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Beta Operators with Jacobi Weights

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We discuss Beta operators with Jacobi weights on $C[0,1]$ for $\alpha, \beta \geq -1$, thus including the discussion of three limiting cases. Emphasis is on the moments and their asymptotic behavior. Extended Voronovskaya-type results and a discussion concerning the over-iteration of the operators is included.

Keywords and Phrases: Beta operator, Jacobi weight, asymptotics, moments, Voronovskaya-type results, over-iteration.

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1. Introduction

Many operators arising in the theory of positive linear operators are compositions of other mappings of this type. Many times the classical Bernstein operator B_n given for $f \in C[0,1]$, $n \in \mathbb{N}$ and $x \in [0,1]$ by

$$B_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\Big(\frac{k}{n}\Big), \qquad 0 \leq k \leq n, \tag{1}$$

is one of the building blocks. Other frequently used factor operators are Beta-type operators $\mathcal{B}_r^{\alpha,\beta}$ of various kinds which will be further discussed in this note.

The best known examples are the genuine Bernstein-Durrmeyer operators U_n, the original Bernstein-Durrmeyer operators M_n, their analogies $M_n^{\alpha,\beta}$ with Jacobi weights, certain Stancu operators S_n^{α}, to name just a few. A complete list will be given in the third author's forthcoming thesis on Bernstein-Euler-Jacobi (BEJ) operators.

Here we focus on the building blocks $\mathcal{B}_r^{\alpha,\beta}$ for natural values of r and $\alpha, \beta \geq -1$, and on their moments of all orders. As is well known, knowledge of their behavior is essential for asymptotic statements as, for example, Voronovskaya-type results. We conclude this paper with a discussion concerning over-iterated operators $\mathcal{B}_n^{\alpha,\beta}$.

2. Definition of Operators $\mathcal{B}_n^{\alpha,\beta}$

Definition 1. For $f \in C[0,1]$, and $x \in [0,1]$ we define

(i) in case $\alpha = \beta = -1$:

$$\mathcal{B}_n^{-1,-1}(f;x) = \begin{cases} f(0), & x = 0; \\ \dfrac{\int\limits_0^1 t^{nx-1}(1-t)^{n-nx-1} f(t)\, dt}{B(nx, n-nx)}, & 0 < x < 1, \\ f(1), & x = 1; \end{cases}$$

(ii) in case $\alpha = -1, \beta > -1$:

$$\mathcal{B}_n^{-1,\beta}(f;x) = \begin{cases} f(0), & x = 0, \\ \dfrac{\int\limits_0^1 t^{nx-1}(1-t)^{n-nx+\beta} f(t)\, dt}{B(nx, n-nx+\beta+1)}, & 0 < x \le 1; \end{cases}$$

(iii) in case $\alpha > -1, \beta = -1$:

$$\mathcal{B}_n^{\alpha,-1}(f;x) = \begin{cases} \dfrac{\int\limits_0^1 t^{nx+\alpha}(1-t)^{n-nx-1} f(t)\, dt}{B(nx+\alpha+1, n-nx)}, & 0 \le x < 1, \\ f(1), & x = 1; \end{cases}$$

(iv) in case $\alpha, \beta > -1$:

$$\mathcal{B}_n^{\alpha,\beta}(f;x) = \frac{\int\limits_0^1 t^{nx+\alpha}(1-t)^{n-nx+\beta} f(t)\, dt}{B(nx+\alpha+1, n-nx+\beta+1)}, \qquad 0 \le x \le 1.$$

Remark 1. When discussing this class of operators one must refer to the papers of Mühlbach [5] and Lupaş in [3] where the first special cases were considered.

Case $\alpha = \beta = -1$. This case can be traced back to a paper by Mühlbach [5] who used a real number $\frac{1}{\lambda} > 0$ instead of the natural n in the definition above. The same case was investigated by Lupaş in [3], where the operator was denoted by $\overline{\mathbb{B}}_n$ (see [3, p. 63]).

Case $\alpha = \beta = 0$. These were called Beta operators by Lupaş (see [3, p. 37]) and denoted by $\mathbb{B}_n$.

