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Semidefinite Extreme Points in a Polynomial Space*

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Let Δ be the standard simplex in $\mathbb{R}^2$. Denote by π_2 the set of all real bivariate algebraic polynomials of total degree at most two. Let B_Δ be the unit ball of the space π_2 endowed with the supremum norm on Δ.

We present with short proofs two results from [13] which describe the semidefinite extreme points of B_Δ. This completes the description of the set E_Δ of all extreme points of B_Δ, initiated in [11] and [12].

In addition, we give graphical illustrations of some typical semidefinite polynomials from E_Δ.

Keywords and Phrases: Polynomials, extreme points, standard simplex.

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1. Introduction

Denote by π_n^d the set of all real algebraic polynomials of d variables and of total degree not exceeding n. Let K be a compact set in $\mathbb{R}^d$ and $\|f\|_{C(K)} := \max_{X\in K} |f(X)|$ be the uniform norm on K.

We use the notation $B_n(K)$ for the unit ball of π_n^d with respect to $\|\cdot\|_{C(K)}$, i.e., $B_n(K) = \left\{p \in \pi_n^d : \|p\|_{C(K)} \le 1\right\}$. The set of all extreme points of $B_n(K)$ will be denoted by $E_n(K)$. Recall that a point p of a convex set B is said to be *extreme* if the equality $p = \lambda p_1 + (1-\lambda)p_2$ for some $p_1, p_2 \in B$ and $\lambda \in (0,1)$ implies $p = p_1 = p_2$.

According to the Krein-Milman theorem, $B_n(K)$ is the convex hull of $E_n(K)$. This result motivates the study of the extreme points of the unit ball of various polynomial spaces. We refer to papers [8, 3, 5, 1, 6, 18], where also related problems in the geometry of polynomials are studied. An important consequence

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of Krein-Milman's theorem is the fact that

$$\max_{p\in B_n(K)} F(p) = \max_{p\in E_n(K)} F(p),$$

provided F is a convex function defined on $B_n(K)$. Therefore, the description of the extreme points of $B_n(K)$ can be useful in deriving the exact constants in certain inequalities for polynomials.

Recently several authors studied generalizations of the inequality of Bernstein for multivariate polynomials on convex bodies (see [20, 9, 10, 14, 16, 15, 2, 19, 4]). If the convex body is non-symmetric, the problem of finding the sharp Bernstein's inequality is still open. In this connection, the description of the extreme points when K is the standard simplex is of special interest.

Let Δ be the standard simplex in $\mathbb{R}^2$, i.e.,

$$\Delta := \big\{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0,\ x+y \leq 1\big\}.$$

The strictly definite and the indefinite extreme points of $B_2(\Delta)$ were described in [11, 12]. In [13] we completed the description of $E_2(\Delta)$, finding its semidefinite elements.

In what follows, we denote the vertices of Δ by $O(0,0)$, $A(1,0)$ and $B(0,1)$. We shall use the abbreviated notations $\pi_2 := \pi_2^2$, $\|\cdot\| := \|\cdot\|_{C(\Delta)}$, $B_\Delta := B_2(\Delta)$, and $E_\Delta := E_2(\Delta)$.

Recall that

$$p(x,y) = a + b\,x + c\,y + d\,x^2 + 2e\,xy + f\,y^2 \tag{1}$$

is a *semidefinite* quadratic polynomial if and only if

$$\det(M) = 0, \qquad \text{where } M := \begin{pmatrix} d & e \\ e & f \end{pmatrix}.$$

Note that a polynomial p of degree at most one is an extreme point of B_Δ if and only if $p \equiv 1$ or $p \equiv -1$ (see Section 2). Therefore, we shall consider only semidefinite polynomials of degree exactly two. For definiteness we shall suppose that the polynomials are negative semidefinite. In other words, we shall describe the elements of $E_\Delta \cap \mathcal{P}_2^-$, where $\mathcal{P}_2^- := \{p \in \pi_2 :\ \lambda_1 < 0,\ \lambda_2 = 0\}$, and $\{\lambda_i\}_1^2$ are the eigenvalues of M. It is known ([7]) that a polynomial of the form (1) belongs to $\mathcal{P}_2^-$ if and only if $\det(M) = 0$ and ($d < 0$ or $f < 0$).

