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Cubature Rules for Harmonic Functions on the Disk Using Line Integrals over Two Sets of Equispaced Chords

Irina Georgieva and Clemens Hofreither

We construct a class of cubature rules for harmonic functions on the unit disk based on line integrals over $4n + 2$ distinct chords which are separated into two groups of equal size. The chords in each group are assumed to have constant distances t_1 and t_2, respectively, to the center of the disk, and equispaced angles over the interval $[0, 2\pi]$. We show that if t_1 and t_2 are chosen properly, these rules integrate exactly all harmonic polynomials of degree up to $8n + 3$. We conclude the paper with some numerical experiments.

Keywords and Phrases: Cubature rules, harmonic functions, Radon projections.

Mathematics Subject Classification 2010: 65D32, 65D05, 41A55.

1. Introduction

In the present work, we study a bivariate cubature problem which has two defining features: first, the functions to be integrated are harmonic functions on the unit disk, and second, the given data comes not in the form of point evaluations, but rather as integrals over certain chords of the unit circle. This Radon projection-like data has been shown to have certain advantages both for interpolation and cubature of multivariate functions. We point in particular to the results of Bojanov and Petrova [2, 3], who showed the existence of a unique cubature formula for the disk that uses n line integrals and is exact for all bivariate polynomials of degree up to $2n - 1$, a result which is not possible using the same number of point evaluations. In [15], a family of cubature rules on the unit disk using Radon projections along symmetrically located chords was found.

Besides the more convincing mathematical theory, we point out that line integrals as input data arise naturally in some real-world problems, e.g., in computer tomography with its many applications in medicine, radiology, geology, etc. The mathematical foundation for these applications is the work of Johann Radon on the so-called Radon transform [16]. Reconstruction of functions from their line integrals can be formulated as an interpolation problem where not the function itself, but its Radon transform is sampled on a discrete set. Early major contributions on the topic of multivariate interpolation using integrals over hyperplanes are due to Marr [14] and Hakopian [13]. Research on this topic was continued in the previous decade by many researchers [1, 4, 10, 11, 8, 12].

Interpolation of harmonic functions using such Radon projections was previously studied in [7, 5, 9]. Based on the interpolation theory, cubature rules for harmonic functions on the unit disk based on line integrals over $2n + 1$ distinct chords were constructed in [6]. These chords were assumed to have constant distance t to the center of the disk. For t properly chosen and equispaced angles, these formulae were shown to integrate exactly all harmonic polynomials of degree up to $4n + 1$, which is the highest achievable degree of precision for this class of cubature formulae.

In the present paper we construct another class of cubature rules for harmonic functions using Radon projections. In contrast to the previous work [6], we now choose two sets of $2n + 1$ equispaced chords each, where the distances to the origin t_1 and t_2 are constant within each set, but different from each other. See Figure 1 for a sketch. We introduce a cubature rule using the integrals along these $4n + 2$ chords with two weights a and b, which we then show how to choose. Finally we derive a family of possible choices for the distances (t_1, t_2) which maximize the degree of precision of the cubature rule for harmonic polynomials.

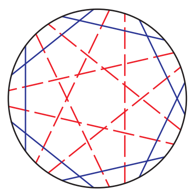

Figure 1. A scheme with two sets of 7 chords each with distances t_1 and t_2.

We point out that the theory both for interpolation and cubature of harmonic functions using Radon projection type of data has thus far only been developed for the case of constant distances t, and the present work is the first result using schemes with chords having different distances.

2. Preliminaries

Let $D \subset \mathbb{R}^2$ denote the open unit disk and ∂D the unit circle. By $I(\theta, t)$ we denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$

from the origin (see Figure 2).

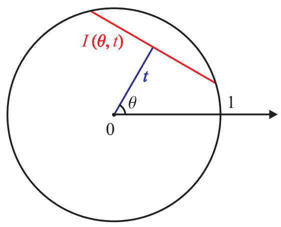

Figure 2. The chord $I(\theta, t)$ of the unit circle.

