,\infty) = \{f \in C[0,\infty) : wf \in L_\infty[0,\infty)\},$$

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$$W^2(w\varphi)(0,\infty) = \{g \in AC_{loc}(0,\infty) : Dg \in AC_{loc}(0,\infty), w\varphi D^2 g \in L_\infty[0,\infty)\},$$
$$W^3(w\varphi^{3/2})(0,\infty) = \{g \in AC_{loc}(0,\infty) : Dg, D^2 g \in AC_{loc}(0,\infty), w\varphi^{3/2} D^3 g \in L_\infty[0,\infty)\},$$

where $AC_{loc}(0,\infty)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,\infty)$. Finally, π_1 stands for the set of algebraic polynomials of first degree.

In [13] the second author introduces the concept of a *natural weight* for an approximation operator.

Definition 1.1. A weight w is called a *natural weight* for approximation by a sequence Q_n of operators in a specified norm if the norm of the weighted approximation error $w(f - Q_n f)$ allows matching direct and strong inverse estimates for the widest reasonable class of functions f.

Here we shall consider weights on $(0,\infty)$ of the form

$$w(x) = w(\gamma_0, \gamma_\infty; x) = \Big(\frac{x}{1+x}\Big)^{\gamma_0} (1+x)^{\gamma_\infty} \tag{1.1}$$

and show that they are natural for the uniform approximation by the Szász-Mirakjan operator for any $\gamma_0 \in [-1,0]$ and $\gamma_\infty \in \mathbb{R}$. More precisely, let the K-functional $K_w(f,t)$ be defined for $f \in C(w)(0,\infty) + \pi_1$ and $t > 0$ by

$$K_w(f,t) = \inf\left\{\|w(f-g)\| + t\|w\varphi D^2 g\| : g \in W^2(w\varphi)(0,\infty)\right\}. \tag{1.2}$$

We establish the following characterization.

Theorem 1.1. *Let $w = w(\gamma_0, \gamma_\infty)$ be given by* (1.1) *with $\gamma_0 \in [-1,0]$ and $\gamma_\infty \in \mathbb{R}$. Then for all $f \in C(w)(0,\infty) + \pi_1$ and $n \geq 1$ there holds*

$$\|w(f - S_n f)\| \sim K_w\Big(f, \frac{1}{n}\Big).$$

The relation $\psi(f,t) \sim \theta(f,t)$ means that there exists a positive constant independent of f and t such that

$$c^{-1}\theta(f,t) \leq \psi(f,t) \leq c\,\theta(f,t).$$

Let us note that the range of γ_0 cannot be reasonably extended. Indeed, if $\gamma_0 < -1$ then we must assume that $f(x) = 0$ in a neighbourhood of 0, otherwise $S_n f$ will not be bounded. On the other hand, if $\gamma_0 > 0$, then $f(x)$ is not generally defined at 0 and hence $S_n f$ is not defined. Moreover, we cannot settle this case by restricting to functions $f \in C[0,\infty)$ because then S_n is not a bounded operator in the weighted uniform norm.

1.2. Characterization of the Weighted K-functionals

The K-functional $K_w(f,t)$ is equivalent to the Ditzian-Totik modulus of smoothness if $\gamma_0 = 0$ (see [4, Theorem 6.1.1]), whereas for $\gamma_0 < 0$ it is only weakly equivalent to the so called weighted main-part modulus (see [4, Section 6.2]). Also, this K-functional with $\gamma_0 = \gamma_\infty = 0$ was shown to be equivalent to a modulus introduced by the second author in [12]. In a series of papers [5, 6, 7, 8] the authors introduced and studied moduli of smoothness which are equivalent to weighted K-functionals like $K_w(f,t)$. They are defined by means of the classical unweighted fixed-step modulus and a continuous linear transform of the function. In particular, we can characterize $K_w(f,t)$ by such moduli by first splitting the singularities

$$K_w(f,t) \sim K_w(f,t)_{[0,2]} + K_w(f,t)_{[1,\infty)},$$

which is valid for all $f \in C(w)(0,\infty)$ and $t \in (0,1]$. The K-functional $K_w(f,t)_I$ is defined as in (1.2) with the supremum taken on the interval I. By [8, Theorem 6.2 and 6.8] we have

$$K_w(f,t)_{[0,2]} \sim \omega_2(\mathcal{A}_{\gamma_0} f,t)_{[0,2]} \tag{1.3}$$

where $\omega_2(f,t)_{[0,2]}$ is the classical modulus of smoothness of second order in uniform norm on the interval $[0,2]$ and

$$\mathcal{A}_{\gamma_0} f(x) = \begin{cases} f(x^2) - x\displaystyle\int_1^x y^{-2} f(y^2)\,dy & \text{for } \gamma_0 = 0, \\ x^{2\gamma} f(x^2) + \dfrac{4\gamma^2-1}{5}\,x\displaystyle\int_1^x y^{2\gamma-2} f(y^2)\,dy & \\ \qquad -\dfrac{4(\gamma_0+2)(\gamma_0+3)}{5}\,x^{-4}\displaystyle\int_0^x y^{2\gamma+3} f(y^2)\,dy & \text{for } -1 < \gamma_0 < 0, \\ x^{-2} f(x^2) + 3x\displaystyle\int_1^x y^{-4} f(y^2)\,dy & \text{for } \gamma_0 = -1, \end{cases}$$

for $x \in [0,2]$. Relation (1.3) holds provided that wf has a limit at 0, which is, moreover, equal to 0 if $\gamma_0 \in (-1,0)$.

Similarly, one can construct a modulus equivalent to $K_w(f,t)_{[1,\infty)}$. We shall present this case in [9].

1.3. Related Works

Many mathematicians investigated the approximation behaviour of the Szász-Mirakjan operator. First, Becker [2] established direct and weak converse estimates for $w(x) = (1+x)^{-N}$, $N \in \mathbb{N}$. Amonov [1] proved a weak equivalence result for $w(x) = (1+x)^{\gamma}$, $\gamma \le 0$. He studied also more general weights. Holhoş [11] introduced a general approach to verifying direct estimates in weighted uniform norm. As an application he obtained a direct result for S_n in the case $w(x) = (1+x)^{\gamma}$, $\gamma \le 0$.

As for strong converse estimates, Totik [15] established the assertion of Theorem 1.1 in the unweighted case $w = 1$. Quite recently Finta [10] developed a general method, which in particular enable him to verify a two-term strong converse inequality for the weights $w(x) = x^\gamma$, $\gamma \in (-1, 0]$.

As can be seen, most researchers were interested in studying the rate of approximation in weighted spaces that are larger than $C[0,\infty)$ and thus allowing for more function to be approximated. In our opinion, it is also worthwhile finding in which weighted spaces the norm of the approximation error can be neatly characterized since that reveals intrinsic properties of the operator.

2. Proof of Theorem 1.1

First, let us recall several basic properties of the Szász-Mirakjan operator:

$$S_n \text{ is a positive linear operator;} \tag{2.1}$$

$$S_n f = f \qquad \forall f \in \pi_1; \tag{2.2}$$

$$S_n\big((\circ - x)^2\big)(x) = \frac{x}{n}, \qquad S_n\big((\circ - x)^3\big)(x) = \frac{x}{n^2}; \tag{2.3}$$

$$S_n\big((\circ - x)^4\big)(x) = d_1\frac{x^2}{n^2} + d_2\frac{x}{n^3}, \tag{2.4}$$

$$S_n\big((\circ - x)^6\big)(x) = d_3\frac{x^3}{n^3} + d_4\frac{x^2}{n^4} + d_5\frac{x}{n^5}; \tag{2.5}$$

$$S_n f = \big(S_1(f_{1/n})\big)_n, \qquad \text{where } F_\nu(x) = F(\nu x),\ \nu > 0. \tag{2.6}$$

Above, $d_1, \dots, d_5$ are constants independent of n. For (2.3)–(2.5) see e.g. [4, (9.1.13) and Lemma 9.5.5].

The proof of Theorem 1.1 is based on [3, Theorem 4.1 and (1.2)–(1.4)]. So, we proceed to the verification of their hypotheses. The first concerns the boundedness of the Szász-Mirakjan operator in weighted uniform norm. Below $c, c_1, c_2, \dots$ denote positive constants which do not depend on f and n.