The coordinates of every point of a local extremum $X_0 = (x_0, y_0)$ of a polynomial $p \in \mathcal{P}_2^-$ satisfy the system

$$2M(x,y)^\top = -(b,c)^\top.$$

Since the rank of M is equal to 1, the above system either has no solutions or has a one-dimensional set of solutions. In the first case p has no local extrema in $\mathbb{R}^2$. In the second case every solution is a global maximum of p on $\mathbb{R}^2$, which follows from the representation

$$p(X) = p(X_0) + (X - X_0)M(X - X_0)^\top,$$

where $X = (x, y)$, and the negative semidefiniteness of M.

Given a polynomial p, we denote by $\mathcal{M}(p)$ the set of all points $X \in \Delta$ such that $|p(X)| = \|p\|$.

The aim of this paper is to present the following results from [13], which give a full description of the semidefinite elements of E_Δ.

Theorem 1. *Suppose that $p \in \mathcal{P}_2^-$ and there exists a point $X_0 = (x_0, y_0) \in \operatorname{int}\Delta$ such that $p(X_0) = 1$. Then p is an extreme point of B_Δ if and only if the following conditions hold:*

(i) $p(x, y) = 1 - \big[\alpha(x - x_0) + \beta(y - y_0)\big]^2$, $(\alpha, \beta) \neq (0, 0)$;

(ii) $\min\{p(O), p(A), p(B)\} = -1$.

Theorem 2. *Suppose that $p \in \mathcal{P}_2^-$ and $p(X) \neq 1$ for every $X \in \operatorname{int}\Delta$. Then p is an extreme point of B_Δ if and only if $p \in \{p_i\}_{i=1}^6$, where*

$$
\begin{aligned}
p_1(x, y) &= 1 - 2(x + y)^2, & p_4(x, y) &= 1 - 2(x + y - 1)^2,\\
p_2(x, y) &= 1 - 2(x - 1)^2, & p_5(x, y) &= 1 - 2x^2,\\
p_3(x, y) &= 1 - 2(y - 1)^2, & p_6(x, y) &= 1 - 2y^2.
\end{aligned}
$$

Note that all positive semidefinite elements of E_Δ have the form $q = -p$, where $p \in E_\Delta \cap \mathcal{P}_2^-$.

Concise proofs of Theorems 1 and 2 are given in Section 2. Section 3 contains graphical illustrations of some typical semidefinite polynomials from E_Δ.

2. Proofs

We shall use a lemma which is proved in [11].

Lemma A. *Suppose that $n, d \in \mathbb{N}$ and K is a convex body in $\mathbb{R}^d$. Let $p \in B_n(K)$ satisfy $\|p\|_{C(K)} = 1$. Suppose that $q \equiv 0$ is the only polynomial from π_n^d, which satisfies the conditions:*

(a) $q(X) = 0$ *for every* $X \in \mathcal{M}(p)$;

(b) $\dfrac{\partial q}{\partial \overrightarrow{XM}}(X) = 0$ *if* $X \in \mathcal{M}(p)$, $M \in K$, *and* $\dfrac{\partial p}{\partial \overrightarrow{XM}}(X) = 0$.

Then p is an extreme point of $B_n(K)$.