Definition 1. Let $u(x, y)$ be a real-valued bivariate function in the unit disk D. The *Radon projection* $\mathcal{R}_\theta(u; t)$ of u in direction θ is defined by the line integral

$$\mathcal{R}_\theta(u; t) := \int_{I(\theta,t)} u(x, y)\, dS.$$

Johann Radon [16] showed in 1917 that a differentiable function u is uniquely determined by the values of its Radon transform,

$$u \;\mapsto\; \big\{\mathcal{R}_\theta(u; t) \;:\; -1 \le t \le 1,\; 0 \le \theta < \pi\big\}.$$

2.1. Radon Projections of Harmonic Polynomials

Let Π_n^2 denote the space of real bivariate polynomials of total degree at most n. In the following, we will often work with the subspace

$$\mathcal{H}_n = \big\{p \in \Pi_n^2 : \Delta p = 0\big\}$$

of real bivariate harmonic polynomials of total degree at most n, which has dimension $2n + 1$. Here and below, $\Delta = \partial_{xx} + \partial_{yy}$.

We use the basis of $\mathcal{H}_n$ given by

$$\phi_0(x, y) = 1, \quad \phi_{k,1}(x, y) = \mathrm{Re}(x + \mathrm{i}y)^k, \quad \phi_{k,2}(x, y) = \mathrm{Im}(x + \mathrm{i}y)^k, \tag{1}$$

where $k = 1, \ldots, n$. In polar coordinates, the basis functions have the representation

$$\phi_{k,1}(r, \theta) = r^k \cos(k\theta), \qquad \phi_{k,2}(r, \theta) = r^k \sin(k\theta).$$

The following result, which gives a closed formula for Radon projections of the basis harmonic polynomials, can be considered a harmonic analogue to the famous Marr's formula [14]. A special case of this harmonic version was first derived using tools from symbolic computation [7]. Later, Georgieva and Hofreither [5] have given an analytic proof in a more general setting.

Theorem 1 ([5]). *The Radon projections of the basis harmonic polynomials are given by*

$$\int_{I(\theta,t)} \phi_{k,1}\, dS = \frac{2}{k+1}\sqrt{1-t^2}\, U_k(t)\cos(k\theta),$$
$$\int_{I(\theta,t)} \phi_{k,2}\, dS = \frac{2}{k+1}\sqrt{1-t^2}\, U_k(t)\sin(k\theta),$$

where $k \in \mathbb{N}$, $\theta \in \mathbb{R}$, $t \in (-1,1)$, *and* $U_k(t) = \frac{\sin((k+1)\arccos t)}{\sin(\arccos t)}$ *is the* k*-th degree Chebyshev polynomial of second kind.*

3. Cubature Rules on Two Sets of Equispaced Chords

For an integrable function u on the unit disk D, we denote

$$I[u] := \iint_D u(x,y)\, dx\, dy.$$

For the integrals of the basis harmonic polynomials over D, it is easy to compute that

$$\begin{aligned} I[\phi_{k,1}] &= \begin{cases} \pi, & k=0, \\ 0, & k \ge 1, \end{cases} \\ I[\phi_{k,2}] &= 0, \qquad k \ge 1. \end{aligned} \tag{2}$$

In the following, we will construct cubature rules for the unit disk D for harmonic functions using Radon projections along $4n+2$ chords $\mathcal{I}$, divided into two sets of $2n+1$ chords each,

$$\mathcal{I} = \{I(\theta_j, t_1) : j = 1, \ldots, 2n+1\} \cup \{I(\psi_j, t_2) : j = 1, \ldots, 2n+1\},$$

where the distances $t_1 \in (-1,1)$ and $t_2 \in (-1,1)$ are constant within each subset.

