CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2013
(K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 165-174
Prof. Marin Drinov Academic Publishing House, Sofia, 2014

On the Path-connectedness of the Set of Extreme Points in a Polynomial Space*

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Let Δ be the standard simplex in $\mathbb{R}^2$. Denote by π_2 the set of all real bivariate algebraic polynomials of total degree at most two. Let B_Δ be the unit ball of the space π_2 endowed with the supremum norm on Δ.

In a recent paper (see [13]) the description of the set E_Δ of all extreme points of B_Δ was completed. We present here a result from [13] concerning the path-connectedness of E_Δ. The conclusion is that $E_\Delta \setminus \{\pm 1\}$ consists of two path-connected components. We also provide graphical illustrations.

Keywords and Phrases: Polynomials, path-connectedness, extreme points.

Mathematics Subject Classification 2010: 26C05, 41A17, 52A21, 54D05.

1. Introduction

Denote by π_n^d the set of all real algebraic polynomials of d variables and of total degree not exceeding n. Let K be a compact set in $\mathbb{R}^d$ and $\|f\|_{C(K)} := \max_{X\in K} |f(X)|$ be the uniform norm on K.

We use the notation $B_n(K)$ for the unit ball of π_n^d with respect to $\|\cdot\|_{C(K)}$, i.e., $B_n(K) = \{p \in \pi_n^d : \|p\|_{C(K)} \le 1\}$. The set of all extreme points of $B_n(K)$ will be denoted by $E_n(K)$. Recall that a point p of a convex set B is said to be *extreme* if the equality $p = \lambda p_1 + (1-\lambda)p_2$ for some $p_1, p_2 \in B$ and $\lambda \in (0,1)$ implies $p = p_1 = p_2$.

An application of the extreme points is the important fact that every convex functional, defined on a convex set B, attains its maximum at some extreme point of B. Therefore, the description of the extreme points of $B_n(K)$ can be useful in solving certain extremal problems for uniformly bounded multivariate polynomials. For example: finding of polynomials with minimal deviation from zero, deriving the exact constants in inequalities of Markov or

*The research was supported by the Bulgarian National Research Fund under Contract DDVU 02/30, and by Sofia University Science Fund under Contract 106/2013.

Bernstein type, etc. Actually our initial motivation comes from studying the multivariate extensions of the Bernstein-Szegő inequality. We refer to papers [18, 8, 9, 14, 16, 15, 2, 17, 4]. Our hope is that the optimal constant in the inequality of the type of Bernstein-Szegő for convex bodies can be obtained for $K = \Delta_d$ – the standard simplex in $\mathbb{R}^d$.

In the sequel, we set $\Delta = \Delta_2$, i.e.

$$\Delta := \big\{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0,\ x+y \leq 1\big\}.$$

We denote the vertices of Δ by $O(0,0)$, $A(1,0)$ and $B(0,1)$. We also use the notations $\pi_2 := \pi_2^2$, $\|\cdot\| := \|\cdot\|_{C(\Delta)}$, $B_\Delta := B_2(\Delta)$, and $E_\Delta := E_2(\Delta)$.

Next we introduce an important subset of Δ, related to a given polynomial p. Namely, let $\mathcal{M}(p)$ be the set of all points $X \in \Delta$ such that $|p(X)| = \|p\|$.

In papers [11, 12, 13] a full description of the set E_Δ was given. The main results are as follows:

I. Strictly definite extreme points ([11]).

A bivariate polynomial p is a strictly concave extreme point of B_Δ if and only if

$$p(x,y) = 1 + \alpha(x-x_0)^2 + 2\beta(x-x_0)(y-y_0) + \gamma(y-y_0)^2, \tag{1}$$

where

$$\alpha = \frac{2(2y_0-1)}{x_0(1-x_0-y_0)}, \quad \beta = -\frac{(1-2x_0)(1-2y_0)}{x_0y_0(1-x_0-y_0)}, \quad \gamma = \frac{2(2x_0-1)}{y_0(1-x_0-y_0)}, \tag{2}$$

and the point (x_0,y_0) belongs to the interior of the triangle Δ_1 with vertices $O_1 = (\frac{1}{2},\frac{1}{2})$, $A_1 = (0,\frac{1}{2})$, and $B_1 = (\frac{1}{2},0)$.

All strictly convex elements of E_Δ have the form $q = -p$, where p is given by (1) and (2).