Proof of Theorem 1. Necessity. Since $p \in E_\Delta$ we have $\|p\| = 1$. The assumption $p(X_0) = 1$ implies that X_0 is a local (and global) extremum of p. Therefore p has a straight line of global maxima, including X_0. Let us denote it by m. We have $p(x, y) \equiv 1$ on m which gives

$$p(x, y) - 1 = m(x, y)\, l(x, y),$$

where $m(x,y)=0$ is the equation of m and $l\in\pi_1^2$. For every $X=(x,y)\in m$ we have

$$\frac{\partial p}{\partial x}(X)=\frac{\partial m}{\partial x}(X)\,l(X)+m(X)\,\frac{\partial l}{\partial x}(X)=\frac{\partial m}{\partial x}(X)\,l(X)=0.$$

Similarly,

$$\frac{\partial m}{\partial y}(X)\,l(X)=0.$$

But $\left(\frac{\partial m}{\partial x}(X),\frac{\partial m}{\partial y}(X)\right)\neq(0,0)$ which implies $l(X)=0$ for every $X\in m$, i.e., $l=c\,m$. Thus $p(x,y)=1+c\,m^2(x,y)$ and the coefficient c has to be negative. The last formula proves the representation (i).

To prove (ii) we note that every concave function on Δ attains its minimum at some vertex of Δ and every non-constant polynomial $p\in E_\Delta$ takes the values ±1 in Δ (see [11, Lemma 1]).

Sufficiency. Suppose that a polynomial p has the form (i) and satisfies (ii). Since $X_0\in\Delta$, conditions (i) and (ii) imply $\|p\|=1$. It remains to show that $p\in E_\Delta$.

We shall apply Lemma A. Let us denote by m the straight line with equation $\alpha(x-x_0)+\beta(y-y_0)=0$. The set $m\cap\Delta$ is a subset of $\mathcal{M}(p)$. Note that $\operatorname{grad}p(X)=\mathbf{0}$ for every $X\in m$. Let X_1 be a vertex of Δ such that $p(X_1)=-1$. Clearly, $X_1\in\mathcal{M}(p)$. According to Lemma A, it is sufficient to prove that if a polynomial $q\in\pi_2$ satisfies the conditions

$$\begin{aligned}&q(X)=0,\quad \operatorname{grad}q(X)=\mathbf{0},\qquad \text{for every } X\in m,\\ &q(X_1)=0,\end{aligned}\tag{2}$$

then $q\equiv0$.

The condition $q(X)=0$ on m implies $q(x,y)=m(x,y)\,l(x,y)$, where $l\in\pi_1^2$. Using the second equality in (2) we obtain

$$\frac{\partial q}{\partial x}(X)=\frac{\partial m}{\partial x}(X)\,l(X)+m(X)\,\frac{\partial l}{\partial x}(X)=0$$

for every $X\in m$, which implies $\alpha\,l(X)=0$ on m. Similarly, $\beta\,l(X)=0$ on m. Since at least one of the coefficients α and β is different from zero, we conclude that $l=c\,m$, where c is a constant.

Finally, it follows from the third condition in (2) that $c\,m^2(X_1)=0$. But $m(X_1)\neq0$ since otherwise we would have $p(X_1)=1$, a contradiction. Consequently $c=0$, i.e., $q\equiv0$. The theorem is proved. □

The following lemmas are needed for the proof of Theorem 2.

Lemma 1. *Suppose that $p\in\mathcal{P}_2^-\cap E_\Delta$ and $p(X_1)=p(X_2)=1$ for some $X_1,X_2\in\Delta$, $X_1\neq X_2$. Then $p\equiv1$ on the line X_1X_2.*

Proof. It is an easy consequence of the concavity of p. $\square$

It follows from Lemma 1 that if a polynomial $p \in \mathcal{P}_2^- \cap E_\Delta$ attains the value one only at $\partial\Delta$ then there are two possibilities:

Case 1. There exists a unique point $X_1 \in \partial\Delta$ such that $p(X_1) = 1$;

Case 2. There exists a side l of Δ such that $p \equiv 1$ on l. (In fact, by Lemma 1, l is unique.)

It turns out that *Case 1* divides into two essentially different subcases, namely X_1 is a vertex of Δ or X_1 is an interior point for a side of Δ. We begin with the first subcase.