3. Moments and Their Recursion

Definition 2. Let $\alpha, \beta \geq -1, n > 1, m \in \mathbb{N}_0$ and $x \in [0,1]$, then the moment of order m is defined by

$$T_{n,m}^{\alpha,\beta}(x) = \mathcal{B}_n^{\alpha,\beta}\big((e_1 - xe_0)^m; x\big).$$

Theorem 1.

$$T_{n,0}^{\alpha,\beta}(x) = 1, \qquad T_{n,1}^{\alpha,\beta}(x) = \frac{\alpha + 1 - (\alpha+\beta+2)x}{n+\alpha+\beta+2} \tag{2}$$

and for $m \geq 1$ we have the following recursion formula

$$(n+m+\alpha+\beta+2)T_{n,m+1}^{\alpha,\beta}(x) = mXT_{n,m-1}^{\alpha,\beta}(x) + [m+\alpha+1-(2m+\alpha+\beta+2)x]T_{n,m}^{\alpha,\beta}(x) \tag{3}$$

where $X = x(1-x)$.

Proof. Below we will repeatedly use the function $\psi(t) = t(1-t)$, $t \in [0,1]$. Let $f \in C^1[0,1]$, $\alpha, \beta \geq -1$, $0 < x < 1$. Then

$$\mathcal{B}_n^{\alpha,\beta}(\psi f'; x) = \frac{\int_0^1 t^{nx+\alpha}(1-t)^{n-nx+\beta} t(1-t) f'(t)\, dt}{B(nx+\alpha+1, n-nx+\beta+1)}.$$

Using integration by parts we obtain

$$\begin{aligned}
\mathcal{B}_n^{\alpha,\beta}(\psi f'; x) &= \frac{1}{B(nx+\alpha+1, n-nx+\beta+1)}\big[t^{nx+\alpha+1}(1-t)^{n-nx+\beta+1} f(t)\big]\Big|_0^1 \\
&\quad - \int_0^1 f(t)\big[(nx+\alpha+1)t^{nx+\alpha}(1-t)^{n-nx+\beta+1} \\
&\qquad - (n-nx+\beta+1)t^{nx+\alpha+1}(1-t)^{n-nx+\beta}\big]\, dt \\
&= \frac{\int_0^1 f(t)t^{nx+\alpha}(1-t)^{n-nx+\beta}[t(n-nx+\beta+1) - (1-t)(nx+\alpha+1)]\, dt}{B(nx+\alpha+1, n-nx+\beta+1)} \\
&= \frac{\int_0^1 f(t)t^{nx+\alpha}(1-t)^{n-nx+\beta}[n(t-x) - (\alpha+1) + t(\alpha+\beta+2)]\, dt}{B(nx+\alpha+1, n-nx+\beta+1)}
\end{aligned}$$

and taking into consideration the identity

$$\begin{aligned}
&n(t-x) - (\alpha+1) + t(\alpha+\beta+2) \\
&\qquad = \big((e_1 - xe_0)(n+\alpha+\beta+2) + [x(\alpha+\beta+2) - (\alpha+1)]e_0\big)(t)
\end{aligned}$$

we can write

$$\mathcal{B}_n^{\alpha,\beta}(\psi f';x) = \mathcal{B}_n^{\alpha,\beta}\big(\big[(e_1-xe_0)(n+\alpha+\beta+2)+(x(\alpha+\beta+2)-(\alpha+1))e_0\big]f;x\big). \tag{4}$$

Now in the last equation (4) we choose $f = (e_1 - xe_0)^m$ and use the identity $t(1-t) = \big(X + X'(e_1 - xe_0) - (e_1 - xe_0)^2\big)(t)$ to obtain:

$$\begin{aligned} m\mathcal{B}_n^{\alpha,\beta}&\big(\big[X(e_1 - xe_0)^{m-1} + X'(e_1 - xe_0)^m - (e_1 - xe_0)^{m+1}\big];x\big) \\ &= \mathcal{B}_n^{\alpha,\beta}\big(\big[(n+\alpha+\beta+2)(e_1-xe_0)^{m+1}-(\alpha+1-(\alpha+\beta+2)x)(e_1-xe_0)^m\big];x\big). \end{aligned}$$

The equality above becomes successively:

$$\begin{aligned} mXT_{n,m-1}^{\alpha,\beta}(x) &+ mX'T_{n,m}^{\alpha,\beta}(x) - mT_{n,m+1}^{\alpha,\beta}(x) \\ &= (n+\alpha+\beta+2)T_{n,m+1}^{\alpha,\beta}(x) - [\alpha+1-(\alpha+\beta+2)x]T_{n,m}^{\alpha,\beta}(x), \end{aligned}$$

$$\begin{aligned} (m+n+\alpha+\beta+2)&T_{n,m+1}^{\alpha,\beta}(x) \\ &= mXT_{n,m-1}^{\alpha,\beta}(x) + [m+\alpha+1-(\alpha+\beta+2+2m)x]T_{n,m}^{\alpha,\beta}(x). \end{aligned}$$

So (3) is established for $0 < x < 1$. Due to the continuity, it is valid also for $x \in \{0,1\}$. □

In particular we have:

Corollary 1. *For $\alpha = \beta = 0$ we have $\mathcal{B}_n^{0,0} = \mathbb{B}_n$ (Lupaş notation) with the corresponding recurrence formula for the moments:*

$$(n+m+2)T_{n,m+1}^{0,0}(x) = mXT_{n,m-1}^{0,0}(x) + (m+1)X'T_{n,m}^{0,0}(x),$$

where $T_{n,0}^{0,0}(x) = 1$, $T_{n,1}^{0,0}(x) = \dfrac{X'}{n+2}$.