II. Indefinite extreme points. ([12])

Let $\{T_i\}_{i=1}^6$ be the affine transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ that map the triangle Δ onto itself. In explicit form,

$$\{T_i(x,y)\}_{i=1}^6 = \{(x,y),(y,x),(1-x-y,y),(x,1-x-y),(1-x-y,x),(y,1-x-y)\}.$$

It is easy to see that if $p \in E_\Delta$, then the polynomials $\{\pm p(T_i(X))\}_{i=1}^6$ also belong to E_Δ. We call them *symmetrical* to p.

1. A polynomial $p \in \pi_2$ is an indefinite element of E_Δ, such that $\mathcal{M}(p)$ is an infinite set, if and only if p is symmetrical to

$$\bar{p}(x,y) = 1 - \frac{4}{\nu}xy + \frac{2}{\nu^2}(1-2\nu)y^2, \tag{3}$$

where $\nu = \frac{\sqrt{2}}{\sqrt{2}+\sqrt{1+\beta}}$ and $\beta \in [-1,1]$.

2. A polynomial $p \in \pi_2$ is an indefinite element of E_Δ, such that $\mathcal{M}(p)$ is a finite set, if and only if p is symmetrical to

$$\bar{p}(x, y) = a + bx + cy + dx^2 + 2exy + fy^2,$$

whose coefficients are given by the formulas

$$\begin{aligned}
a &= \gamma, \\
b &= 2\sqrt{1-\gamma}(\sqrt{1-\alpha} + \sqrt{1-\gamma}), \\
c &= 2\sqrt{1-\gamma}(\sqrt{1-\beta} + \sqrt{1-\gamma}), \\
d &= -(\sqrt{1-\alpha} + \sqrt{1-\gamma})^2, \\
e &= -\frac{1}{2}\big[(\sqrt{1+\alpha} + \sqrt{1+\beta})^2 + (\sqrt{1-\alpha} + \sqrt{1-\gamma})^2 \\
&\qquad + (\sqrt{1-\beta} + \sqrt{1-\gamma})^2\big], \\
f &= -(\sqrt{1-\beta} + \sqrt{1-\gamma})^2,
\end{aligned} \tag{4}$$

and parameters (α, β, γ) belong to $\mathcal{P} = \cup_{i=1}^4 \mathcal{P}_i$, where

$$\begin{aligned}
\mathcal{P}_1 &= \{(\alpha, \beta, \gamma) : \ \alpha, \beta, \gamma \in (-1, 1), \ \alpha \neq \beta\}, \\
\mathcal{P}_2 &= \{(\pm 1, \beta, \gamma) : \ \beta, \gamma \in (-1, 1)\} \cup \{(\alpha, \pm 1, \gamma) : \ \alpha, \gamma \in (-1, 1)\} \\
&\quad \cup \ \{(\alpha, \beta, -1) : \ \alpha, \beta \in (-1, 1)\}, \\
\mathcal{P}_3 &= \{(\alpha, \pm 1, -1) : \ \alpha \in (-1, 1)\} \cup \{(\pm 1, \beta, -1) : \ \beta \in (-1, 1)\} \\
&\quad \cup \ \{(\pm 1, \mp 1, \gamma) : \ \gamma \in (-1, 1)\}, \\
\mathcal{P}_4 &= \{(\pm 1, \mp 1, -1), \ (1, 1, -1)\}.
\end{aligned}$$

Remark 1. Note that the polynomials (3) can be obtained from (4) for $\alpha = \gamma = 1$, $\beta \in [-1, 1]$.

III. Semidefinite extreme points. ([13] or [10])

Suppose that p is a negative semidefinite polynomial from π_2.

1. If there exists a point $X_0 = (x_0, y_0) \in \operatorname{int} \Delta$ such that $p(X_0) = 1$, then p is an extreme point of B_Δ if and only if the following conditions hold:

(i) $p(x, y) = 1 - [\alpha(x - x_0) + \beta(y - y_0)]^2$, $(\alpha, \beta) \neq (0, 0)$;

(ii) $\min\{p(O), p(A), p(B)\} = -1$.