Proposition 2.1. *Let $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) with $\gamma_0 \in [-1, 0]$ and $\gamma_\infty \in \mathbb{R}$. Then*

$$\|wS_n f\| \le c_1 \|wf\| \qquad \forall f \in C(w)(0,\infty) \quad \forall n \ge 1.$$

Proof. Using the positivity of S_n, we get

$$|S_n f(x)| \le \big(S_n(w^{-1})\big)(x)\|wf\|.$$

Thus it remains to show that

$$\big(S_n(w^{-1})\big)(x) \le c\, w(x)^{-1}. \tag{2.7}$$

To this end we apply a routine technique based on Hölder's inequality. If $\gamma_0 \neq \gamma_\infty$, we fix $r \in \mathbb{Z}$ such that $r/(\gamma_0 - \gamma_\infty) \geq 1$; if $\gamma_0 = \gamma_\infty$, we set $r = 1$. Then consecutive applications of Hölder's inequality yield

$$\begin{aligned}
&\sum_{k\geq 0}\Big(\frac{k}{n}\Big)^{-\gamma_0}\Big(1+\frac{k}{n}\Big)^{\gamma_0-\gamma_\infty} s_{n,k}(x)\\
&\leq \left\{\sum_{k\geq 0}\frac{k}{n}\Big(1+\frac{k}{n}\Big)^{\gamma_0-\gamma_\infty} s_{n,k}(x)\right\}^{-\gamma_0}\left\{\sum_{k\geq 0}\Big(1+\frac{k}{n}\Big)^{\gamma_0-\gamma_\infty} s_{n,k}(x)\right\}^{1+\gamma_0}\\
&\leq \left\{\sum_{k\geq 0}\frac{k}{n}\Big(1+\frac{k}{n}\Big)^{r} s_{n,k}(x)\right\}^{\gamma_0(\gamma_\infty-\gamma_0)/r}\left\{\sum_{k\geq 0}\frac{k}{n}\, s_{n,k}(x)\right\}^{-\gamma_0(1+(\gamma_\infty-\gamma_0)/r)}\\
&\times \left\{\sum_{k\geq 0}\Big(1+\frac{k}{n}\Big)^{r} s_{n,k}(x)\right\}^{(1+\gamma_0)(\gamma_0-\gamma_\infty)/r}\left\{\sum_{k\geq 0} s_{n,k}(x)\right\}^{(1+\gamma_0)(1+(\gamma_\infty-\gamma_0)/r)}.
\end{aligned} \tag{2.8}$$

It is known that (cf. [4, p. 163]),

$$\sum_{k\geq 0}\Big(1+\frac{k}{n}\Big)^{r} s_{n,k}(x) \leq c\,(1+x)^r, \qquad r \in \mathbb{Z}; \tag{2.9}$$

hence also

$$\sum_{k\geq 0}\frac{k}{n}\Big(1+\frac{k}{n}\Big)^{r} s_{n,k}(x) \leq c\,x\sum_{k\geq 0}\Big(1+\frac{k}{n}\Big)^{r} s_{n,k}(x) \leq c\,x(1+x)^r. \tag{2.10}$$

Now, (2.7) readily follows from (2.2) and (2.8)-(2.10). □

Next, we establish a Jackson-type inequality.

Proposition 2.2. *Let $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) with $\gamma_0 \in [-1, 0]$ and $\gamma_\infty \in \mathbb{R}$. Then*

$$\|w(g - S_n g)\| \leq \frac{c_2}{n}\,\|w\varphi D^2 g\| \qquad \forall g \in W^2(w\varphi)(0,\infty) \quad \forall n \geq 1.$$

Proof. Since $g \in W^2(w\varphi)(0,\infty)$, then $g \in C[0,\infty)$. By Taylor's formula we have for $x > 0$ and $t \geq 0$ that

$$g(t) = g(x) + (t - x)Dg(x) + \int_x^t (t-u)D^2 g(u)\,du.$$

We apply S_n to both sides of this identity with regard to the variable t. Then, taking into account that S_n preserves the linear functions (see (2.2)), we get

$$S_n g(x) = g(x) + S_n\left(\int_x^{\circ}(\circ - u)D^2 g(u)\,du\right)(x).$$

Next, by the positivity of S_n, we derive the estimate

$$\begin{aligned} |g(x) - S_n g(x)| &\le S_n \left(\int_x^{\circ} (\circ - u) |D^2 g(u)| \, du \right)(x) \\ &\le R_{1,n}(x) \, \|w\varphi D^2 g\|, \end{aligned} \tag{2.11}$$

where we have set

$$R_{1,n}(w;x) = S_n \left(\int_0^{\infty} Q_1(\circ, u, x) w(u)^{-1} \, du \right)(x)$$

and $Q_1(t,u,x) = |t-u|u^{-1}$ for u between t and x and $Q_1(t,u,x) = 0$ otherwise.

Thus it is sufficient to show that

$$R_{1,n}(w;x) \le \frac{c}{n} \, w(x)^{-1}. \tag{2.12}$$

Just as in the proof of Proposition 2.1 we get

$$\begin{aligned} R_{1,n}(w(\gamma_0,\gamma_\infty);x) &\le R_{1,n}(w(-1,-r-1);x)^{\gamma_0(\gamma_\infty-\gamma_0)/r} \\ &\quad \times R_{1,n}(w(-1,0);x)^{-\gamma_0(1+(\gamma_\infty-\gamma_0)/r)} \\ &\quad \times R_{1,n}(w(0,-r);x)^{(1+\gamma_0)(\gamma_0-\gamma_\infty)/r} \\ &\quad \times R_{1,n}(w(0,0);x)^{(1+\gamma_0)(1+(\gamma_\infty-\gamma_0)/r)}. \end{aligned} \tag{2.13}$$

We estimate separately each of the $R_{1,n}$'s above. By (2.3) we immediately get

$$R_{1,n}(w(-1,0);x) \le S_n\big((\circ - x)^2\big)(x) = \frac{x}{n}. \tag{2.14}$$

Next, we observe that for u between t and x there hold

$$\frac{|t-u|}{u} \le \frac{|t-x|}{x}, \qquad (1+u)^r \le (1+x)^r + (1+t)^r. \tag{2.15}$$

The first of these inequalities is verified directly and the second one is trivial. The first inequality above and (2.2) yield

$$R_{1,n}(w(0,0);x) \le \frac{1}{x} \, S_n\big((\circ - x)^2\big)(x) = \frac{1}{n}. \tag{2.16}$$

Further, by means of the second inequality in (2.15) we arrive at

$$\begin{aligned} R_{1,n}(w(-1,-r-1);x) &\le (1+x)^r S_n\big((\circ - x)^2\big)(x) \\ &\quad + S_n\big((\circ - x)^2 (1+\circ)^r\big)(x). \end{aligned} \tag{2.17}$$

Let us consider $S_n\big((\circ - x)^2(1+\circ)^r\big)(x)$. For $nx \ge 1$ we apply the Cauchy inequality, (2.4) and (2.9) to get

$$\begin{aligned} S_n((\circ - x)^2(1+\circ)^r)(x) &\le \sqrt{S_n\big((\circ - x)^4\big)(x)} \sqrt{S_n\big((1+\circ)^{2r}\big)(x)} \\ &\le \frac{c\,x(1+x)^r}{n}. \end{aligned}$$

To verify this inequality for $nx \le 1$ we proceed as follows:

$$\begin{aligned}
S_n\big((\circ - x)^2(1+\circ)^r\big)(x) &= \sum_{k\ge 0}\Big(\frac{k}{n}-x\Big)^2\Big(1+\frac{k}{n}\Big)^r s_{n,k}(x)\\
&= \sum_{k\ge 0}\Big(\frac{k}{n}\Big)^2\Big(1+\frac{k}{n}\Big)^r s_{n,k}(x) - 2x\sum_{k\ge 0}\frac{k}{n}\Big(1+\frac{k}{n}\Big)^r s_{n,k}(x)\\
&\quad + x^2\sum_{k\ge 0}\Big(1+\frac{k}{n}\Big)^r s_{n,k}(x)\\
&\le c\,x\sum_{k\ge 0}\frac{k}{n}\Big(1+\frac{k}{n}\Big)^r s_{n,k}(x) + c\Big(\frac{x}{n}+x^2\Big)\sum_{k\ge 0}\Big(1+\frac{k}{n}\Big)^r s_{n,k}(x)\\
&\le c\Big(\frac{x}{n}+x^2\Big)(1+x)^r \le \frac{c\,x(1+x)^r}{n},
\end{aligned}$$

as at the last but one step we have applied (2.9) and (2.10).

Thus we have established the estimate

$$S_n\big((\circ - x)^2(1+\circ)^r\big)(x) \le \frac{c\,x(1+x)^r}{n}, \qquad x \ge 0. \tag{2.18}$$

Now, (2.17), (2.3) and (2.18) imply

$$R_{1,n}(w(-1,-r-1);x) \le \frac{c\,x(1+x)^r}{n}. \tag{2.19}$$

Finally, by means of (2.15) we get

$$R_{1,n}(w(0,-r);x) \le \frac{(1+x)^r}{x}\, S_n\big((\circ - x)^2\big)(x) + \frac{1}{x}\, S_n\big((\circ - x)^2(1+\circ)^r\big)(x),$$

which along with (2.3) and (2.18), gives

$$R_{1,n}(w(0,-r);x) \le \frac{c(1+x)^r}{n}, \qquad x \ge 0. \tag{2.20}$$

The estimate (2.12) follows from (2.13), (2.14), (2.16), (2.19) and (2.20). The proof of the proposition is completed. □

We shall also need the following Voronovskaya-type inequality.