Lemma 2. *Suppose that $p \in \mathcal{P}_2^- \cap E_\Delta$, $p(O) = 1$, and $p(X) < 1$ for every $X \in \Delta \setminus \{O\}$. Then $p \equiv -1$ on the closed segment $[AB]$.*

Proof. The concavity of p and the fact that p attains the value -1 in Δ (see [11, Lemma 1]) imply that

$$\min_{X\in\Delta} p(X) = \min\{p(O), p(A), p(B)\} = -1.$$

Let us suppose for definiteness that $p(A) = -1$. It remains to prove that $p(X) = -1$ for every X belonging to the semi-closed segment $(AB]$. Assume the contrary, i.e., there exists a point $X_0 \in (AB]$ such that $p(X_0) > -1$. We shall construct a nonzero polynomial $q \in \pi_2$ such that $f := p \pm \varepsilon q \in B_\Delta$ for every sufficiently small positive number ε. This will be in contradiction to $p \in E_\Delta$.

It is easily seen that $\mathcal{M}(p) \subset \{O, A, B\}$. By virtue of Lemma A, we choose a nonzero $q \in \pi_2$ which satisfies the conditions:

$$q(O) = q(A) = q(B) = 0 \quad \text{and} \quad \frac{\partial q}{\partial x}(O) = \frac{\partial q}{\partial y}(O) = 0.$$

For definiteness, we take $q(x, y) = xy$. We establish the following properties of f, provided ε is sufficiently small:

(a) $\dfrac{\partial f}{\partial \overrightarrow{OM}}(O) \le 0$ for every $M \in \Delta$, $M \ne O$;

(b) $\dfrac{\partial f}{\partial \overrightarrow{AM}}(A) > 0$ for every $M \in \Delta$, $M \ne A$.

Using (a) and (b), we prove the inequality $|f(X)| \le 1$ first for $X \in \partial\Delta$ and then for $X \in \operatorname{int}\Delta$. $\square$

Lemma 3. *Suppose that $p \in \mathcal{P}_2^-$, $p(O) = 1$, and $p(X) < 1$ for every $X \in \Delta \setminus \{O\}$. Then $p \in E_\Delta$ if and only if $p(x, y) = 1 - 2(x + y)^2$.*

Proof. Necessity. Suppose that $p \in E_\Delta$. Using Lemma 2 and the semidefiniteness of p, we obtain that p has the form $p_a(x, y) = [a(x+y)-2](x+y-1)-1$.

Furthermore, it is seen that $p_a \in \mathcal{P}_2^- \cap B_\Delta$ if and only if $a \in [-2,0)$. The representation $p_a = \lambda p_-2 + (1-\lambda)p_0$, $\lambda = -\frac{a}{2} \in (0,1)$, shows that $p_a \notin E_\Delta$ for $a \in (-2,0)$, hence $p = p_{-2} = 1 - 2(x+y)^2$.

Sufficiency. It is proved by applying Lemma A to the polynomial p_{-2}. □

Recall that the second possibility is that X_1 belongs to the interior of a side of Δ. We prove the following

Lemma 4. *Suppose that* $p \in \mathcal{P}_2^-$, $\|p\| = 1$, $p(X_1) = 1$ *for some point* $X_1 \in (AB)$, *and* $p(X) < 1$ *for every* $X \in \Delta \setminus \{X_1\}$. *Then* p *is not an extreme point of* B_Δ.

Proof. Note first that p cannot be identically equal to -1 on $[OA] \cup [OB]$. Without loss of generality we can suppose that $p\big|_{OB} \not\equiv -1$. We set $q(x,y) := y(x+y-1)$ and shall prove that $f := p \pm \varepsilon q \in B_\Delta$ for every sufficiently small positive number ε, which implies $p \notin E_\Delta$. To this end, we show that $\operatorname{grad} p(X) \neq \mathbf{0}$ for every $X \in \Delta$. Hence, the same holds true for f, provided ε is sufficiently small. Therefore $\|f\| = \|f\|_{C(\partial\Delta)}$ and a careful examination of f on $\partial\Delta$ completes the proof. □

Next we consider *Case 2* for p, namely p is identically equal to one on a side of Δ (see the remark after Lemma 1).