Let $n \in \mathbb{N}_0$, fix some $a, b \in \mathbb{R} \setminus \{0\}$ and $t_1, t_2 \in (-1,1)$, $t_1 \neq t_2$ and define the cubature rule

$$Q[u] := a \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u, t_1) + b \sum_{j=1}^{2n+1} \mathcal{R}_{\psi_j}(u, t_2). \tag{3}$$

Using Theorem 1, we obtain

$$\begin{aligned} Q[\phi_{k,1}] = a\frac{2}{k+1}\sqrt{1-t_1^2}\, U_k(t_1) \sum_{j=1}^{2n+1} \cos(k\theta_j) \\ + b\frac{2}{k+1}\sqrt{1-t_2^2}\, U_k(t_2) \sum_{j=1}^{2n+1} \cos(k\psi_j), \end{aligned}$$

$$Q[\phi_{k,2}] = a\frac{2}{k+1}\sqrt{1-t_1^2}\,U_k(t_1)\sum_{j=1}^{2n+1}\sin(k\theta_j)$$
$$+ b\frac{2}{k+1}\sqrt{1-t_2^2}\,U_k(t_2)\sum_{j=1}^{2n+1}\sin(k\psi_j).$$

Let us denote $A := a\sqrt{1-t_1^2}$ and $B := b\sqrt{1-t_2^2}$ for the sake of shortness of notation.

Using Euler's formula, we can rewrite the sums over trigonometric functions in the above formulas as the real and imaginary parts, respectively, of $\sum_{j=1}^{2n+1} e^{\mathrm{i}k\theta_j}$ and $\sum_{j=1}^{2n+1} e^{\mathrm{i}k\psi_j}$. With the special choice of equispaced angles

$$\theta_j = \frac{2j\pi}{2n+1}, \quad \psi_j = \alpha + \frac{2j\pi}{2n+1}, \quad j = 1, 2, \ldots, 2n+1, \tag{4}$$

where $\alpha \in \mathbb{R}$ is an arbitrary offset, we have

$$\sum_{j=1}^{2n+1} e^{\mathrm{i}k\psi_j} = \sum_{j=1}^{2n+1} e^{\mathrm{i}k(\frac{2j\pi}{2n+1}+\alpha)} = \sum_{l=0}^{2n} e^{\mathrm{i}k\alpha} e^{\mathrm{i}k(l+1)\frac{2\pi}{2n+1}} = e^{\mathrm{i}k\alpha} e^{\mathrm{i}k\frac{2\pi}{2n+1}} \sum_{l=0}^{2n} e^{\mathrm{i}k\frac{2\pi}{2n+1}l}$$
$$= \begin{cases} e^{\mathrm{i}k\alpha}(2n+1), & k \in \mathbb{N}_0 \cdot (2n+1), \\ e^{\mathrm{i}k\alpha} e^{\mathrm{i}k\frac{2\pi}{2n+1}} \frac{1-e^{\mathrm{i}k2\pi}}{1-e^{\mathrm{i}k\frac{2\pi}{2n+1}}} = 0, & \text{otherwise.} \end{cases}$$

Hence,

$$Q[\phi_{k,1}] = \begin{cases} \frac{2(2n+1)}{k+1}\big(AU_k(t_1) + BU_k(t_2)\cos(k\alpha)\big), & k \in \mathbb{N}_0 \cdot (2n+1), \\ 0, & \text{otherwise,} \end{cases}$$
$$Q[\phi_{k,2}] = 0, \quad k \geq 1.$$

For the time being, we restrict ourselves to the case $\alpha = 0$, and thus, $\theta_j = \psi_j$.