Proposition 2.3. *Let $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) with $\gamma_0 \in [-1, 0]$ and $\gamma_\infty \in \mathbb{R}$. Then*

$$\Big\| w\Big(S_n g - g - \frac{1}{2n}\,\varphi D^2 g\Big)\Big\| \le \frac{c_3}{n^{3/2}}\,\|w\varphi^{3/2}D^3 g\| \ \ \forall g \in W^3(w\varphi^{3/2})(0,\infty) \ \ \forall n \ge 1.$$

Proof. The proof is quite similar to the one of the previous proposition. We first use Taylor's formula to get the representation

$$g(t) = g(x) + (t-x)Dg(x) + \frac{(t-x)^2}{2}D^2 g(x) + \frac{1}{2}\int_x^t (t-u)^2 D^3 g(u)\,du.$$

We apply S_n to both sides of this identity with regard to the variable t. Then, taking into account (2.2) and (2.3), we get

$$S_n g(x) = g(x) + \frac{1}{2n}\,\varphi(x) D^2 g(x) + \frac{1}{2}\,S_n\left(\int_x^{\circ} (\circ - u)^2 D^3 g(u)\,du\right)(x).$$

Next, by the positivity of S_n, we derive the estimate

$$\begin{aligned}\Big|S_n g(x) - g(x) - \frac{1}{2n}\,\varphi(x)D^2 g(x)\Big| \\ &\le \frac{1}{2}\,S_n\left(\Big|\int_x^{\circ}(\circ - u)^2 |D^3 g(u)|\,du\Big|\right)(x) \qquad (2.21)\\ &\le \frac{1}{2}\,R_{2,n}(x)\,\|w\varphi^{3/2}D^3 g\|,\end{aligned}$$

where we have set

$$R_{2,n}(w;x) = S_n\left(\int_0^{\infty} Q_2(\circ, u, x) w(u)^{-1}\,du\right)(x)$$

and $Q_2(t,u,x) = (t-u)^2 u^{-3/2}$ for u between t and x and $Q_2(t,u,x) = 0$ otherwise.

We shall show that

$$R_{2,n}(w(\gamma_0,\gamma_\infty);x) \le \frac{c}{n^{3/2}}\,w(x)^{-1}. \qquad (2.22)$$

As in the proof of the previous propositions we get

$$\begin{aligned}R_{2,n}(w(\gamma_0,\gamma_\infty);x) &\le R_{2,n}(w(-1,-r-1);x)^{\gamma_0(\gamma_\infty-\gamma_0)/r}\\ &\times R_{2,n}(w(-1,0);x)^{-\gamma_0(1+(\gamma_\infty-\gamma_0)/r)}\\ &\times R_{2,n}(w(0,-r);x)^{(1+\gamma_0)(\gamma_0-\gamma_\infty)/r}\\ &\times R_{2,n}(w(0,0);x)^{(1+\gamma_0)(1+(\gamma_\infty-\gamma_0)/r)}.\end{aligned}$$

Thus it is sufficient to establish (2.22) only for each of the $R_{2,n}$'s on the right above. This is done by arguments similar to those in the proof of the previous proposition based on the following relations:

$$\frac{(t-u)^2}{u^{1/2}} \le \frac{(t-x)^2}{x^{1/2}}, \quad \frac{(t-u)^2}{u^{3/2}} \le \frac{(t-x)^2}{x^{3/2}} \quad \text{for } u \text{ between } t \text{ and } x, \qquad (2.23)$$

$$S_n\big(|\circ - x|^3 (1+\circ)^i\big)(x) \le \frac{c}{n^{3/2}}\,x^{3/2}(1+x)^i, \qquad i\in\mathbb{Z}, \quad nx\ge 1, \qquad (2.24)$$

$$\sum_{k\ge 2}\Big(\frac{k}{n}-x\Big)^3\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) \le \frac{c}{n^{3/2}}\,x^{3/2}(1+x)^i, \qquad i\in\mathbb{Z}, \quad nx\le 1, \ (2.25)$$

and

$$\Big|\int_x^{k/n} \frac{\left(\frac{k}{n}-u\right)^2}{u^{3/2}}\, u^{\alpha}(1+u)^{\beta}\, du\Big| s_{n,k}(x) \le \frac{c}{n^{3/2}}\, w(-\alpha,-\beta;x)^{-1}, \quad nx \le 1, \quad (2.26)$$

for $k = 0, 1$ and each of the couples

$$(\alpha,\beta) = (0,0), \qquad (\alpha,\beta) = (1,0), \qquad (\alpha,\beta) = (0,r), \qquad (\alpha,\beta) = (1,r).$$

The inequalities (2.23) are verified directly.

To prove (2.24) we apply the Cauchy inequality, (2.5) and (2.9) and thus get

$$\begin{aligned} S_n\big(|\circ - x|^3(1+\circ)^i\big)(x) &\le \sqrt{S_n\big((\circ - x)^6\big)(x)}\,\sqrt{S_n\big((1+\circ)^{2i}\big)(x)} \\ &\le \frac{c}{n^{3/2}}\, x^{3/2}(1+x)^i \end{aligned}$$

since $nx \ge 1$.

To establish (2.25) we observe that

$$\begin{aligned} &\sum_{k\ge 2}\Big(\frac{k}{n}-x\Big)^3\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) \\ &= \sum_{k\ge 2}\frac{k}{n}\Big(\frac{k}{n}-x\Big)^2\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) - x\sum_{k\ge 2}\Big(\frac{k}{n}-x\Big)^2\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) \\ &\le c\,x\sum_{k\ge 1}\Big(\frac{k+1}{n}-x\Big)^2\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) \\ &= c\,x\Bigg[\sum_{k\ge 1}\Big(\frac{k}{n}-x\Big)^2\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) + \frac{2}{n}\sum_{k\ge 1}\Big(\frac{k}{n}-x\Big)\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) \\ &\qquad + \frac{1}{n^2}\sum_{k\ge 1}\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x)\Bigg] \\ &\le \frac{c\,x^2(1+x)^i}{n} \le \frac{c\,x^{3/2}(1+x)^i}{n^{3/2}}, \qquad nx \le 1, \end{aligned}$$

as at the last but one step we have used (2.9), (2.10), (2.18) and the estimate

$$\begin{aligned} \frac{1}{n^2}\sum_{k\ge 1}\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) &= \frac{x}{n}\sum_{k\ge 1}\frac{1}{k}\Big(1+\frac{k}{n}\Big)^i s_{n,k-1}(x) \\ &\le \frac{c\,x}{n}\sum_{k\ge 0}\Big(1+\frac{k}{n}\Big)^i s_{n,k}(x) \le \frac{c\,x(1+x)^i}{n}. \end{aligned}$$

It remains to prove (2.26). Set

$$A_{n,k}(\alpha,\beta;x) = \left|\int_x^{k/n} \frac{\left(\frac{k}{n}-u\right)^2}{u^{3/2}}\, u^{\alpha}(1+u)^{\beta}\, du\right| s_{n,k}(x).$$

For $\alpha = \beta = 0$ we have by straightforward calculations

$$A_{n,0}(0,0;x) = \int_0^x \frac{u^2}{u^{3/2}}\,du\, s_{n,0}(x) = \frac{2}{3}\, x^{3/2}\, e^{-nx} \le \frac{c}{n^{3/2}}\, w(0,0;x)^{-1}$$

and

$$\begin{aligned} A_{n,1}(0,0;x) &= \int_x^{1/n} \frac{\left(\frac{1}{n}-u\right)^2}{u^{3/2}}\,du\, s_{n,1}(x) \\ &= e^{-nx}\left(2(1-nx)^2\frac{x^{1/2}}{n} - 4nx\int_x^{1/n}\frac{\frac{1}{n}-u}{u^{1/2}}\,du\right) \\ &\le \frac{c}{n^{3/2}}\, w(0,0;x)^{-1}. \end{aligned}$$

For $\alpha = 1,\ \beta = 0$ we calculate

$$A_{n,0}(1,0;x) = \int_0^x \frac{u^2}{u^{3/2}}\,u\,du\, s_{n,0}(x) = \frac{2}{5}\, x^{5/2}\, e^{-nx} \le \frac{c}{n^{3/2}}\, w(-1,0;x)^{-1}$$

and

$$\begin{aligned} A_{n,1}(1,0;x) &= \int_x^{1/n} \frac{\left(\frac{1}{n}-u\right)^2}{u^{3/2}}\,u\,du\, s_{n,1}(x) \\ &= e^{-nx}\left(-2(1-nx)^2\frac{x^{3/2}}{n} + \frac{8}{3}\,(1-nx)x^{5/2} - \frac{8}{3}\,nx\int_x^{1/n} u^{3/2}\,du\right) \\ &\le \frac{c}{n^{3/2}}\, w(-1,0;x)^{-1}. \end{aligned}$$

In the other two cases (2.26) follows from the inequality

$$A_{n,k}(\alpha,r;x) \le c\,A_{n,k}(\alpha,0;x)(1+x)^r, \qquad x\in[0,1],$$

and the estimates we have already established. □

The last two propositions contain Bernstein-type inequalities.