Lemma 5. *Suppose that* $p \in \mathcal{P}_2^-$, $\|p\| = 1$, *and* $p(X) \equiv 1$ *for all* $X \in [AB]$. *If* $\max_{X\in\mathbb{R}^2} p(X) > 1$ *or* p *does not have local extrema in* $\mathbb{R}^2$, *then* p *is not an extreme point of* B_Δ.

Proof. It is similar to that of Lemma 4. Here $\mathcal{M}(p) = \{O\} \cup [AB]$. By virtue of Lemma A, we consider the polynomial $q(x,y) := x(x+y-1)$ which vanishes on $\mathcal{M}(p)$ and prove that $f := p \pm \varepsilon q \in B_\Delta$ for every sufficiently small $\varepsilon > 0$. □

Lemma 6. *Suppose that a polynomial* $p \in \mathcal{P}_2^-$ *satisfies* $p(X) = 1$ *for every* $X \in [AB]$. *Then* $p \in E_\Delta$ *if and only if* $p(x,y) = 1 - 2(x+y-1)^2$.

Proof. Necessity. Using Lemma 5 and a reasoning similar to that in the beginning of the proof of Theorem 1, we obtain $p(x,y) = 1 + c(x+y-1)^2$. The concavity of p and [11, Lemma 1] imply $p(O) = -1$, hence $p(x,y) = 1 - 2(x+y-1)^2$.

Sufficiency. It is based on Lemma A. We have $\mathcal{M}(p) = \{O\} \cup [AB]$. Therefore, the conditions of the type (a) for q are

$$q(X) = 0, \qquad \text{for every } X \in [AB], \tag{3}$$

and

$$q(O) = 0. \tag{4}$$

Since $\operatorname{grad} p(X) = \mathbf{0}$ for every $X \in [AB]$ a condition of the type (b) for q is

$$\operatorname{grad} q(X) = \mathbf{0}, \qquad \text{for every } X \in [AB]. \tag{5}$$

We prove that $q \equiv 0$ is the only polynomial from π_2 which satisfies (3)–(5). By Lemma A, the proof is completed. □

Proof of Theorem 2. Necessity. Assume that $p \in \mathcal{P}_2^- \cap E_\Delta$ and $p(X) < 1$ for every $X \in \operatorname{int}\Delta$. As it was mentioned above, either there exists a unique point $X_1 \in \partial\Delta$ such that $p(X_1) = 1$, or there exists a unique side l of Δ such that $p \equiv 1$ on l.

In the first case, by Lemma 4, X_1 is a vertex of Δ. If $X_1 = O$ the explicit form of p is given by Lemma 3 and this is $p_1(x, y) = 1 - 2(x + y)^2$. The polynomials $p_2(x, y) = 1 - 2(x - 1)^2$ and $p_3(x, y) = 1 - 2(y - 1)^2$ correspond to $X_1 = A$ and $X_1 = B$, respectively.

In the second case, if $l = [AB]$ then the explicit form of $p(x, y) = p_4(x, y) = 1 - 2(x + y - 1)^2$ is obtained in Lemma 6. The remaining two polynomials (p_5 and p_6) correspond to $l = [OB]$ and $l = [OA]$.

Sufficiency. It follows form Lemmas 3 and 6 that $p_i \in \mathcal{P}_2^- \cap E_\Delta$, for every $i = 1, \ldots, 6$. Theorem 2 is proved. □

Remark. ([13, Section 3]) The only extreme points of B_Δ of degree at most one are the constants ± 1.

Indeed, note first that if $p \in \pi_0^2 \cap E_\Delta$ then $p = \pm 1$. It remains to prove that E_Δ does not contain affine functions. To this end, let $p \in (\pi_1^2 \setminus \pi_0^2) \cap B_\Delta$. We set $\alpha = p(A)$, $\beta = p(B)$, $\gamma = p(O)$. If $\alpha = \beta = \gamma$ then p is a constant, which is a contradiction. Therefore, without loss of generality we can suppose that $\alpha \neq \beta$. We consider the indefinite polynomials $f_{\pm\varepsilon} := p \pm \varepsilon q$, where $q(x, y) := xy$ and $\varepsilon > 0$ and prove that $f_{\pm\varepsilon} \in B_\Delta$ provided ε is sufficiently small. Consequently, p is not an extreme point of B_Δ.