For $\alpha = \beta = -1$ we have $\mathcal{B}_n^{-1,-1} = \overline{\mathbb{B}}_n$ (Lupaş notation). Then the recurrence formula becomes

$$(n+m)T_{n,m+1}^{-1,-1}(x) = mXT_{n,m-1}^{-1,-1}(x) + mX'T_{n,m}^{-1,-1}(x),$$

where $T_{n,0}^{-1,-1}(x) = 1$, $T_{n,1}^{-1,-1}(x) = 0$.

In the sequel we denote by $(a)^{\overline{r}} = a(a+1)\cdots(a+r-1)$ the rising factorial function. The next proposition contains another kind of recurrence formula for the moments.

Proposition 1. *Let $i \geq 0$ and $j \geq 0$ be integers. Then*

$$T_{n,m}^{\alpha+i,\beta+j}(x) = \frac{(n+\alpha+\beta+2)^{\overline{i+j}}}{(nx+\alpha+1)^{\overline{i}}(nx+\beta+1)^{\overline{j}}}\sum_{k=0}^{i+j}\frac{[x^i(1-x)^j]^{(k)}}{k!}T_{n,m+k}^{\alpha,\beta}(x). \tag{5}$$

Proof. Using the definition of the Beta operator it is easy to show that

$$\mathcal{B}_n^{\alpha,\beta}\big(t^i(1-t)^j f(t);x\big) = \frac{(nx+\alpha+1)^{\overline{i}}(nx+\beta+1)^{\overline{j}}}{(n+\alpha+\beta+2)^{\overline{i+j}}}\,\mathcal{B}_n^{\alpha+i,\beta+j}(f(t);x). \quad (6)$$

The following equation

$$t^i(1-t)^j = \sum_{k=0}^{i+j}\frac{[x^i(1-x)^j]^{(k)}}{k!}(t-x)^k \quad (7)$$

is a consequence of Taylor's formula.

Next, using (7) and the fact that the Beta operator is linear, we get

$$\mathcal{B}_n^{\alpha,\beta}\big(t^i(1-t)^j f(t);x\big) = \sum_{k=0}^{i+j}\frac{[x^i(1-x)^j]^{(k)}}{k!}\,\mathcal{B}_n^{\alpha,\beta}\big((t-x)^k f(t);x\big). \quad (8)$$

Combining (6) and (8) we arrive at

$$\begin{aligned}\mathcal{B}_n^{\alpha+i,\beta+j}(f(t);x) &= \frac{(n+\alpha+\beta+2)^{\overline{i+j}}}{(nx+\alpha+1)^{\overline{i}}(nx+\beta+1)^{\overline{j}}}\\ &\quad\times\sum_{k=0}^{i+j}\frac{[x^i(1-x)^j]^{(k)}}{k!}\,\mathcal{B}_n^{\alpha,\beta}\big((t-x)^k f(t);x\big).\end{aligned}$$

For $f(t)=(t-x)^m$ we obtain (5). □

Remark 2. Another recurrence formula for the moments of $\mathcal{B}_n^{-1,-1}$ can be found in [5, Satz 3].

4. The Moments of Order Two

Since the second moment controls to a certain extent the approximation properties of $\mathcal{B}_n^{\alpha,\beta}$, it is useful to have a closer look at it. From Theorem 1 we obtain

$$\begin{aligned}T_{n,2}^{\alpha,\beta}(x) &= \frac{(\alpha+1)(\alpha+2)+\big(n-2(\alpha+1)(\alpha+\beta+3)\big)x}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)}\\ &\qquad+\frac{\big(-n+6+(\alpha+\beta)(\alpha+\beta+5)\big)x^2}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)}.\end{aligned} \quad (9)$$

(I). First, let us remark that

$$\lim_{\alpha\to\infty} T_{n,2}^{\alpha,\beta}(x) = (1-x)^2 \quad \text{uniformly on } [0,1], \quad (10)$$

and

$$\lim_{\beta\to\infty} T_{n,2}^{\alpha,\beta}(x) = x^2 \quad \text{uniformly on } [0,1]. \tag{11}$$

Roughly speaking, a large value of α (with a fixed β) suggests a better approximation near 1, and we draw a similar conclusion from (11).