Proposition 2.4. *Let $w = w(\gamma_0,\gamma_\infty)$ be given by* (1.1) *with $\gamma_0 \in [-1,0]$ and $\gamma_\infty \in \mathbb{R}$. Then*

$$\|w\varphi D^2(S_nf)\| \le c_4 n\,\|wf\| \qquad \forall f\in C(w)(0,\infty) \quad \forall n\ge 1.$$

Proof. Analogously to the assertions we proved so far we shall show that

$$|\varphi(x)D^2(S_nf)(x)| \le c\,n\,\|wf\| w(x)^{-1}. \tag{2.27}$$

To this end, we shall consider the cases $nx\le 1$ and $nx \ge 1$ separately.

For $nx \le 1$ we use the representation

$$D^2 S_n f(x) = n^2\, S_n(\Delta^2_{1/n} f)(x). \tag{2.28}$$

It follows from

$$D S_n f(x) = n\, S_n(\Delta_{1/n} f)(x), \tag{2.29}$$

which is established straightforwardly. By (2.28), we have for $nx \le 1$

$$\begin{aligned} &|\varphi(x) D^2 S_n f(x)| \\ &\le n^2 x \sum_{k\ge 0} \left(w\Big(\frac{k+2}{n}\Big)^{-1} + 2w\Big(\frac{k+1}{n}\Big)^{-1} + w\Big(\frac{k}{n}\Big)^{-1} \right) s_{n,k}(x)\, \|wf\|. \end{aligned} \tag{2.30}$$

But since $nx \le 1$ and $\gamma_0 \in [-1, 0]$

$$\Big(\frac{k+i}{n}\Big)^{-\gamma_0} \le \Big(\frac{k}{n}\Big)^{-\gamma_0} + c\, n^{\gamma_0} \le \Big(\frac{k}{n}\Big)^{-\gamma_0} + \frac{c}{n}\, x^{-1-\gamma_0}, \qquad i = 1, 2.$$

Consequently,

$$w\Big(\frac{k+i}{n}\Big)^{-1} \le c\, w\Big(\frac{k}{n}\Big)^{-1} + \frac{c}{n}\, x^{-1-\gamma_0}\Big(1 + \frac{k}{n}\Big)^{\gamma_0 - \gamma_\infty}, \qquad i = 1, 2. \tag{2.31}$$

Relations (2.30), (2.31) and (2.7) imply (2.27) for $nx \le 1$.

Let now $nx \ge 1$. In this case we use that

$$s'_{n,k}(x) = \frac{n}{x}\Big(\frac{k}{n} - x\Big) s_{n,k}(x). \tag{2.32}$$

Therefore

$$D S_n f(x) = \frac{n}{x}\, S_n\big((\circ - x) f(\circ)\big)(x);$$

hence

$$D^2 S_n f(x) = \frac{n^2}{x^2}\, S_n\big((\circ - x)^2 f(\circ)\big)(x) - \frac{n}{x^2}\, S_n\big((\circ - x) f(\circ)\big)(x).$$

Consequently,

$$\begin{aligned} &|\varphi(x) D^2 S_n f(x)| \\ &\qquad \le \frac{n}{x}\big[n\, S_n\big((\circ - x)^2 w(\circ)^{-1}\big)(x) + S_n\big(|\circ - x| w(\circ)^{-1}\big)(x)\big]\, \|wf\|. \end{aligned} \tag{2.33}$$

For the first summand on the right above we have by Hölder's inequality the estimate (cf. (2.8))

$$\begin{aligned} S_n((\circ - x)^2 w(\circ)^{-1})(x) \le{}& S_n((\circ - x)^2 w(1, r+1; \circ))(x)^{\gamma_0(\gamma_\infty - \gamma_0)/r} \\ &\times S_n\big((\circ - x)^2 w(1, 0; \circ)\big)(x)^{-\gamma_0(1 + (\gamma_\infty - \gamma_0)/r)} \\ &\times S_n\big((\circ - x)^2 w(0, r; \circ)\big)(x)^{(1+\gamma_0)(\gamma_0 - \gamma_\infty)/r} \\ &\times S_n\big((\circ - x)^2 w(0, 0; \circ)\big)(x)^{(1+\gamma_0)(1 + (\gamma_\infty - \gamma_0)/r)}. \end{aligned} \tag{2.34}$$

To estimate $S_n\big((\circ - x)^2 w(1,0;\circ)\big)(x)$ we observe that by (2.3)

$$\begin{aligned} S_n\big((\circ - x)^2 w(1,0;\circ)\big)(x) &= x\, S_n\big((\circ - x)^2\big)(x) + S_n\big((\circ - x)^3\big)(x) \\ &= \frac{x^2}{n} + \frac{x}{n^2} \le \frac{2x^2}{n}. \end{aligned} \tag{2.35}$$

Similarly, we have

$$\begin{aligned} &S_n\big((\circ - x)^2 w(1,r+1;\circ)\big)(x) \\ &\quad \le c\,x \sum_{k\ge 0} \Big(\frac{k+1}{n} - x\Big)^2 \Big(1 + \frac{k}{n}\Big)^r s_{n,k}(x) \\ &\quad = c\,x \Bigg[\sum_{k\ge 0} \Big(\frac{k}{n} - x\Big)^2 \Big(1 + \frac{k}{n}\Big)^r s_{n,k}(x) \\ &\qquad + \frac{2}{n} \sum_{k\ge 0} \Big(\frac{k}{n} - x\Big) \Big(1 + \frac{k}{n}\Big)^r s_{n,k}(x) + \frac{1}{n^2} \sum_{k\ge 0} \Big(1 + \frac{k}{n}\Big)^r s_{n,k}(x)\Bigg] \\ &\quad \le c \Big(\frac{x^2}{n} + \frac{x}{n^2}\Big)(1+x)^r \le \frac{c\,x^2(1+x)^r}{n}, \end{aligned} \tag{2.36}$$

as at the last but one step we have used (2.9), (2.10) and (2.18).

Now, (2.34), (2.3), (2.18), (2.35) and (2.36) yield for $nx \ge 1$

$$S_n\big((\circ - x)^2 w(\circ)^{-1}\big)(x) \le \frac{c\,x}{n}\, w(x)^{-1}. \tag{2.37}$$

For the second summand on the right of (2.33) we have by the Cauchy inequality, (2.37) and (2.7)

$$\begin{aligned} S_n\big(|\circ - x| w(\circ)^{-1}\big)(x) &\le \sqrt{S_n\big((\circ - x)^2 w(\circ)^{-1}\big)(x)}\,\sqrt{S_n(w^{-1})(x)} \\ &\le c\sqrt{\frac{x}{n}}\, w(x)^{-1} \le c\,x\, w(x)^{-1}, \end{aligned}$$

as $nx \ge 1$.

The inequalities (2.33), (2.37) and the last estimate above imply (2.27) for $nx \ge 1$. This completes the proof of the proposition. □

The next proposition is the analogue of a result of Knoop and Zhou concerning the Bernstein operator [14, Theorem 2.1]. We prove it following their argument. Perhaps it is worth noting that it fits even more naturally and is easier to apply for the Szász-Mirakjan operator than for the Bernstein operator, for which it was originally developed.

Proposition 2.5. *Let $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) with $\gamma_0 \in [-1, 0]$ and $\gamma_\infty \in \mathbb{R}$. Let $r \in \mathbb{Z}$ be such that $r/(\gamma_0 - \gamma_\infty) > 1$ if $\gamma_0 \ne \gamma_\infty$, and let $r = 1$ if $\gamma_0 = \gamma_\infty$. We set $\gamma = (1 + \gamma_0)(\gamma_0 - \gamma_\infty)/r$. Let also $m \in \mathbb{N}$ as $m \ge 2$. Then*

$$\|w\varphi^{3/2}D^3(S_n^m g)\| \le c_5\sqrt{m^{\gamma-1}(\log m)^{1+\gamma_0}}\,\sqrt{n}\,\|w\varphi D^2 g\| \qquad \forall g\in W^2(w\varphi)(0,\infty) \quad \forall n\ge m^2.$$

The value of the constant c_5 depends only on r.