3. Graphical Illustrations

We provide some graphs which illustrate the results of Theorem 1 and Theorem 2.

Figure 1 presents the graph of a semidefinite extreme polynomial of the type described in Theorem 1. This polynomial corresponds to the parameters $X_0 = (x_0, y_0) = (0.6, 0.1) \in \operatorname{int}\Delta$, $\alpha = 2.232$, $\beta = 0.744$, and satisfies the condition $p(O) = -1$. The straight line of global maxima m has the equation $\alpha(x - x_0) + \beta(y - y_0) = 0$. Let us note that $[PQ] = \{(x, y, 1) : \ (x, y) \in m \cap \Delta\}$.

Figures 2 and 3 show the graphs of the polynomials p_1 and p_4 of Theorem 2. The remaining four polynomials can be obtained from p_1 and p_4 by affine transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ that map the triangle Δ onto itself. The

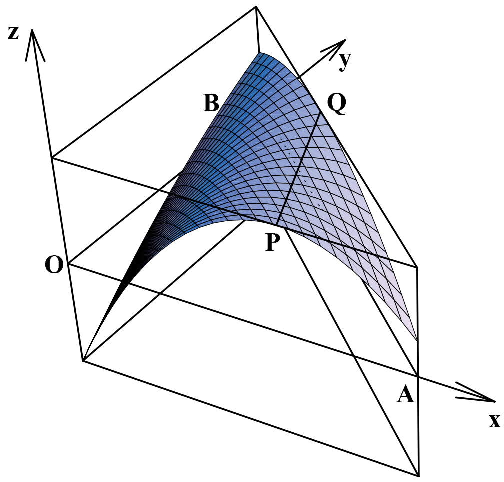

Figure 1. The polynomial $p(x, y)$ of the type in Theorem 1 with $x_0 = 0.6$, $y_0 = 0.1$, $\alpha = 2.232$, $\beta = 0.744$.

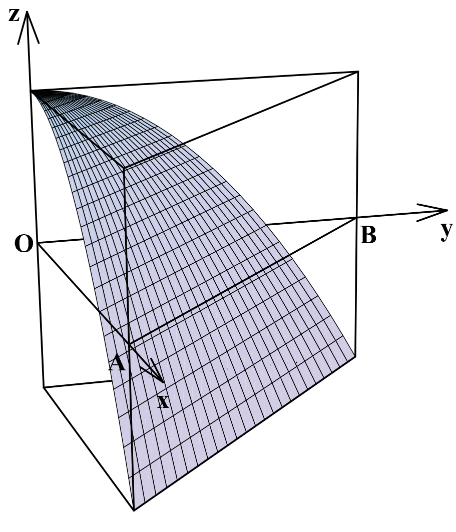

Figure 2. The polynomial $p_1(x, y) = 1 - 2(x + y)^2$ in Theorem 2.

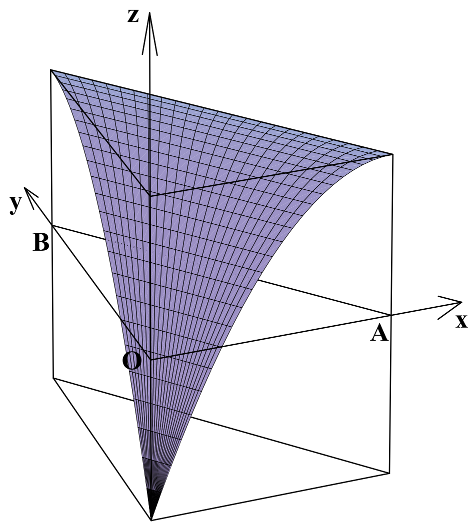

Figure 3. The polynomial $p_4(x, y) = 1 - 2(x + y - 1)^2$ in Theorem 2.

polynomials p_1 and p_4 represent the first and the second case in the proof of Theorem 2, respectively.