Comparing with (2), for the cubature formula Q to be exact for all the harmonic polynomials of degree up to $2n$, i.e., $Q[\phi_{k,j}] = I[\phi_{k,j}]$, $k = 0, \ldots, 2n$, we only have to require

$$I[\phi_{0,1}] = \pi = 2(2n+1)(A+B) = Q[\phi_{0,1}].$$

This gives us

$$\frac{\pi}{2(2n+1)} = A + B. \tag{5}$$

For the cubature formula Q to be exact for all the harmonic polynomials of degree up to $4n+1$, we have to additionally satisfy $Q[\phi_{2n+1,1}] = I[\phi_{2n+1,1}]$, i.e.,

$$0 = AU_{2n+1}(t_1) + BU_{2n+1}(t_2). \tag{6}$$

For exactness up to degree $6n+2$, we get additionally ($k = 4n+2$):

$$0 = AU_{4n+2}(t_1) + BU_{4n+2}(t_2). \tag{7}$$

For exactness up to degree $8n+3$, we get additionally ($k = 6n+3$):

$$0 = AU_{6n+3}(t_1) + BU_{6n+3}(t_2). \tag{8}$$

In the following, we construct a family of solutions to the system of equations (5)–(8). From (6), we get that

$$B = -A\frac{U_{2n+1}(t_1)}{U_{2n+1}(t_2)},$$

provided that $U_{2n+1}(t_2) \neq 0$. Inserting this in (5), we get that

$$A = \frac{\pi}{2(2n+1)} \frac{U_{2n+1}(t_2)}{U_{2n+1}(t_2) - U_{2n+1}(t_1)} \tag{9}$$

under the additional condition that $U_{2n+1}(t_1) \neq U_{2n+1}(t_2)$. Hence from (5) it follows that

$$B = \frac{-\pi}{2(2n+1)} \frac{U_{2n+1}(t_1)}{U_{2n+1}(t_2) - U_{2n+1}(t_1)} \tag{10}$$

under the additional condition that $U_{2n+1}(t_1) \neq 0$. Inserting A and B in (7), we get that the equality

$$U_{4n+2}(t_1)U_{2n+1}(t_2) = U_{4n+2}(t_2)U_{2n+1}(t_1) \tag{11}$$

should hold true. Analoguously, inserting A and B in (8), the equality

$$U_{6n+3}(t_1)U_{2n+1}(t_2) = U_{6n+3}(t_2)U_{2n+1}(t_1) \tag{12}$$

should hold true.

We summarize the conditions obtained so far in a lemma.

Lemma 1. *Assume that a pair* (t_1, t_2) *satisfying* $t_1 \neq t_2$, $U_{2n+1}(t_1) \neq 0$, $U_{2n+1}(t_2) \neq 0$, *and* $U_{2n+1}(t_2) - U_{2n+1}(t_1) \neq 0$ *satisfies also the equations* (11) *and* (12), *A and B are defined according to* (9) *and* (10), *respectively, and the angles* θ_i, ψ_i *are equispaced as in* (4) *with* $\alpha = 0$. *Then the corresponding cubature rule* (3) *is precise for all harmonic polynomials of degree up to* $8n+3$.

In order to construct such pairs(t_1, t_2), we first prove a technical lemma.

Lemma 2. *Let* t, t' *be zeros of* U_{4n+2}. *Then we have*

$$U_{2n+1}(t) \neq 0, \tag{13}$$

$$U_{6n+3}(t) = -U_{2n+1}(t), \tag{14}$$

$$U_{8n+4}(t) = -1, \tag{15}$$

$$U_{2n+1}(t) = U_{2n+1}(t') \quad \Leftrightarrow \quad t = t'. \tag{16}$$

Proof. Since t is a zero of U_{4n+2}, it can be written as $t = \cos\frac{k\pi}{4n+3}$ with some $k \in \{1, 2, \ldots, 4n+2\}$.

We can only have $U_{2n+1}(t) = 0$ if $\sin\frac{(2n+2)k\pi}{4n+3} = 0$, or in other words, if $\frac{2n+2}{4n+3}k \in \mathbb{Z}$. However, by Euclid's algorithm, we have

$$\gcd(4n+3, 2n+2) = \gcd(2n+2, 2n+1) = \gcd(2n+1, 1) = 1.$$

Hence there are no solutions k to this equation between $k = 0$ and $k = 4n+3$, which proves (13).