Proof. For $g\in W^2(w\varphi)(0,\infty)$ the formulae (2.28) and (2.29) give

$$DS_n g(x) = \sum_{k\ge 0} n\int_0^{1/n} Dg\Big(\frac{k}{n}+u\Big)\,du\, s_{n,k}(x) \tag{2.38}$$

and

$$D^2 S_n g(x) = \sum_{k\ge 0} n^2\int_0^{1/n}\int_0^{1/n} D^2 g\Big(\frac{k}{n}+u+v\Big)\,du\,dv\, s_{n,k}(x). \tag{2.39}$$

Iterating the latter we arrive at

$$D^2 S_n^m g(x) = \sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} n^2\int_0^{1/n}\int_0^{1/n} D^2 g\Big(\frac{k_1}{n}+u+v\Big)\,du\,dv\,\mathcal{S}_{n,\bar k}\, s_{n,k_m}(x), \tag{2.40}$$

where we have set $\bar k=(k_1,\dots,k_m)$,

$$\mathcal{S}_{n,\bar k} = \prod_{j=1}^{m-1} s_{n,2,k_j}\Big(\frac{k_{j+1}}{n}\Big)$$

and

$$s_{n,i,k}(x) = n^i\int_0^{1/n}\cdots\int_0^{1/n} s_{n,k}(x+t_1+\dots+t_i)\,dt_1\cdots dt_i.$$

Next, just as in [14, pp. 319-320], using (2.38) and (2.40), we arrive at the following $m-1$ representations of $D^3S_n^m g$

$$D^3 S_n^m g(x) = \sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} n^2\int_0^{1/n}\int_0^{1/n} D^2 g\Big(\frac{k_1}{n}+u+v\Big)\,du\,dv \times \mathcal{S}_{n,\bar k}\, Q_{n,j,\bar k}\, s_{n,k_m}(x), \qquad j=1,\dots,m-1, \tag{2.41}$$

where

$$Q_{n,j,\bar k} = \ell^*_{n,k_j}\Big(\frac{k_{j+1}}{n}\Big)\ell_{n,k_{j+1}}\Big(\frac{k_{j+2}}{n}\Big)\cdots\ell_{n,k_{m-1}}\Big(\frac{k_m}{n}\Big), \qquad j=1,\dots,m-2,$$

$$Q_{n,m-1,\bar k} = \ell^*_{n,k_{m-1}}\Big(\frac{k_m}{n}\Big)$$

and

$$\ell^*_{n,k}(x) = \frac{n \displaystyle\int_0^{1/n} s'_{n,2,k}(x+t)\,dt}{s_{n,2,k}(x)}, \qquad \ell_{n,k}(x) = \frac{s_{n,3,k}(x)}{s_{n,2,k}(x)}.$$

Summing the relations in (2.41) we get

$$D^3 S_n^m g(x) = \frac{1}{m-1} \sum_{\substack{k_j \ge 0 \\ j=1,\dots,m}} n^2 \int_0^{1/n}\int_0^{1/n} D^2 g\Big(\frac{k_1}{n}+u+v\Big)\,du\,dv \times \mathcal{S}_{n,\bar{k}}\, Q_{n,\bar{k}}\, s_{n,k_m}(x) \quad (2.42)$$

with

$$Q_{n,\bar{k}} = \sum_{j=1}^{m-1} Q_{n,j,\bar{k}}.$$

Further, let us observe that

$$\Big|n^2 \int_0^{1/n}\int_0^{1/n} D^2 g\Big(\frac{k}{n}+u+v\Big)\,du\,dv\Big| \le n^2 \int_0^{1/n}\int_0^{1/n} w\Big(\frac{k}{n}+u+v\Big)^{-1} \varphi\Big(\frac{k}{n}+u+v\Big)^{-1} du\,dv\, \|w\varphi D^2 g\| \quad (2.43)$$

and

$$n^2 \int_0^{1/n}\int_0^{1/n} w\Big(\frac{k}{n}+u+v\Big)^{-1} \varphi\Big(\frac{k}{n}+u+v\Big)^{-1} du\,dv \le c\, w\Big(\frac{k+1}{n}\Big)^{-1} \varphi\Big(\frac{k+1}{n}\Big)^{-1}. \quad (2.44)$$

The first of these estimates is obvious. To verify the second one we proceed as follows:

$$\begin{aligned} n^2 &\int_0^{1/n}\int_0^{1/n} w\Big(\frac{k}{n}+u+v\Big)^{-1} \varphi\Big(\frac{k}{n}+u+v\Big)^{-1} du\,dv \\ &= n^2 \int_{-1/n}^{1/n} \Big(\frac{1}{n}-|u|\Big) w\Big(\frac{k+1}{n}+u\Big)^{-1} \varphi\Big(\frac{k+1}{n}+u\Big)^{-1} du \\ &\le c\left(1+\frac{k+1}{n}\right)^{\gamma_0-\gamma_\infty} n^2 \int_{-1/n}^{1/n} \Big(\frac{1}{n}-|u|\Big)\Big(\frac{k+1}{n}+u\Big)^{-\gamma_0-1} du \\ &\le c\, w\Big(\frac{k+1}{n}\Big)^{-1} \varphi\Big(\frac{k+1}{n}\Big)^{-1}, \end{aligned}$$

as at the last estimate we have used that the function $(1/n-|u|)[(k+1)/n+u]^{-\gamma_0-1}$ is increasing on $[-1/n,0]$ and decreasing on $[0,1/n]$. We derive from

(2.42) by the Cauchy inequality and (2.43)-(2.44) the estimate

$$|D^3 S_n^m g(x)| \le \frac{1}{m-1} \sqrt{\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} \mathcal{S}_{n,\bar{k}}\, Q_{n,\bar{k}}^2\, s_{n,k_m}(x)}$$
$$\times \sqrt{\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} \left| n^2 \int_0^{1/n}\int_0^{1/n} D^2 g\Big(\frac{k_1}{n}+u+v\Big)\,du\,dv\right|^2 \mathcal{S}_{n,\bar{k}}\, s_{n,k_m}(x)}$$
$$\le \frac{c}{m} \sqrt{\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} \mathcal{S}_{n,\bar{k}}\, Q_{n,\bar{k}}^2\, s_{n,k_m}(x)}$$
$$\times \sqrt{\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} w\Big(\frac{k_1+1}{n}\Big)^{-2} \varphi\Big(\frac{k_1+1}{n}\Big)^{-2} \mathcal{S}_{n,\bar{k}}\, s_{n,k_m}(x)}\ \|w\varphi D^2 g\|.$$

Thus we can finish the proof of the proposition if we show that

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} \mathcal{S}_{n,\bar{k}}\, Q_{n,\bar{k}}^2\, s_{n,k_m}(x) \le c\, nm\varphi(x)^{-1} \tag{2.45}$$

and

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} w\Big(\frac{k_1+1}{n}\Big)^{-2} \varphi\Big(\frac{k_1+1}{n}\Big)^{-2} \mathcal{S}_{n,\bar{k}}\, s_{n,k_m}(x)$$
$$\le c\, w(x)^{-2}\varphi(x)^{-2} m^{\gamma} (\log m)^{1+\gamma_0}, \qquad n \ge m^2. \tag{2.46}$$

In order to simplify these two assertions we take into account that $s_{n,i,k}(x) = s_{1,i,k}(nx)$. Consequently,

$$\begin{aligned} &\mathcal{S}_{n,\bar{k}} = \mathcal{S}_{1,\bar{k}},\\ &\ell^*_{n,k}(x) = n\ell^*_{1,k}(nx), \qquad \ell_{n,k}(x) = \ell_{1,k}(nx),\\ &Q_{n,j,\bar{k}} = nQ_{1,j,\bar{k}}. \end{aligned}$$

Also, we have (cf. [14, pp. 327–328])

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} \mathcal{S}_{n,\bar{k}}\, Q_{n,j',\bar{k}}\, Q_{n,j'',\bar{k}}\, s_{n,k_m}(x) \equiv 0, \qquad j' \ne j''.$$

Hence (2.45) and (2.46) reduce respectively to

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} \mathcal{S}_{1,\bar{k}}\, Q_{1,j,\bar{k}}^2\, s_{1,k_m}(x) \le c\,\varphi(x)^{-1}, \qquad j = 1,\dots,m-1,$$

and

$$\sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} (1+k_1)^{-2(1+\gamma_0)}(n+k_1)^{2(\gamma_0-\gamma_\infty)}\mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x)$$
$$\leq c\, x^{-2(1+\gamma_0)}(n+x)^{2(\gamma_0-\gamma_\infty)} m^{\gamma}(\log m)^{1+\gamma_0}, \qquad n \geq m^2. \quad (2.47)$$

The first estimate is verified in Lemma 2.1 below. As for for the second we use Hölder's inequality as in the proof of the previous propositions to get

$$\sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} (1+k_1)^{-2(1+\gamma_0)}(n+k_1)^{2(\gamma_0-\gamma_\infty)}\mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x)$$
$$\leq \left\{ \sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} (1+k_1)^{-4}\mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x) \right\}^{(1+\gamma_0)(\gamma_0-\gamma_\infty)/(2r)}$$
$$\times \left\{ \sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} (1+k_1)^{-2}\mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x) \right\}^{(1+\gamma_0)(1+(\gamma_\infty-\gamma_0)/r)}$$
$$\times \left\{ \sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} (n+k_1)^{2r}\mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x) \right\}^{\gamma_0(\gamma_\infty-\gamma_0)/r}$$
$$\times \left\{ \sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} (n+k_1)^{4r}\mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x) \right\}^{(1+\gamma_0)(\gamma_0-\gamma_\infty)/(2r)}$$
$$\times \left\{ \sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} \mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x) \right\}^{-\gamma_0(1+(\gamma_\infty-\gamma_0)/r)}.$$

Now, (2.47) follows from Lemma 2.2 and the fact that the last term above is equal to 1. □

Now we shall establish the two lemmas used in the proof above.