2. If $p(X) \neq 1$ for every $X \in \operatorname{int} \Delta$, then p is an extreme point of B_Δ if and only if $p \in \{p_i\}_{i=1}^6$, where

$$\begin{aligned}
p_1(x, y) &= 1 - 2(x + y)^2, & p_4(x, y) &= 1 - 2(x + y - 1)^2, \\
p_2(x, y) &= 1 - 2(x - 1)^2, & p_5(x, y) &= 1 - 2x^2, \\
p_3(x, y) &= 1 - 2(y - 1)^2, & p_6(x, y) &= 1 - 2y^2.
\end{aligned}$$

All positive semidefinite elements of E_Δ have the form $q = -p$, where p is a negative semidefinite extreme point.

In addition, it was proved in [13] that the only extreme points of B_Δ of degree not exceeding one are the constants ± 1.

Here we present some results from [13] concerning the path-connectedness of E_Δ. Recall that a set A in a metric space M is *path-connected* if every two points in the set can be joined by a continuous path lying in A. A *path component* of a set A is a path-connected subset $A_0 \subset A$ such that there is no path-connected set in A containing A_0 other than A_0 itself. Our main result is the following:

Theorem 1. *The set $E_\Delta \backslash \{\pm 1\}$ consists of two path-connected components, E^+ and E^-, which contain the positive semidefinite and negative semidefinite extreme points, respectively.*

We hope that the above result can be applied to construct effective algorithms for numerical solution of extremal problems for multivariate polynomials.

The proof of Theorem 1 is based on a detailed analysis of the interrelations between the different parts of E_Δ. Its main steps are given in Section 2.

Section 3 contains graphical illustrations of some typical elements of E_Δ.

2. Path-connectedness of E_Δ

We shall use the following properties of the path-connected sets, which follow easily from the definition.

(P_1) If a set A in a metric space M has the form

$$A = \{f(t_1, \dots, t_n) : \ (t_1, \dots, t_n) \in D\},$$

where D is a path-connected subset of $\mathbb{R}^n$ and $f : D \mapsto M$ is a continuous mapping, then A is a path-connected set, too.

Next we introduce a useful notation. Let p and q be two elements of M, $E \subset M$ and $p \in E$. We say that *p is path-connected in E with q* and write $p \stackrel{E}{\to} q$ if there is a continuous path $\varphi : [a, b] \to M$ such that $\varphi(a) = p$, $\varphi(b) = q$, and $\varphi(x) \in E$ for every $x \in [a, b)$. Obviously, if E is a path-connected set, then the same is true for $E \cup \{q\}$.

(P_2) Let A and B be subsets of M and let A be path-connected. Suppose that for every $b \in B$ there exists $a \in A$ such that $b \stackrel{B}{\to} a$. Then $A \cup B$ is a path-connected set.

(P_3) Let A and B be two path-connected sets in a metric space M. If $\overline{A} \cap \overline{B} = \emptyset$, then $A \cup B$ is not a path-connected set.

We denote by E^-_{III} (resp., E^+_{III}) the set of all negative (resp., positive) semidefinite extreme points of B_Δ.

Proposition 1. *E^-_{III} and E^+_{III} are path-connected sets in the space π_2 endowed with the supremum norm on Δ.*

Proof. It relies on the property (P_1) and the following continuous representation of the elements from $E^-_{III,1}$, which is the part of E^-_{III}, described in III.1:

$$p(x,y) = 1 - \rho^2[(x-x_0)\cos\theta + (y-y_0)\sin\theta]^2, \tag{5}$$

with $\rho > 0$ and $\theta \in [0, 2\pi)$. The condition (ii) from III.1 is equivalent to

$$\rho = \sqrt{\frac{2}{\max\{d_O^2(x_0,y_0,\theta), d_A^2(x_0,y_0,\theta), d_B^2(x_0,y_0,\theta)\}}}, \tag{6}$$

where $d_X(x_0,y_0,\theta) := |(x-x_0)\cos\theta + (y-y_0)\sin\theta|$ is the distance from the point $X = (x,y)$ to the straight line m consisting of the global maxima of p.

Every polynomial from $E^-_{III,2} := \{p_1,\ldots,p_6\}$, where $\{p_i\}_{i=1}^6$ are the polynomials from III.2, can be continuously joined with a polynomial from $E^-_{III,1}$. Then, by (P_2), $E^-_{III} = E^-_{III,1} \cup E^-_{III,2}$ is a path-connected set. The proof for E^+_{III} is similar. □

We denote by E_{II} the set of all indefinite elements of E_Δ.