Statement (14) follows immediately from

$$\sin\frac{(6n+4)k\pi}{4n+3} = \sin\Big(2k\pi - \frac{(2n+2)k\pi}{4n+3}\Big) = -\sin\frac{(2n+2)k\pi}{4n+3}$$

and the trigonometric definition of the Chebyshev polynomials. Similarly, the statement (15) follows from the observation that

$$\sin\frac{(8n+5)k\pi}{4n+3} = \sin\Big(2k\pi - \frac{k\pi}{4n+3}\Big) = -\sin\frac{k\pi}{4n+3}.$$

For proving (16), we use the relation

$$T_p(t)U_q(t) = \frac{1}{2}\big(U_{p+q}(t) + U_{q-p}(t)\big) \tag{17}$$

which can be easily proved using the "product to sum" formula

$$\cos\alpha\sin\beta = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\beta-\alpha)\big)$$

for $\alpha := p\arccos t$, $\beta := (q+1)\arccos t$, and the definitions for the Chebyshev polynomials $T_p(t) = \cos(p\arccos t)$ and $U_q(t) = \frac{\sin(q+1)\arccos t}{\sin(\arccos t)}$. Setting $p = q = 2n+1$ in (17), we get

$$T_{2n+1}(t)U_{2n+1}(t) = \frac{1}{2}\big(U_{4n+2}(t) + U_0(t)\big).$$

Inserting the expression for t, we obtain

$$U_{2n+1}\Big(\cos\frac{k\pi}{4n+3}\Big) = \frac{1}{2T_{2n+1}\big(\cos\frac{k\pi}{4n+3}\big)}.$$

Let $t' = \cos\frac{j\pi}{4n+3}$. Then $U_{2n+1}(t) = U_{2n+1}(t')$ if and only if

$$\cos\frac{2n+1}{4n+3}k\pi = \cos\frac{2n+1}{4n+3}j\pi.$$

The latter holds true if and only if

$$\frac{(2n+1)\pi}{4n+3}(k \pm j) = 2\pi m, \qquad j,k \in \{1,\ldots,4n+2\}, \quad m \in \mathbb{Z}.$$

Since

$$\gcd(4n+3, 2n+1) = \gcd(2n+1, 1) = 1,$$

then $k \pm j = 2(4n+3)s$ for some $s \in \mathbb{Z}$, which can only happen when $k = j$, i.e., when $t = t'$. This completes the proof of the lemma. $\square$

According to the above lemma, we construct pairs (t_1, t_2) which satisfy the conditions of Lemma 1 by choosing t_1 and t_2 as two different zeros of U_{4n+2}. Then equation (11) is satisfied. Statement (14) of the above lemma proves that also equation (12) is satisfied for this choice. Statements (13) and (16) take care of three of the additional assumptions in Lemma 1.

The third statement (15) allows us to show sharpness of the obtained degree of precision of the cubature rule Q. Indeed, if we wish to increase the degree of precision beyond $8n+3$, we have to satisfy the additional condition $Q[\phi_{8n+4,1}] = I[\phi_{8n+4,1}]$, i.e.,

$$0 = A\,U_{8n+4}(t_1) + B\,U_{8n+4}(t_2).$$

However, due to (15), this is equivalent to

$$0 = A + B,$$

a contradiction to (5).

All in all, we have proved the following theorem.

Theorem 2. *Consider the cubature rule*

$$\begin{aligned} I[u] \approx Q[u] = {} & \frac{\pi}{2(2n+1)} \frac{1}{\sqrt{1-t_1^2}} \frac{U_{2n+1}(t_2)}{U_{2n+1}(t_2) - U_{2n+1}(t_1)} \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u, t_1) \\ & - \frac{\pi}{2(2n+1)} \frac{1}{\sqrt{1-t_2^2}} \frac{U_{2n+1}(t_1)}{U_{2n+1}(t_2) - U_{2n+1}(t_1)} \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u, t_2) \end{aligned}$$

with equispaced angles θ_j as in (4)*. If we choose roots of* U_{4n+2}

$$t_1 = \sin\frac{j\pi}{4n+3}, \quad t_2 = \sin\frac{k\pi}{4n+3}, \qquad j, k \in \{1, \dots, 4n+2\}$$

with $j \neq k$, then the cubature rule Q is exact for all harmonic polynomials of degree up to $8n+3$ and there exist harmonic polynomials of degree $8n+4$ for which it is not exact.