(II). Now let $\beta = \alpha \geq -1$. Consider the sequence $s_n := \dfrac{\sqrt{4n+1}-5}{4}$, $n \geq 1$. In this case,

$$T_{n,2}^{\alpha,\alpha}(x) = \frac{(\alpha+1)(\alpha+2) - \big(-n+6+2\alpha(2\alpha+5)\big)x(1-x)}{(n+2\alpha+2)(n+2\alpha+3)}. \tag{12}$$

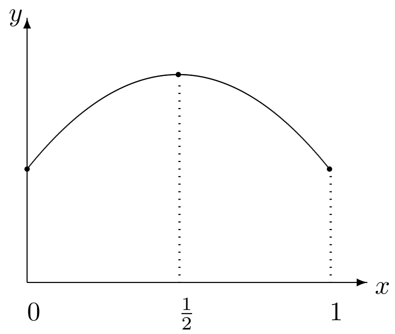
Therefore,

$$T_{n,2}^{\alpha,\alpha}(0) = T_{n,2}^{\alpha,\alpha}(1) = \frac{(\alpha+1)(\alpha+2)}{(n+2\alpha+2)(n+2\alpha+3)}, \tag{13}$$

and

$$T_{n,2}^{\alpha,\alpha}\Big(\frac{1}{2}\Big) = \frac{1}{4(n+2\alpha+3)}. \tag{14}$$

(i) If $-1 \leq \alpha < s_n$, the graph of $T_{n,2}^{\alpha,\alpha}$ has the following form:

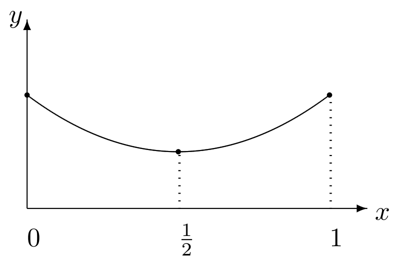


This suggests a better approximation near the end points.

(ii) If $\alpha = s_n, T_{n,2}^{\alpha,\alpha}$ is a constant function, namely

$$T_{n,2}^{s_n,s_n}(x) = \Big(\frac{\sqrt{4n+1}-1}{4n}\Big)^2, \qquad x \in [0,1]. \tag{15}$$

(iii) For $\alpha > s_n$, the graph looks like



and indicates a better approximation near $\frac{1}{2}$.

(iv) In the extreme cases, when $\alpha = -1$, respectively $\alpha \to \infty$, we have $T_{n,2}^{-1,-1}(x) = \dfrac{x(1-x)}{n+1}$, respectively $\lim\limits_{\alpha\to\infty} T_{n,2}^{\alpha,\alpha}(x) = \left(\dfrac{1-2x}{2}\right)^2$.

5. Asymptotic Formulae

Here we present first two asymptotic formulae for higher order moments of $\mathcal{B}_n^{\alpha,\beta}$ in order to arrive at Voronovskaya-type results.

Theorem 2. *For $\alpha, \beta \geq -1$ and all $l \geq 1$ one has*

$$(P_l): \begin{cases} \lim\limits_{n\to\infty} n^l T_{n,2l}^{\alpha,\beta}(x) = (2l-1)!!X^l, \\ \lim\limits_{n\to\infty} n^l T_{n,2l-1}^{\alpha,\beta}(x) = X^{l-1}\Big[(l-1)!2^{l-1}X' \sum\limits_{k=1}^{l-1} \dfrac{(2k-1)!!}{(2k-2)!!} \\ \qquad\qquad +(2l-1)!!(\alpha+1-(\alpha+\beta+2)x)\Big]. \end{cases} \tag{16}$$

The convergence is uniform on $[0,1]$.