Lemma 2.1. *Let $m \in \mathbb{N}$ as $m \geq 2$. Then there hold*

$$\sum_{\substack{k_j \geq 0 \\ j=1,\dots,m}} \mathcal{S}_{1,\bar{k}}\, Q_{1,j,\bar{k}}^2\, s_{1,k_m}(x) \leq c\,\varphi(x)^{-1}, \qquad x > 0,$$

for $j = 1, \dots, m-1$. The value of the constant c does not depend on x or m.

Proof. We follow the argument of Knoop and Zhou [14]. We start with the equality

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} \mathcal{S}_{1,\bar k}\, Q^2_{1,j,\bar k}\, s_{1,k_m}(x) = \sum_{k_m\ge 0} s_{1,k_m}(x) \sum_{k_{m-1}\ge 0} s_{1,2,k_{m-1}}(k_m)\,\ell_{1,k_{m-1}}(k_m)^2$$
$$\cdots \sum_{k_{j+1}\ge 0} s_{1,2,k_{j+1}}(k_{j+2})\,\ell_{1,k_{j+1}}(k_{j+2})^2 \sum_{k_j\ge 0} s_{1,2,k_j}(k_{j+1})\,\ell^*_{1,k_j}(k_{j+1})^2. \quad (2.48)$$

Next, we establish that

$$\sum_{k\ge 0} s_{1,2,k}(i)\,\ell^*_{1,k}(i)^2 \le \frac{c}{1+i}, \qquad i\in\mathbb{N}_0, \quad (2.49)$$

and

$$\sum_{k\ge 0} s_{1,2,k}(i)\,\ell_{1,k}(i)^2 \frac{1}{1+k} \le \frac{1}{1+i}, \qquad i\in\mathbb{N}_0. \quad (2.50)$$

Then the lemma follows from (2.48)-(2.50) and (2.9) (with $r=-1$).

To prove (2.49) for $i=0$ we use that $s'_{n,k}(x) = n(s_{n,k-1}(x) - s_{n,k}(x))$, $k>0$, to derive the estimate

$$\Big|\int_0^1\int_0^1\int_0^1 s'_{1,k}(t_1+t_2+t_3)\,dt_1\,dt_2\,dt_3\Big| \le \frac{3^k(k+1)}{k!}.$$

Since, also, we have

$$\int_0^1\int_0^1 s_{1,k}(t_1+t_2)\,dt_1\,dt_2 \ge \frac{1}{e^2 k!}\int_0^1 t^k\,dt = \frac{1}{e^2(k+1)!},$$

we arrive at

$$s_{1,2,k}(0)\,\ell^*_{1,k}(0)^2 = \frac{\left(\displaystyle\int_0^1\int_0^1\int_0^1 s'_{1,k}(t_1+t_2+t_3)\,dt_1\,dt_2\,dt_3\right)^2}{\displaystyle\int_0^1\int_0^1 s_{1,k}(t_1+t_2)\,dt_1\,dt_2} \le \frac{e^2 3^{2k}(k+1)^3}{k!};$$

hence (2.49) follows for $i=0$.

To prove (2.49) for $i\ge 1$ we apply the formula (2.32) and the Cauchy inequality to get

$$\begin{aligned} s_{1,2,k}(i)\,\ell^*_{1,k}(i)^2 &= \frac{\left(\displaystyle\int_0^1\int_0^1\int_0^1 s'_{1,k}(i+t_1+t_2+t_3)\,dt_1\,dt_2\,dt_3\right)^2}{\displaystyle\int_0^1\int_0^1 s_{1,k}(i+t_1+t_2)\,dt_1\,dt_2} \\ &\le \int_0^1\int_0^1\int_0^1 \Big(\frac{k-(i+t_1+t_2+t_3)}{i+t_1+t_2+t_3}\Big)^2 \frac{s_{1,k}(i+t_1+t_2+t_3)^2}{s_{1,k}(i+t_1+t_2)}\,dt_1\,dt_2\,dt_3 \\ &\le \frac{c}{i^2}\big([k-(i+5)]^2+1\big)s_{1,k}(i+5) \end{aligned} \quad (2.51)$$

since for $t_1, t_2, t_3 \in [0,1]$

$$\left(\frac{k-(i+t_1+t_2+t_3)}{i+t_1+t_2+t_3}\right)^2 \le \frac{c}{i^2}\,\left([k-(i+5)]^2+1\right)$$

and

$$\frac{(i+t_1+t_2+t_3)^2}{i+t_1+t_2} \le i+5, \qquad i \ge 1.$$

Now, (2.49) follows for $i \ge 1$ from (2.51), (2.2) and (2.3).

The inequality (2.50) is derived by similar though more precise estimates. First, let $i = 0$. We observe that

$$\begin{aligned}\int_0^1\int_0^1 s_{1,k}(t_1+t_2)\,dt_1\,dt_2 &= \frac{1}{k!}\int_0^1\int_0^1 e^{-t_1-t_2}(t_1+t_2)^k\,dt_1\,dt_2\\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\int_0^1 e^{-t_1}t_1^l\,dt_1\int_0^1 e^{-t_2}t_2^{k-l}\,dt_2\\ &\ge \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\int_0^1 e^{-t_1}t_1^l\,dt_1\frac{1}{e(k-l+1)}\\ &\ge \frac{1}{e(k+1)}\,\frac{1}{k!}\int_0^1 e^{-t_1}\sum_{l=0}^{k}\binom{k}{l}t_1^l\,dt_1\\ &= \frac{1}{k+1}\int_0^1 s_{1,k}(1+t_1)\,dt_1.\end{aligned}$$

Then by the Cauchy inequality we get

$$\begin{aligned}s_{1,2,k}(0)\,\ell_{1,k}(0)^2\frac{1}{1+k} &= \frac{1}{1+k}\frac{\left(\displaystyle\int_0^1\int_0^1\int_0^1 s_{1,k}(t_1+t_2+t_3)\,dt_1\,dt_2\,dt_3\right)^2}{\displaystyle\int_0^1\int_0^1 s_{1,k}(t_1+t_2)\,dt_1\,dt_2}\\ &\le \int_0^1\int_0^1\int_0^1 \frac{s_{1,k}(t_1+t_2+t_3)^2}{s_{1,k}(1+t_1)}\,dt_1\,dt_2\,dt_3\\ &= \int_0^1\int_0^1\int_0^1 e^{1-t_1-2t_2-2t_3}\frac{1}{k!}\left[\frac{(t_1+t_2+t_3)^2}{1+t_1}\right]^k dt_1\,dt_2\,dt_3.\end{aligned}$$

Now, to estimate the sum on the left hand-side of (2.50) for $i = 0$ we calculate separately its term for $k = 0$ whereas for the terms with $k \ge 1$ we apply the inequality above. Thus we get

$$\sum_{k\ge 0} s_{1,2,k}(0)\,\ell_{1,k}(0)^2\frac{1}{1+k}$$
$$\le (1-e^{-1})^4+\int_0^1\int_0^1\int_0^1\Big(e^{\frac{(1-t_2-t_3)^2}{1+t_1}}-e^{1-t_1-2t_2-2t_3}\Big)\,dt_1\,dt_2\,dt_3$$
$$=\int_0^1\int_0^1\int_0^1 e^{\frac{(1-t_2-t_3)^2}{1+t_1}}\,dt_1\,dt_2\,dt_3-\frac{e}{4}(1-e^{-1})(1-e^{-2})^2+(1-e^{-1})^4;$$

hence, in view of [14, (4.9)], (2.50) follows for $i=0$.