Proposition 2. *Every polynomial from E_{II} can be path-connected in E_{II} with a polynomial from $E_{III} := E^-_{III} \cup E^+_{III}$.*

Proof. a) Let $F \subset E_{II}$ be the set of all polynomials

$$p(\alpha,\beta,\gamma;x,y) = a + bx + cy + dx^2 + 2exy + fy^2,$$

where the coefficients are given by (4) and $(\alpha,\beta,\gamma) \in \mathcal{P}$. It is easily seen that the parametric set $\mathcal{P}$ is path-connected. Then according to (P_1), F is path-connected, too.

Next we prove that every polynomial $p = p(\alpha_0,\beta_0,\gamma_0;\cdot) \in F$ can be path-connected in F with $p_1 = p(-1,-1,1;\cdot) \in E^-_{III}$.

b) Let G be the subset of all polynomials p from E_{II}, given by (3). Since G is a subset of ∂F, we easily obtain the relation $p \overset{F\cup G}{\longrightarrow} p_1 \in E^-_{III}$.

c) Suppose now that q is an arbitrary polynomial from E_{II}. It has the form $q(X) = \sigma p(T_i(X))$, where $\sigma \in \{-1,1\}$, $p \in F \cup G$, and T_i is defined in Section 1. Then Proposition 2 follows from

$$q \overset{E_{II}}{\longrightarrow} q_1 := \sigma p_1(T_i(\cdot)) \in E_{III},$$

using the invariance of E_Δ with respect to the symmetries. □

Let us set

$$E_{II}^{\mp} := \{f = \pm p(T_i(\cdot)) : \quad p \in F \cup G,\ i \in \{1, \ldots, 6\}\}.$$

It follows from the proof of Proposition 2 that every $f \in E_{II}^{-}$ can be path-connected in E_{II}^{-} with a polynomial from E_{III}^{-}. Since, by Proposition 1, E_{III}^{-} is a path-connected set, (P_2) implies that $E_{II}^{-} \cup E_{III}^{-}$ is path-connected, too. The same conclusion holds true for $E_{II}^{+} \cup E_{III}^{+}$.

We denote by E_I^{-} (resp., E_I^{+}) the set of all strictly negative (resp., positive) definite extreme points. Formulas (1), (2) and (P_1) imply that E_I^{-} and E_I^{+} are path-connected sets.

Proposition 3. *Every polynomial from E_I^{-} (E_I^{+}) can be path-connected in E_I^{-} (E_I^{+}) with a polynomial from E_{III}^{-} (E_{III}^{+}).*

Proof. Let $p(x_0, y_0; \cdot)$ be a polynomial of the form (1). If $(x_0, y_0) \to (\frac{1}{2}, y_0) \in \partial\Delta_1$ then $p(x_0, y_0; \cdot)$ tends to $1 - 8(x - \frac{1}{2})^2$, which belongs to E_{III}^{-}. □

Let us set

$$E^{\pm} := E_I^{\pm} \cup E_{II}^{\pm} \cup E_{III}^{\pm}.$$

The above results immediately imply the following

Corollary 1. *We have*

$$E_\Delta \setminus \{\pm 1\} = E^{+} \cup E^{-}.$$

Moreover, E^{+} and E^{-} are path-connected sets.

Our next goal is to prove that $E^{+} \cup E^{-}$ is not a path-connected set. To this end we shall need an important additional result.

Proposition 4. *$\overline{E^{+}}$ and $\overline{E^{-}}$ are disjoint sets.*

The proof is based on the following Lemmas 1–4.

Lemma 1.
(i) *Every polynomial from $\overline{E_I^{-}}$ is strictly concave or negative semidefinite.*
(ii) *Every polynomial from $\overline{E_{III}^{-}}$ is negative semidefinite.*

Remark 2. Similarly, the elements of $\overline{E_I^{+}}$ are strictly convex or positive semidefinite polynomials, while the elements of $\overline{E_{III}^{+}}$ are positive semidefinite.

Lemma 2. *Every polynomial from $\overline{E_{II}^{-}}$ is indefinite or negative semidefinite.*

Remark 3. The elements of $\overline{E_{II}^{+}}$ are indefinite or positive semidefinite polynomials.

Remark 4. It follows from Lemmas 1 and 2 that the polynomials from $\overline{E^+} \cup \overline{E^-}$ do not belong to π_1^2.

We set $l_1 = [OA]$, $l_2 = [OB]$, $l_3 = [AB]$, $\vec{l_1} = \overrightarrow{OA}$, $\vec{l_2} = \overrightarrow{OB}$, $\vec{l_3} = \overrightarrow{AB}$.