Proof. We shall prove the proposition by induction on $l \geq 1$. The moments $T_{n,1}^{\alpha,\beta}$ and $T_{n,2}^{\alpha,\beta}$ are given by (2), respectively (9), and it is easy to prove that (P_1) is true. Suppose that (P_l) is true. According to (3) and (16),

$$\begin{aligned}
&\lim_{n\to\infty} n^{l+1} T_{n,2l+1}^{\alpha,\beta}(x) = \lim_{n\to\infty} n^{l+1} \frac{2lX}{n+2l+\alpha+\beta+2} T_{n,2l-1}^{\alpha,\beta}(x) \\
&\quad + \lim_{n\to\infty} n^{l+1} \frac{2l+\alpha+1-(4l+\alpha+\beta+2)x}{n+2l+\alpha+\beta+2} T_{n,2l}^{\alpha,\beta}(x) \\
&= 2lX^l\Big[(l-1)!2^{l-1}X'\sum_{k=1}^{l-1}\frac{(2k-1)!!}{(2k-2)!!} + (2l-1)!!(\alpha+1-(\alpha+\beta+2)x)\Big] \\
&\quad + [2l+\alpha+1-(4l+\alpha+\beta+2)x](2l-1)!!X^l \\
&= X^l\Big[2^l l! X' \sum_{k=1}^{l-1}\frac{(2k-1)!!}{(2k-2)!!} \\
&\quad + (2l-1)!!(2l(\alpha+1) - 2l(\alpha+\beta+2)x + 2l+\alpha+1-(4l+\alpha+\beta+2)x)\Big] \\
&= X^l\Big[2^l l! X' \sum_{k=1}^{l}\frac{(2k-1)!!}{(2k-2)!!} - (2l)!!X'\frac{(2l-1)!!}{(2l-2)!!} \\
&\quad + (2l-1)!!((2l+1)(\alpha+1-(\alpha+\beta+2)x) + 2l - 4lx\Big] \\
&= X^l\Big[2^l l! X' \sum_{k=1}^{l}\frac{(2k-1)!!}{(2k-2)!!} + (2l+1)!!(\alpha+1-(\alpha+\beta+2)x\Big]
\end{aligned}$$

and this proves the first formula in (16) for $l+1$ instead of l. Similarly,

$$\begin{aligned}\lim_{n\to\infty} n^{l+1} T^{\alpha,\beta}_{n,2l+2}(x) &= \lim_{n\to\infty} n^{l+1} \frac{(2l+1)X}{n+2l+1+\alpha+\beta+2} T^{\alpha,\beta}_{n,2l}(x) \\ &+ \lim_{n\to\infty} n^{l+1} \frac{2l+1+\alpha+1-(4l+2+\alpha+\beta+2)x}{n+2l+1+\alpha+\beta+2} T^{\alpha,\beta}_{n,2l+1}(x) \\ &= (2l+1)X(2l-1)!!X^l \\ &= (2l+1)!!X^{l+1},\end{aligned}$$

which is the second formula in (16) for $l+1$ instead of l. This concludes the proof by induction. □

The following result of Sikkema (see [7, p. 241]) will be used below. Note also the 1962 result of Mamedov [4] dealing with a similar problem.

Theorem 3. *Let $L_n : B[a,b] \to C[c,d]$, $[c,d] \subseteq [a,b]$, be a sequence of positive linear operators. Let the function $f \in B[a,b]$ be $q-$times differentiable at $x \in [c,d]$, where $q \geq 2$ is a natural number. Let $\varphi : \mathbb{N} \to \mathbb{R}$ be a function such that*

(i) $\lim_{n\to\infty} \varphi(n) = \infty$;

(ii) $L_n((e_1-x)^q;x) = \dfrac{c_q(x)}{\varphi(n)} + o\big(\frac{1}{\varphi(n)}\big)$ *as $n \to \infty$, where $c_q(x)$ does not depend on n;*

(iii) there exists an even number $m > q$ such that $L_n\big((e_1-x)^m;x\big) = o\big(\frac{1}{\varphi(n)}\big)$ *as $n \to \infty$.*

Then

$$\lim_{n\to\infty} \varphi(n)\Big\{L_n(f;x) - \sum_{r=0}^{q} \frac{L_n\big((e_1-x)^r;x\big)}{r!} f^{(r)}(x)\Big\} = 0.$$

Corollary 2. *(i) Theorem 3 can be rewritten in the form*

$$\lim_{n\to\infty} \varphi(n)\Big\{L_n(f;x) - \sum_{r=0}^{q-1} \frac{L_n\big((e_1-x)^r;x\big)}{r!} f^{(r)}(x)\Big\} = c_q(x)\frac{f^{(q)}(x)}{q!}.$$

(ii) If in addition to the assumption of Theorem 3, one assumes that

$$L_n\big((e_1-x)^r;x\big) = \frac{c_r(x)}{\varphi(n)} + o\Big(\frac{1}{\varphi(n)}\Big), \qquad n\to\infty, \quad r = 1,2,\ldots,q,$$

where the functions c_r are independent of n, then one also has

$$\lim_{n\to\infty} \varphi(n)\big\{L_n(f;x) - f(x)L_n(e_0;x)\big\} = \sum_{r=1}^{q} c_r(x)\frac{f^{(r)}(x)}{q!}.$$

That is, all derivatives now appear on the right-hand side which is independent of n.