Remark 1. Theorem 2 states that with integrals along $M = 4n+2$ properly chosen chords, we can integrate exactly all harmonic polynomials from the space $\mathcal{H}_{2M-1}$, which has dimension $4M-1$. The result is thus of the same quality as the cubature rule using constant distances t from [6], where $N = 2n+1$ chords were sufficient for exact integration in $\mathcal{H}_{2N-1}$ with dimension $4N-1$.

4. Examples

4.1. Example 1

We test the cubature rule from Theorem 2 on the harmonic function

$$u(x,y) = \log \sqrt{(x-1)^2 + (y-1)^2}.$$

In Figure 3 we plot the cubature errors for varying degree n (x-axis) with the choice $t_1 = \cos \frac{6\pi}{4n+3}$ and $t_2 = \cos \frac{12\pi}{4n+3}$ according to Theorem 2.

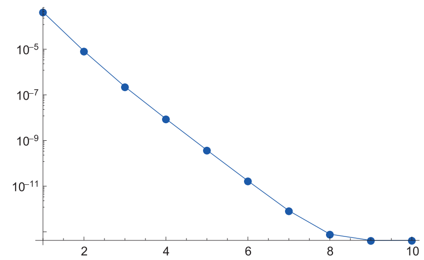

Figure 3. Cubature errors for varying degree n (x-axis) with the choice $t_1 = \cos \frac{6\pi}{4n+3}$ and $t_2 = \cos \frac{12\pi}{4n+3}$.

We observe that the errors decay exponentially with the degree n until machine accuracy is reached.

In Figure 4 we plot the cubature errors for varying distances $t_1, t_2 \in [0,1]$ for $n = 2$. Although the diagonal $t_1 = t_2$ is disallowed and leads to division by zero, we see that choices very close to the diagonal still lead to well-performing cubature rules.

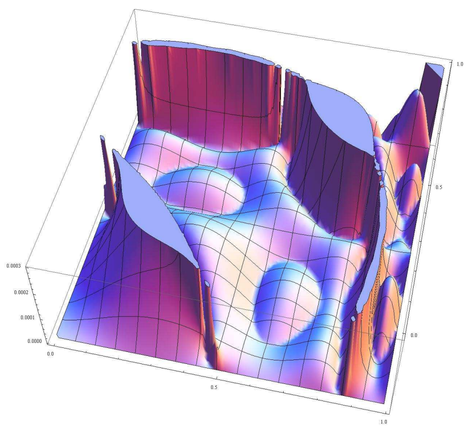

Figure 4. Cubature errors for varying distances t_1 (x-axis) and t_2 (y-axis). Errors are truncated at $3 \cdot 10^{-4}$.

A cross section through the error plot with $n = 2$ and fixed $t_1 = \cos\frac{5\pi}{4n+3}$ is shown in Figure 5. The positive roots of U_{4n+2} are marked with diamonds. Again we see that the limit $t_2 \to t_1$ exists, although $t_1 = t_2$ is disallowed. However, choosing t_2 as a different root leads to smaller errors. The peaks in the error plot are avoided by our choice.

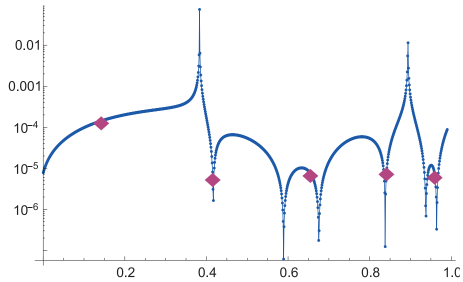

Figure 5. Cubature errors for fixed distance $t_1 = \cos\frac{5\pi}{4n+3}$ and t_2 varying (x-axis). The positive roots of U_{4n+2} are shown as diamonds. The root t_1 corresponds to the left-most diamond.