Graphs of strictly definite and indefinite polynomials from E_Δ are given in [17].

Bibliography

[1] R. M. Aron and M. Klimek, Supremum norms for quadratic polynomials, *Arch. Math. (Basel)* **76** (2001), 73–80.

[2] D. Burns, N. Levenberg, S. Mau and Sz. Révész, Monge-Ampère measures for convex bodies and Bernstein-Markov type inequalities, *Trans. Amer. Math. Soc.* **362** (2010), 6325–6340.

[3] Y. S. Choi and S. G. Kim, The unit ball of $\mathcal{P}(^2l_2^2)$, *Arch. Math. (Basel)* **71** (1998), 472–480.

[4] J. L. Gámez-Merino, G. A. Muñoz-Fernández, V. M. Sánchez and J. B. Seoane-Sepúlveda, Inequalities for polynomials on the unit square via the Krein-Milman theorem, *J. Convex Anal.* **20** (2013), 125–142.

[5] B. C. Grecu, Geometry of three-homogeneous polynomials on real Hilbert spaces, *J. Math. Anal. Appl.* **246** (2000), 217–229.

[6] B. C. Grecu, Geometry of homogeneous polynomials on two-dimensional real Hilbert spaces, *J. Math. Anal. Appl.* **293** (2004), 578–588.

[7] G. H. Hardy, J. E. Littlewood and G. Pólya, "Inequalities", Cambridge University Press, 1952.

[8] A. G. Konheim and T. J. Rivlin, Extreme points of the unit ball in a space of real polynomials, *Amer. Math. Monthly* **73** (1966), 505–507.

[9] A. Kroó, Classical polynomial inequalities in several variables, *in* "Constructive Theory of Functions, Varna 2002" (B. Bojanov, Ed.), pp. 19–32, DARBA, Sofia, 2003.

[10] A. Kroó and Sz. Révész, On Bernstein and Markov-type inequalities for multivariate polynomials on convex bodies, *J. Approx. Theory* **99** (1999), 134–152.

[11] L. Milev and N. Naidenov, Strictly definite extreme points of the unit ball in a polynomial space, *C. R. Acad. Bulgare Sci.* **61** (2008), no. 11, 1393–1400.

[12] L. Milev and N. Naidenov, Indefinite extreme points of the unit ball in a polynomial space, *Acta Sci. Math. (Szeged)* **77** (2011), 409–424.

[13] L. Milev and N. Naidenov, Semidefinite extreme points of the unit ball in a polynomial space, *J. Math. Anal. Appl.* **405** (2013), 631–641.

[14] L. B. Milev and Sz. Gy. Révész, Bernstein's inequality for multivariate polynomials on the standard simplex, *J. Inequal. Appl.* **2005** (2005), no. 2, 145–163.

[15] G. A. Mûnoz-Fernández, Sz. Gy. Révész and J. B. Seoane-Sepúlveda, Geometry of homogeneous polynomials on non-symmetric convex bodies, *Math. Scand.* **104** (2008), 1–14.

[16] N. Naidenov, Note on Bernstein type inequalities for multivariate polynomials, *Anal. Math.* **33**, (2007), 55–62.

[17] N. Naidenov, On the path-connectedness of the set of extreme points in a polynomial space, *in* "Constructive Theory of Functions, Sozopol 2013" (K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 165–174, Prof. Marin Drinov Academic Publishing House, Sofia, 2014.

[18] S. Neuwirth, The maximum modulus of a trigonometric trinomial, *J. Anal. Math.* **104** (2008), 371–396.

[19] Sz. Gy. Révész, Conjectures and results on the multivariate Bernstein inequality on convex bodies, *in* "Constructive Theory of Functions, Sozopol 2010" (G. Nikolov and R. Uluchev, Eds.), pp. 318–353, Prof. Marin Drinov Academic Publishing House, Sofia, 2012.

[20] Y. Sarantopoulos, Bounds on the derivatives of polynomials on Banach spaces, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), 307–312.

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