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Interpolating Basis in the Space $C^\infty[-1,1]^d$

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An interpolating Schauder basis in the space $C^\infty[-1,1]^d$ is suggested. This gives a unified approach for constructing bases in spaces of infinitely differentiable functions and their traces on compact sets. In the construction we use Newton's interpolation of functions at the sequence that was found recently by Jean-Paul Calvi and Phung Van Manh.

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1. Introduction

There is a variety of different topological bases in the space $C^\infty[-1,1]$. The classical work here is [14], where Mityagin found the first such basis, namely the Chebyshev polynomials. Later it was proven in [12] and [1] that other classical orthogonal polynomials have the basis property in this space as well. If we apply the result by Zeriahi [18] to the set $[-1,1]$ then we obtain a basis from polynomials that are orthogonal with respect to more general measures. Following Triebel [17] (see also [2]) a basis from eigenvectors of a certain differential operator can be constructed in this space. A special basis in $C^\infty[0,1]$ was used in [10] to construct a basis in the space of C^∞-functions on a graduated sharp cusp with arbitrary sharpness. Recently it was shown in [9] that the wavelets system suggested by Kilgor and Prestin in [13] also forms a basis in the space $C^\infty[-1,1]$.

In view of the isomorphism

$$C^\infty[-1,1]^d \simeq \underbrace{C^\infty[-1,1]\,\hat{\otimes}\,C^\infty[-1,1]\,\hat{\otimes}\,\cdots\,\hat{\otimes}\,C^\infty[-1,1]}_{d}$$

(see [11, Ch. 2, Theorem 13]) these results can be extended to the multivariate case.

Here we present an interpolating topological basis in the space $C^\infty[-1,1]^d$. Together with [7] it gives a unified approach for constructing bases in spaces of infinitely differentiable functions and their traces on compact sets.

A polynomial basis $(P_n)_{n=0}^\infty$ in a functional space is called a Faber (or strict polynomial) basis if $\deg P_n = n$ for all n. Due to the classical result of Faber [6], the space $C[a,b]$ does not possess such a basis.

Here we use the Newton interpolation, so the basis presented for $C^\infty[-1,1]$ is a Faber basis. The crucial aspect in the proof is the existence of the sequence $(x_n)_{n=1}^\infty \subset [-1,1]$ with a moderate growth of the corresponding Lebesgue constants. A sequence of this type was found recently by Calvi and Manh in [4]. This essentially improves the author's result [8] where, for the sequence of the Lebesgue constants, only the asymptotic behavior $\exp(\log^2 n)$ was achieved.

2. Interpolating Topological Basis in $C^\infty[-1,1]^d$

Let $\mathcal{X}$ be a linear topological space over the field $\mathbb{K}$. By $\mathcal{X}'$ we denote the topological dual space. A sequence $(e_n)_{n=0}^\infty \subset \mathcal{X}$ is a (topological) basis for $\mathcal{X}$ if for each $f \in \mathcal{X}$ there is a unique sequence $(\xi_n(f))_{n=0}^\infty \subset \mathbb{K}$ such that the series $\sum_{n=0}^\infty \xi_n(f)\, e_n$ converges to f in the topology of $\mathcal{X}$. The sequence $(\xi_n)_{n=0}^\infty$ of linear functionals $\xi_n : \mathcal{X} \longrightarrow \mathbb{K} : f \mapsto \xi_n(f)$ for $n \in \mathbb{N}_0 := \{0,1,\dots\}$ is biorthogonal to $(e_n)_{n=0}^\infty$ and total over $\mathcal{X}$. The latter indicates that $\xi_n(f) = 0$ for all $n \in \mathbb{N}_0$ implies $f = 0$.

Given a compact set $K \subset \mathbb{R}$ and a sequence of distinct points $(x_n)_{n=1}^\infty \subset K$, let $e_0 \equiv 1$ and $e_n(x) = \prod_{k=1}^n (x - x_k)$ for $n \in \mathbb{N}$. Let $\mathcal{X}(K)$ be a Fréchet space of continuous functions on K, containing all polynomials. By ξ_n we denote, by means of the divided differences, the linear functional $\xi_n(f) = [x_1, x_2, \dots, x_{n+1}]f$ with $f \in \mathcal{X}(K)$ and $n \in \mathbb{N}_0$. Properties of the divided differences (see e.g. [5]) imply the following evident result.