4.2. Example 2

In order to test the cubature rule for functions with less smoothness, we construct the harmonic extension of the boundary function $f(\theta) = \theta^2$ on the unit circle in polar coordinates, where the argument θ is chosen in the interval $[-\pi, \pi]$. This function is only C^0 on the unit circle, but analytic within the unit disk. By expanding f into its Fourier series, it can be shown that the corresponding harmonic function has the representation

$$u(x,y) = \mathrm{Re}\Big(\frac{\pi^2}{3} + 2\big(\mathrm{Li}_2(-x-\mathrm{i}y) + \mathrm{Li}_2(-x+\mathrm{i}y)\big)\Big),$$

where

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

is the dilogarithm or Spence's function. See Figure 6 for a plot of the harmonic function u.

For the chord distances, we choose $t_1 = \cos\frac{\pi}{4n+3}$ and $t_2 = \cos\frac{4\pi}{4n+3}$, which satisfy the conditions of Theorem 2.

We compare the cubature rule Q from Theorem 2 to the cubature rule $\bar{Q}[u]$ derived in [6], which uses $2n+1$ equispaced angles as in (4) and a constant distance $\bar{t}$ chosen as a zero of U_{2n+1}. In this example we choose $\bar{t} = \cos\frac{n\pi}{2n+2}$ and thus

$$\bar{Q}[u] = \frac{\pi}{(4n+2)\sqrt{1-\bar{t}^2}} \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u, \bar{t}).$$

This rule is exact for all harmonic polynomials of degree at most $4n + 3$, as shown in [6].

In Figure 7 we plot the errors for the cubature rules Q and $\bar{Q}$ for varying number of used chords (x-axis). We observe that both cubatures rules show convergence which is exponential in the number of chords. Furthermore, the cubature rule Q yields smaller errors than the cubature rule $\bar{Q}$ in this example. However, we note that this is dependent on the chosen distances. In general, the rules seem to produce errors which are of similar order of magnitude.

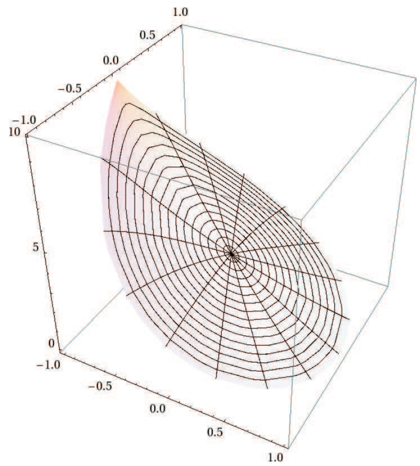

Figure 6. The harmonic function u with C^0 boundary data.

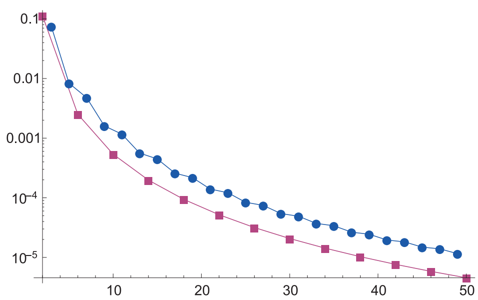

Figure 7. Errors for the cubature rules Q (circles) and $\bar{Q}$ (squares) for varying the number of the used chords (x-axis).

5. Conclusions

We have constructed cubature rules for harmonic functions on the disk using Radon projections along chords with two different distances to the origin. We have given a construction for pairs of distances (t_1, t_2) which leads to high degree of precision. The numerical results show that these rules perform well and yield similar errors, for the same number of chords, as the previous rules with only one constant distance of the chords.

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