For $i\ge 1$ we have by the Cauchy inequality

$$s_{1,2,k}(i)\,\ell_{1,k}(i)^2\frac{1}{1+k}=\frac{1}{1+k}\,\frac{\Big(\int_0^1\int_0^1\int_0^1 s_{1,k}(i+t_1+t_2+t_3)\,dt_1\,dt_2\,dt_3\Big)^2}{\int_0^1\int_0^1 s_{1,k}(i+t_1+t_2)\,dt_1\,dt_2}$$
$$\le\frac{1}{1+k}\int_0^1\int_0^1\int_0^1\frac{s_{1,k}(i+t_1+t_2+t_3)^2}{s_{1,k}(i+t_1+t_2)}\,dt_1\,dt_2\,dt_3$$
$$=\int_0^1\int_0^1\int_0^1\frac{i+t_1+t_2}{(i+t_1+t_2+t_3)^2}\,e^{\frac{t_3^2}{i+t_1+t_2}}\,s_{1,k+1}\left(\frac{(i+t_1+t_2+t_3)^2}{i+t_1+t_2}\right)dt_1\,dt_2\,dt_3.$$

Summing on $k\ge 0$ we arrive at the estimate

$$\sum_{k\ge 0} s_{1,2,k}(i)\,\ell_{1,k}(i)^2\frac{1}{1+k}\le\int_0^1\int_0^1\int_0^1\frac{i+t_1+t_2}{(i+t_1+t_2+t_3)^2}\,e^{\frac{t_3^2}{i+t_1+t_2}}\,dt_1\,dt_2\,dt_3.$$

Now, (2.50) follows for $i\ge 1$ from [14, (4.10)]. □

Lemma 2.2. *Let $m\in\mathbb{N}$ as $m\ge 2$.*

(a) For $i\in\mathbb{Z}_-$ we have

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}}(1+k_1)^i\mathcal{S}_{1,\bar k}\,s_{1,k_m}(x)\le c\,x^i\,m^{-i-2}\log m,\qquad x>0.$$

(b) For $i\in\mathbb{Z}_-$ we have

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}}(n+k_1)^i\mathcal{S}_{1,\bar k}\,s_{1,k_m}(x)\le c\,(n+x)^i,\qquad x\ge 0,\quad n\ge m^2.$$

(c) For $i\in\mathbb{N}_0$ we have

$$\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}}(n+k_1)^i\mathcal{S}_{1,\bar k}\,s_{1,k_m}(x)\le c\,(n+x)^i,\qquad x\ge 0,\quad n\ge m.$$

The value of the constant c depends only on i.

Proof. We again follow the argument of Knoop and Zhou [14]. For $i \le -1$ we set $\ell = -i$. We shall use the representation

$$(n+k)^i = \int_0^1 \cdots \int_0^1 (\tau_1 \cdots \tau_\ell)^{n+k-1}\, d\tau_1 \cdots d\tau_\ell$$

and the formula

$$\sum_{k\ge 0} a^k s_{1,2,k}(x) = e^{-(1-a)x}\left(\frac{1-e^{-(1-a)}}{1-a}\right)^2 .$$

The latter is checked directly by means of the argument

$$\begin{aligned}\sum_{k\ge 0} a^k s_{1,2,k}(x) &= \int_0^1\int_0^1 e^{-x-t_1-t_2}\sum_{k\ge 0}\frac{[a(x+t_1+t_2)]^k}{k!}\, dt_1\, dt_2\\ &= \int_0^1\int_0^1 e^{(a-1)(x+t_1+t_2)}\, dt_1\, dt_2\\ &= e^{(a-1)x}\left(\int_0^1 e^{(a-1)t}\, dt\right)^2 .\end{aligned}$$

Now, just as in [14, pp. 322-323], we arrive at

$$\begin{aligned}&\sum_{\substack{k_j\ge 0\\ j=1,\dots,m}} (n+k_1)^i \mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x)\\ &\quad = \int_0^1 \cdots \int_0^1 (\tau_1\cdots\tau_\ell)^{n-1}\frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell,\end{aligned}$$

where

$$F_0(\tau) = 1-\tau, \quad F_\rho(\tau) = 1-e^{-F_{\rho-1}(\tau)}, \qquad \rho = 1,2\dots .$$

Let $\mathcal{D} \subset [0,1]^\ell$ be a rectangle with at least one side of the form $[0,1/2]$. Then, clearly,

$$F_0(\tau_1\cdots\tau_\ell) \ge 1/2, \qquad (\tau_1,\dots,\tau_\ell) \in \mathcal{D}. \tag{2.52}$$

It can be shown by induction that

$$\frac{1-\tau}{m} \le F_{m-1}(\tau) \le 1, \qquad 0 \le \tau \le 1. \tag{2.53}$$

Also, since $y^\ell e^{-y} \le c$, $y \ge 0$, then

$$F_{m-1}(\tau)^\ell e^{-F_{m-1}(\tau)x} \le c\, x^i, \qquad \tau \ge 0, \quad x > 0. \tag{2.54}$$

Now, (2.52), the left inequality in (2.53) and (2.54) imply

$$\int_{\mathcal{D}} \frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell \le c\, m^{-i-2} x^i. \tag{2.55}$$

Similarly, (2.52) and the right inequality in (2.53) directly imply the relation

$$\int_{\mathcal{D}} (\tau_1\cdots\tau_\ell)^{n-1}\frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell \le c\,n^i, \qquad n\ge 1. \tag{2.56}$$

For $n\ge m$ (2.52), the left inequality in (2.53) and (2.54) yield

$$\begin{aligned}\int_{\mathcal{D}} (\tau_1\cdots\tau_\ell)^{n-1}\frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell&\\ \le c\,m^{\ell-2}(nx)^i \le c\,x^i, \qquad n\ge m.&\end{aligned} \tag{2.57}$$

Now, by means of (2.55) we derive

$$\begin{aligned}&\int_0^1\cdots\int_0^1 \frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell\\ &\quad\le \int_{1/2}^1\cdots\int_{1/2}^1 \frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell + c\,m^{-i-2}x^i;\end{aligned}$$

and similarly by (2.56)–(2.57) we get

$$\begin{aligned}&\int_0^1\cdots\int_0^1 (\tau_1\cdots\tau_\ell)^{n-1}\frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell\\ &\quad\le \int_{1/2}^1\cdots\int_{1/2}^1 (\tau_1\cdots\tau_\ell)^{n-1}\frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell\\ &\qquad + c\,(n+x)^i, \qquad n\ge m.\end{aligned}$$

So, to complete the proof of (a) and (b) it is enough to show respectively that that

$$\begin{aligned}\int_{1/2}^1\cdots\int_{1/2}^1 \frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell&\\ \le c\,x^i\, m^{-i-2}\log m,&\end{aligned} \tag{2.58}$$

and

$$\begin{aligned}\int_{1/2}^1\cdots\int_{1/2}^1 (\tau_1\cdots\tau_\ell)^{n-1}\frac{F_{m-1}(\tau_1\cdots\tau_\ell)^2}{F_0(\tau_1\cdots\tau_\ell)^2}\, e^{-F_{m-1}(\tau_1\cdots\tau_\ell)x}\, d\tau_1\cdots d\tau_\ell&\\ \le c\,(n+x)^i, \qquad n\ge m^2.&\end{aligned} \tag{2.59}$$

We set

$$F_n(\tau,x) = \tau^{n-1}\frac{F_{m-1}(\tau)^2}{F_0(\tau)^2}\, e^{-F_{m-1}(\tau)x}.$$

In the integral in (2.59) we make the change of the variables, defined by the formulae $\sigma_\rho = \tau_1 \cdots \tau_\rho$, $\rho = 1, \dots, \ell$, and arrange the order of integration from σ_1 to σ_ℓ to get the estimate

$$\begin{aligned}
&\int_{1/2}^{1} \cdots \int_{1/2}^{1} F_n(\tau_1 \cdots \tau_\ell, x)\, d\tau_1 \cdots d\tau_\ell \\
&\le \int_{2^i}^{1} \left[F_n(\sigma_\ell, x) \int_{\sigma_\ell}^{1} \left(\frac{1}{\sigma_{\ell-1}} \cdots \left(\frac{1}{\sigma_3} \int_{\sigma_3}^{1} \left(\frac{1}{\sigma_2} \int_{\sigma_2}^{1} \frac{1}{\sigma_1}\, d\sigma_1 \right) d\sigma_2 \right) \cdots \right) d\sigma_{\ell-1} \right] d\sigma_\ell \\
&\le c \int_{2^i}^{1} \left[F_n(\sigma_\ell, x) \int_{\sigma_\ell}^{1} \left(\cdots \left(\int_{\sigma_3}^{1} \left(\int_{\sigma_2}^{1} d\sigma_1 \right) d\sigma_2 \right) \cdots \right) d\sigma_{\ell-1} \right] d\sigma_\ell \\
&\le c \int_{2^i}^{1} F_n(\sigma_\ell, x)(1-\sigma_\ell)^{\ell-1} d\sigma_\ell \\
&\le c \int_{0}^{1} (1-v)^{n-1} v^{\ell-3} G_{m-1}(v)^2\, e^{-G_{m-1}(v)x}\, dv, \qquad (2.60)
\end{aligned}$$

as at the last step we have made the change of the variable $\sigma_\ell = 1 - v$ and set

$$G_0(v) = v, \quad G_\rho(v) = 1 - e^{-G_{\rho-1}(v)}, \qquad \rho = 1, 2, \dots .$$

Routine considerations show that

$$v - \frac{m}{2}\, v^2 \le G_{m-1}(v) \le v, \quad 0 \le v \le 1. \qquad (2.61)$$

For $n = 1$ (2.60) reads

$$\begin{aligned}
&\int_{1/2}^{1} \cdots \int_{1/2}^{1} \frac{F_{m-1}(\tau_1 \cdots \tau_\ell)^2}{F_0(\tau_1 \cdots \tau_\ell)^2}\, e^{-F_{m-1}(\tau_1 \cdots \tau_\ell)x}\, d\tau_1 \cdots d\tau_\ell \\
&\qquad\qquad \le c \int_0^1 v^{\ell-3} G_{m-1}(v)^2\, e^{-G_{m-1}(v)x}\, dv.
\end{aligned}$$

We split the integral on the right by means of the intermediate point $1/m$. For the one between 0 and $1/m$ we apply (2.61) to get

$$\int_0^{1/m} v^{\ell-3} G_{m-1}(v)^2\, e^{-G_{m-1}(v)x}\, dv \le c \int_0^1 v^{\ell-1} e^{-vx}\, dv \le c\, x^i, \qquad (2.62)$$

as the last estimate is verified by integration by parts.