As a consequence of Corollary 2 (ii) we have the following Voronovskaya-type relation.

Corollary 3. *Let* $f \in C^2[0,1]$. *Then*

$$\lim_{n\to\infty} n\big\{\mathcal{B}_n^{\alpha,\beta}(f;x)-f(x)\big\} = \frac{x(1-x)}{2}f''(x)+[\alpha+1-(\alpha+\beta+2)x]f'(x), \quad (17)$$

uniformly on $[0,1]$.

Proof. For $\varphi(n)=n$ and $q=2$ as given in Corollary 2 (ii),

$$\lim_{n\to\infty} n\big\{\mathcal{B}_n^{\alpha,\beta}(f;x)-f(x)\big\} = \sum_{r=1}^{2} c_r(x)\frac{f^{(r)}(x)}{r!} = c_1(x)\frac{f'(x)}{1!}+c_2(x)\frac{f''(x)}{2!}$$

where $c_r(x)=\lim_{n\to\infty} nT_{n,r}^{\alpha,\beta}(x)$. By using Lemma 2 with $l=1$, we get

$$c_1(x) = \alpha+1-(\alpha+\beta+2)x,$$
$$c_2(x) = X,$$

and this concludes the proof. □

Remark 3. As a consequence of Lemma 2 and Corollary 2 (i) we deduce similarly that for $f\in C^{2l}[0,1]$,

$$\lim_{n\to\infty} n^l\Big\{\mathcal{B}_n^{\alpha,\beta}(f(t);x)-\sum_{k=0}^{2l-1}\frac{f^{(k)}(x)}{k!}T_{n,k}^{\alpha,\beta}(x)\Big\} = \frac{(2l-1)!!}{(2l)!}X^l f^{(2l)}(x), \qquad l\geq 1. \quad (18)$$

From this we get also

$$\begin{aligned}\lim_{n\to\infty} n^l\Big\{\mathcal{B}_n^{\alpha,\beta}(f(t);x)-\sum_{k=0}^{2l-2}\frac{f^{(k)}(x)}{k!}T_{n,k}^{\alpha,\beta}(x)\Big\} &= \frac{(2l-1)!!}{(2l)!}X^l f^{(2l)}(x)\\ &+\frac{X^{l-1}}{(2l-1)!}\Big[(l-1)!2^{l-1}X'\sum_{k=1}^{l-1}\frac{(2k-1)!!}{(2k-2)!!}\\ &\quad+(2l-1)!!(\alpha+1-(\alpha+\beta+2)x)\Big]f^{(2l-1)}(x).\end{aligned} \quad (19)$$

Remark 4. In order to compare the above with a special previous result for the case $\alpha=\beta=-1$ we manipulate the left hand side of (19) for $l=2$ by writing

$$\begin{aligned}&\lim_{n\to\infty} n\Big[n(\mathcal{B}_n^{\alpha,\beta}(f(t);x)-f(x))-(\alpha+1-(\alpha+\beta+2)x)f'(x)-\frac{X}{2}f''(x)\Big]\\ &=\lim_{n\to\infty} n^2\Big[(\mathcal{B}_n^{\alpha,\beta}(f(t);x)-f(x)-T_{n,1}^{\alpha,\beta}(x)f'(x)-T_{n,2}^{\alpha,\beta}(x)\frac{f''(x)}{2}\Big]\\ &\quad+\lim_{n\to\infty} n\big[nT_{n,1}^{\alpha,\beta}(x)-(\alpha+1-(\alpha+\beta+2)x)\big]f'(x)\\ &\quad+\frac{1}{2}\lim_{n\to\infty} n\big[nT_{n,2}^{\alpha,\beta}(x)-X\big]f''(x).\end{aligned}$$

By using (2), (16) and (19) with $l = 2$, we get

$$\begin{aligned}
&\lim_{n\to\infty} n\Big[n(\mathcal{B}_n^{\alpha,\beta}\big(f(t);x) - f(x)\big) - \frac{X}{2}\, f''(x) - (\alpha+1-(\alpha+\beta+2)x)f'(x)\Big] \\
&\quad = \frac{1}{8}\, X^2 f^{IV}(x) + \frac{1}{6}\, X(3\alpha+5-(3\alpha+3\beta+10)x)f'''(x) \\
&\quad - (\alpha+\beta+2)(\alpha+1-(\alpha+\beta+2)x)f'(x) \\
&\quad + \frac{1}{2}\, f''(x)[(\alpha+1)(\alpha+2) - (2\alpha^2+2\alpha\beta+10\alpha+4\beta+11)x \\
&\qquad\qquad + x^2((\alpha+\beta)(\alpha+\beta+7)+11)].
\end{aligned}$$