Lemma 1. *If a sequence $(x_n)_{n=1}^\infty$ of distinct points is dense on a perfect compact set $K \subset \mathbb{R}$, then the system $(e_n, \xi_n)_{n=0}^\infty$ is biorthogonal and the sequence of functionals $(\xi_n)_{n=0}^\infty$ is total on $\mathcal{X}(K)$.*

Here the partial sum of the expansion with respect to the system $(e_n, \xi_n)_{n=0}^\infty$ is the Lagrange interpolating polynomial of f, so $L_n(f,x) = \sum_{k=0}^n \xi_k(f)\, e_k$, and $L_n : \mathcal{X}(K) \to \mathcal{P}_n : f \mapsto L_n(f,\cdot)$ is the corresponding projection on the space of all polynomials of degree at most n.

We proceed to present an interpolating basis in the space $\mathcal{X} = C^\infty[-1,1]$ equipped with the topology defined by the sequence of norms

$$\|f\|_p = \sup\{|f^{(i)}(x)| : |x| \le 1,\ 0 \le i \le p\}, \qquad p \in \mathbb{N}_0.$$

Let $(x_n)_{n=1}^\infty$ be the sequence in $[-1,1]$ suggested in [4]. Then, by [4, Theorem 3.1], the sequence of uniform norms of L_n, which are the Lebesgue constants corresponding to the sequence $(x_n)_{n=1}^\infty$, is polynomially bounded:

$$\|L_n\|_0 \le C\, n^3 \log n \tag{1}$$

for some constant C.

Theorem 1. *The functions* $(e_n)_{n=0}^{\infty}$ *form a topological basis in the space* $C^\infty[-1,1]$.

Proof. Since the space under consideration is complete, it is enough to show that, given $f \in C^\infty[-1,1]$, the series $\sum_{n=0}^{\infty} \xi_n(f)\, e_n$ converges absolutely, that is, the series $\sum_{n=0}^{\infty} |\xi_n(f)| \cdot \|e_n\|_p$ converges for each $p \in \mathbb{N}$.

By the Markov inequality (see e.g. [5, p. 98]),

$$|\xi_n(f)| \cdot \|e_n\|_p = \|L_n(f) - L_{n-1}(f)\|_p \le n^{2p} \|L_n(f) - L_{n-1}(f)\|_0. \qquad (2)$$

Let Q_n be the polynomial of best uniform approximation to f on $[-1,1]$ and $E_n(f) = \|f - Q_n\|_0$. By the Jackson theorem (see e.g. [5, p. 219]), the sequence $(E_n(f))_{n=0}^{\infty}$ is rapidly decreasing, that is, $n^q\, E_n(f) \to 0$ as $n \to \infty$ for any fixed q. Thus, for each q there is a constant C_q such that $E_n(f) \le C_q\, n^{-q}$ for all $n \in \mathbb{N}$.

Applying (1) and a standard argument we have

$$\|L_n(f) - f\|_0 \le \|L_n(f) - L_n(Q_n)\|_0 + \|Q_n - f\|_0 \le (C\, n^3\, \log n + 1)\, C_q\, n^{-q}.$$

Therefore,

$$\|L_n(f) - L_{n-1}(f)\|_0 \le (C\, n^3\, \log n + 1)\, (C_q + C_{q-1})\, (n-1)^{-q}.$$

From (2) we conclude that the value $q = 2p + 5$ provides the desired result. □

Corollary 1. *The space* $C^\infty[-1,1]^d$ *possesses an interpolating topological basis.*

Indeed, by [3, Theorem 16], there is a sequence of points in $[-1,1]^d$ for a multivariate Newton interpolation with a polynomial grows of the corresponding Lebesgue constants. The tensor products of ordinary divided differences work now as biorthogonal functionals. Since for the set $[-1,1]^d$ both Markov's type estimation and Jackson's theorem are valid (see e.g. [15] and [16, 5.3.2]), we can repeat the proof for the univariate case.

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