For the other integral we again use (2.53) and (2.54) to arrive at

$$\begin{aligned}
&\int_{1/m}^{1} v^{\ell-3} G_{m-1}(v)^2\, e^{-G_{m-1}(v)x}\, dv \\
&\qquad\qquad \le c\, m^{\ell-2} x^i \int_{1/m}^{1} \frac{dv}{v} = c\, x^i\, m^{-i-2} \log m. \qquad (2.63)
\end{aligned}$$

Now, (2.62)-(2.63) imply (2.58), which completes the proof of assertion (a).

The estimate (2.59) is established in a similar way. Using the right inequality in (2.61), we get

$$\int_0^1 (1-v)^{n-1} v^{\ell-3} G_{m-1}(v)^2\, e^{-G_{m-1}(v)x}\, dv \\ \le \int_0^1 (1-v)^{n-1} v^{\ell-1}\, dv = \frac{(\ell-1)!\,(n-1)!}{(\ell+n-1)!} \le c\, n^i. \quad (2.64)$$

As above we get

$$\int_0^{1/m} (1-v)^{n-1} v^{\ell-3} G_{m-1}(v)^2\, e^{-G_{m-1}(v)x}\, dv \\ \le c \int_0^1 v^{\ell-1} e^{-vx}\, dv \le c\, x^i. \quad (2.65)$$

To estimate the integral between $1/m$ and 1, we first observe that

$$m^\ell \Big(1 - \frac{1}{m}\Big)^{m^2} \le m^\ell e^{-m} \le c, \quad (2.66)$$

which follow from $(1-y)^n \le e^{-ny}$, $y \in [0,1]$, and $y^\ell e^{-y} \le c$, $y \ge 0$. Now, by means of the first inequality in (2.53), (2.54) and (2.66), we get for $n \ge m^2$

$$\int_{1/m}^1 (1-v)^{n-1} v^{\ell-3} G_{m-1}(v)^2\, e^{-G_{m-1}(v)x}\, dv \\ \le c \Big(1 - \frac{1}{m}\Big)^{n-1} m^{\ell-1} x^i \le c\, m^\ell e^{-m} x^i \le c\, x^i, \qquad n \ge m^2. \quad (2.67)$$

The estimates (2.60), (2.64), (2.65) and (2.67) imply (2.59). Thus assertion (b) is established.

For the proof of (c) we first observe that for $\theta \ge 0$ there holds

$$\sum_{k\ge 0} (\theta+k)^i s_{1,k}(x) \le (\theta + x) \sum_{k\ge 0} (\theta+1+k)^{i-1} s_{1,k}(x).$$

Iterating it we arrive at

$$\sum_{k\ge 0} (\theta+k)^i s_{1,k}(x) \le (\theta+x)(\theta+1+x)\cdots(\theta+i-1+x) \le (\theta+i+x)^i. \quad (2.68)$$

Similarly,

$$\sum_{k\ge 0} (\theta+k)^i s_{1,2,k}(x) \le (\theta+2+x) \sum_{k\ge 0} (\theta+1+k)^{i-1} s_{1,2,k}(x);$$

hence

$$\sum_{k\ge 0}(\theta+k)^i s_{1,2,k}(x) \le (\theta+2+x)(\theta+3+x)\cdots(\theta+i+1+x) \\ \le (\theta+i+1+x)^i. \tag{2.69}$$

Now, applying (2.69) consecutively with $\theta = n, n+i+1, \ldots, n+(m-2)(i+1)$ and then (2.68) with $\theta = n+(m-1)(i+1)$ we get

$$\begin{aligned}
&\sum_{\substack{k_j\ge 0\\ j=1,\ldots,m}} (n+k_1)^i \mathcal{S}_{1,\bar{k}}\, s_{1,k_m}(x) \\
&\quad = \sum_{k_m\ge 0} s_{1,k_m}(x) \sum_{k_{m-1}\ge 0} s_{1,2,k_{m-1}}(k_m)\cdots \sum_{k_1\ge 0}(n+k_1)^i s_{1,2,k_1}(k_2) \\
&\quad \le \sum_{k_m\ge 0} s_{1,k_m}(x) \sum_{k_{m-1}\ge 0} s_{1,2,k_{m-1}}(k_m)\cdots \sum_{k_2\ge 0}(n+i+1+k_2)^i s_{1,2,k_2}(k_3) \\
&\quad \le \sum_{k_m\ge 0} [n+(m-1)(i+1)+k_m]^i s_{1,k_m}(x) \\
&\quad \le [n+(m-1)(i+1)+i+x]^i,
\end{aligned}$$

which, in view of $n \ge m$, implies assertion (c) of the lemma. □

We are ready to proof our main result.

Proof of Theorem 1.1. It is enough to prove the theorem for $f\in C(w)(0,\infty)$ since $\|w(f-S_nf)\|$ and $K_w(f,n^{-1})$ are invariant under addition of a liner function to f.

The upper error estimate of the theorem follows from Propositions 2.1 and 2.2 via a direct and well-known argument.

To verify the strong converse inequality we apply [3, Theorem 4.1] as we set $Q_\alpha = S_n$, $\alpha = n^{-1}$, $X = C(w)[0,\infty)$, $Y = W^2(w\varphi)(0,\infty)$, $Z = W^3(w\varphi^{3/2})(0,\infty)$, the differential operator D of [3, Theorem 4.1] is φD^2, $\Phi(g) = \|w\varphi^{3/2}D^3 g\|$, $M = c_1$, $\lambda(\alpha) = (2n)^{-1}$, $\lambda_1(\alpha) = c_3 n^{-3/2}$, $\ell = 1$, $B = c_4/2$ and m replaced with $m+1$. Then

$$A = 2c_3c_5\sqrt{m^{\gamma-1}(\log m)^{1+\gamma_0}}.$$

We fix $m\in\mathbb{N}$, $m\ge 2$, so large that $A<1$. Note that $\gamma<1$. Then [3, Theorem 4.1], Propositions 2.1, 2.3, 2.4 and 2.5 imply the left hand-side inequality of Theorem 1.1 for any $n\ge m^2$. To extend the latter for $1\le n\le m^2-1$, we first observe that the property (2.6) of the Szász-Mirakjan operator yields

$$S_n f = \left[S_{m'}(f_{m'/n})\right]_{n/m'},$$

where we have set $m' = m^2$.

Consequently,

$$\|w(f - S_n f)\| = \|w_{m'/n}(f_{m'/n} - S_{m'}(f_{m'/n}))\| \\ \geq c\,\|w(f_{m'/n} - S_{m'}(f_{m'/n}))\| \quad (2.70)$$

with a constant $c > 0$ whose value is independent of f and n because $1 \leq m'/n \leq m'$ and

$$w_\nu(x) \geq \nu^{\min\{\gamma_0,\gamma_\infty\}} w(x), \qquad \nu \geq 1.$$

On the other hand,

$$K_w(f_\nu, t) = K_{w_{1/\nu}}(f, \nu t) \geq \nu^{-\max\{\gamma_0,\gamma_\infty\}} K_w(f, \nu t), \qquad \nu \geq 1, \qquad (2.71)$$

because $(g_\nu)' = \nu(g')_\nu$ and

$$w_{1/\nu}(x) \geq \nu^{-\max\{\gamma_0,\gamma_\infty\}} w(x), \qquad \nu \geq 1.$$

Now, combining the relation

$$K_w\left(f, \frac{1}{m'}\right) \leq c\,\|w(f - S_{m'}f)\| \qquad \forall f \in C(w)(0,\infty)$$

with (2.70) and (2.71) with $\nu = m'/n \in [1, m']$ and $t = 1/m'$, we get

$$K_w\Big(f, \frac{1}{n}\Big) \leq c\,\|w(f - S_n f)\| \qquad \forall f \in C(w)(0,\infty)$$

for $1 \leq n \leq m^2 - 1$ as well. □

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B. R. Draganov

Department of Mathematics and Informatics
The Sofia University
5 James Bourchier Blvd.
1164 Sofia
BULGARIA

and

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
BULGARIA
E-mail: bdraganov@fmi.uni-sofia.bg

K. G. Ivanov

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
BULGARIA
E-mail: kamen@math.bas.bg