For $\alpha = \beta = -1$ this reduces to

$$\begin{aligned}
&\lim_{n\to\infty} n\Big[n\big(\mathcal{B}_n^{-1,-1}(f;x) - f(x)\big) - \frac{X}{2}\, f''(x)\Big] \\
&\qquad = \frac{1}{24}\big(3X^2 f^{IV}(x) + 8X(1-2x)f'''(x) - 12Xf''(x)\big).
\end{aligned}$$

This result can be also deduced from [1, Remark 3].

6. Iterates of $\mathcal{B}_n^{\alpha,\beta}$

1. $\alpha = \beta = -1$. In this case $\mathcal{B}_n^{-1,-1}$ are positive linear operators preserving linear functions, and $\mathcal{B}_n^{-1,-1}e_2(x) = \dfrac{nx(nx+1)}{n(n+1)} > e_2(x)$, for $0 < x < 1$. Consequently

$$\lim_{m\to\infty} \big(\mathcal{B}_n^{-1,-1}\big)^m f(x) = (1-x)f(0) + xf(1), \qquad f \in C[0,1],$$

uniformly on $[0,1]$ (see [6]).

2. $\alpha > -1$, $\beta = -1$. Then $\mathcal{B}_n^{\alpha,-1}$ are positive linear operators preserving constant functions, $\mathcal{B}_n^{\alpha,-1}f(1) = f(1)$ for all $f \in C[0,1]$, and

$$\mathcal{B}_n^{\alpha,-1}e_2(x) = \frac{(nx+\alpha+1)(nx+\alpha+2)}{(n+\alpha+1)(n+\alpha+2)} > e_2(x), \qquad 0 \le x < 1.$$

Therefore

$$\lim_{m\to\infty} \big(\mathcal{B}_n^{\alpha,-1}\big)^m f(x) = f(1), \qquad f \in C[0,1],$$

uniformly on $[0,1]$ (see [6]).

3. $\alpha = -1$, $\beta > -1$. As in the previous case, one proves that

$$\lim_{m\to\infty} \big(\mathcal{B}_n^{-1,\beta}\big)^m f(x) = f(0), \qquad f \in C[0,1].$$

4. $\alpha > -1$, $\beta > -1$. In this case we have for all $k \geq 0$,

$$\mathcal{B}_n^{\alpha,\beta} e_k(x) = \frac{(nx + \alpha + 1)^{\overline{k}}}{(n + \alpha + \beta + 2)^{\overline{k}}}, \qquad x \in [0, 1].$$

From this we get

$$\mathcal{B}_n^{\alpha,\beta} e_k(x) = \frac{1}{(n + \alpha + \beta + 2)^{\overline{k}}} \sum_{j=0}^{k} s_{k-j}(k, \alpha) n^j x^j, \tag{20}$$

where $s_{k-j}(k, \alpha)$ are elementary symmetric sums of the numbers $\alpha + 1, \alpha + 2, \ldots, \alpha + k$; in particular $s_0(k, \alpha) = 1$ and

$$s_1(k, \alpha) = (\alpha + 1) + \cdots + (\alpha + k) = k\alpha + \frac{k(k+1)}{2}. \tag{21}$$

It follows that the numbers

$$\lambda_{n,k} := \frac{n^k}{(n + \alpha + \beta + 2)^{\overline{k}}}, \qquad k \geq 0,$$

are eigenvalues of $\mathcal{B}_n^{\alpha,\beta}$, and to each of them there corresponds a monic eigenpolynomial $p_{n,k}$ with $\deg p_{n,k} = k$. Let $p \in \Pi$ and $d = \deg p$. Then p has a decomposition

$$p = a_{n,0}(p) p_{n,0} + a_{n,1}(p) p_{n,1} + \cdots + a_{n,d}(p) p_{n,d}$$

with some coefficients $a_{n,k}(p) \in \mathbb{R}$. Since $\lambda_{n,0} = 1$ and $p_{n,0} = e_0$ we get

$$\left(\mathcal{B}_n^{\alpha,\beta}\right)^m p = a_{n,0}(p) e_0 + \sum_{k=1}^{d} a_{n,k}(p) \lambda_{n,k}^m p_{n,k}, \qquad m \geq 1$$

and so

$$\lim_{m \to \infty} \left(\mathcal{B}_n^{\alpha,\beta}\right)^m p = a_{n,0}(p) e_0, \qquad p \in \Pi. \tag{22}$$

Consider the linear functional $\mu_n : \Pi \to \mathbb{R}$, $\mu_n(p) = a_{n,0}(p)$ and the linear operator $P_n : \Pi \to \Pi$,

$$P_n p = \mu_n(p) e_0, \qquad p \in \Pi.$$

Then (22) becomes

$$\lim_{m \to \infty} \left(\mathcal{B}_n^{\alpha,\beta}\right)^m p = P_n p, \qquad p \in \Pi. \tag{23}$$

Obviously P_n is positive, and so μ_n is positive; moreover, $\|\mu_n\| = 1$ because $\mu_n(e_0) = 1$. By the Hahn-Banach theorem, μ_n can be extended to a norm-one linear functional on $C[0, 1]$. Since Π is dense in $C[0, 1]$, the extension is unique and the extended functional $\mu_n : C[0,1] \to \mathbb{R}$ is also positive. Now P_n can be extended from Π to $C[0,1]$ by setting $P_n : C[0,1] \to \Pi$, $P_n f = \mu_n(f)e_0$, $f \in C[0,1]$. Remark that

$$\|\big(\mathcal{B}_n^{\alpha,\beta}\big)^m\| = \|P_n\| = 1, \qquad m \geq 1. \tag{24}$$

Using again the fact that Π is dense in $C[0,1]$, we get from (23) and (24)

$$\lim_{m\to\infty} \big(\mathcal{B}_n^{\alpha,\beta}\big)^m f = P_n f, \qquad f \in C[0,1]. \tag{25}$$

On the other hand, from (20) we deduce the following recurrence formula for the computation of $P_n e_k, k \geq 1$:

$$\big((n+\alpha+\beta+2)^{\overline{k}} - n^k\big)P_n e_k = \sum_{j=0}^{k-1} s_{k-j}(k,\alpha) n^j P_n e_j.$$

Since $P_n e_k = \mu_n(e_k)e_0$, we get for $n \geq 1$ and $k \geq 1$

$$\mu_n(e_k) = \sum_{j=0}^{k-1} s_{k-j}(k,\alpha) \frac{n^j}{(n+\alpha+\beta+2)^{\overline{k}} - n^k}\, \mu_n(e_j). \tag{26}$$

Using (26) it is easy to prove by induction on k that there exists

$$\mu(e_k) := \lim_{n\to\infty} \mu_n(e_k), \qquad k \geq 0, \tag{27}$$

and, moreover,

$$\mu(e_k) = \frac{s_1(k,\alpha)}{(\alpha+\beta+2)+\cdots+(\alpha+\beta+k+1)}\, \mu(e_{k-1}),$$

i.e., taking (21) into account,

$$\mu(e_k) = \frac{2\alpha+k+1}{2\alpha+2\beta+k+3}\, \mu(e_{k-1}), \qquad k \geq 1.$$

Since $\mu(e_0) = 1$, it follows that

$$\mu(e_k) = \frac{(2\alpha+2)^{\overline{k}}}{(2\alpha+2\beta+4)^{\overline{k}}}, \qquad k \geq 0.$$

This can be rewritten as

$$\mu(e_k) = \frac{B(2\alpha+k+2, 2\beta+2)}{B(2\alpha+2, 2\beta+2)} = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} e_k(t)\, dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1}\, dt},$$

so that

$$\mu(p) = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} p(t)\,dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1}\,dt}, \qquad p \in \Pi.$$

Consider the extension of μ to $C[0,1]$, i.e.,

$$\mu(f) = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} f(t)\,dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1}\,dt}, \qquad f \in C[0,1],$$

and the positive linear operator $P : C[0,1] \to \Pi$, $Pf = \mu(f)e_0$, $f \in C[0,1]$. According to (27), $\lim_{n\to\infty} \mu_n(p) = \mu(p)$, $p \in \Pi$, i.e.,

$$\lim_{n\to\infty} P_n p = Pp, \qquad p \in \Pi. \tag{28}$$

Since $\|P_n\| = \|P\| = 1$, $n \geq 1$, we conclude from (28) that $\lim_{n\to\infty} P_n f = Pf$, $f \in C[0,1]$. Thus, for the operators P_n described in (25) we have proved:

Theorem 4. *Let $\alpha > -1$, $\beta > -1$. Then for each $f \in C[0,1]$ and $n \geq 1$,*

$$\lim_{n\to\infty} P_n f = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} f(t)\,dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1}\,dt}\, e_0.$$

For $\alpha = \beta = 0$, this result was obtained, with different methods, in [2].

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