Definition. Let $p \in \pi_2$. Suppose that there exist points $X_i \in l_i$, $i = 1, 2, 3$, such that $|p(X_i)| = 1$ and $\frac{\partial p}{\partial \vec{l_i}}(X_i) = 0$ for $i = 1, 2, 3$. The *signature* of p is the vector $S(p) = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_i := \operatorname{sign} p(X_i)$, $i = 1, 2, 3$.

Note that if a polynomial from π_2 has a signature, it is uniquely determined. Indeed, the assumption that for some i there exist points $X_i, Y_i \in l_i$ such that $p(X_i) = \pm 1$, $p(Y_i) = \mp 1$, $\frac{\partial p}{\partial \vec{l_i}}(X_i) = \frac{\partial p}{\partial \vec{l_i}}(Y_i) = 0$ leads to an inconsistent system for the restriction $p\big|_{l_i}$. In addition, if $X_i, Y_i \in l_i$ satisfy the conditions $p(X_i) = p(Y_i) = \pm 1$ and $\frac{\partial p}{\partial \vec{l_i}}(X_i) = \frac{\partial p}{\partial \vec{l_i}}(Y_i) = 0$, then either $X_i = Y_i$ or $p \equiv \pm 1$ on l_i.

It was proved in [12, Lemma 4] that every indefinite element of E_Δ has a signature.

We say that a signature $S(p) = (\sigma_1, \sigma_2, \sigma_3)$ is *positive* (*negative*) if exactly two of its components are positive (negative). We denote by $E_{II}(-)$ (resp., $E_{II}(+)$) the set of all indefinite extreme points, whose signature is negative (resp., positive).

Lemma 3. *We have* $E_{II}^- = E_{II}(+)$ *and* $E_{II}^+ = E_{II}(-)$.

Proof. Let $f \in E_{II}^-$. By the definition, $f = p(T_i(\cdot))$ where $p \in F \cup G$ and $i \in \{1, \ldots, 6\}$. The analysis in [12, Sections 3,4] shows that p has a positive signature. This can also be checked directly. For example, if $p \in F$ one can use (4) and the points $X_1 = (\lambda, 0)$, $X_2 = (0, \mu)$, $X_3 = (1 - \nu, \nu)$, where

$$\lambda = \frac{\sqrt{1-\gamma}}{\sqrt{1-\gamma} + \sqrt{1-\alpha}}, \quad \mu = \frac{\sqrt{1-\gamma}}{\sqrt{1-\gamma} + \sqrt{1-\beta}}, \quad \nu = \frac{\sqrt{1+\alpha}}{\sqrt{1+\alpha} + \sqrt{1+\beta}}.$$

We just proved that $E_{II}^- \subset E_{II}(+)$. Analogously, $E_{II}^+ \subset E_{II}(-)$. The uniqueness of the signature shows that $E_{II}(-) \cap E_{II}(+) = \emptyset$. Since by definition $E_{II} = E_{II}^+ \cup E_{II}^-$, we conclude that $E_{II}^- = E_{II}(+)$ and $E_{II}^+ = E_{II}(-)$. The proof is completed. □

A limiting process yields the validity of the next

Lemma 4. *If* $p \in \overline{E_{II}^-}$ *(resp.,* $\overline{p \in E_{II}^+}$*), then* p *has positive (resp., negative) signature.*

Corollary 2. *The sets* $\overline{E_{II}^-}$ *and* $\overline{E_{II}^+}$ *are disjoint.*

Proof of Proposition 4. Clearly $\overline{E^\pm} = \overline{E_I^\pm} \cup \overline{E_{II}^\pm} \cup \overline{E_{III}^\pm}$. Let us consider a polynomial $p \in \overline{E^-}$. It follows from Lemmas 1 and 2 that there are three cases.

Case 1: p is a strictly concave polynomial. Then $p \notin \overline{E^+}$ since the elements of $\overline{E^+}$ are either strictly convex, or indefinite, or positive semidefinite.

Case 2: p is a negative semidefinite polynomial. As in *Case 1*, we conclude that $p \notin \overline{E^+}$.

Case 3: p is an indefinite polynomial. This implies $p \in \overline{E_{II}^-}$. The indefiniteness of p shows that $p \notin \overline{E_I^+} \cup \overline{E_{III}^+}$. In addition, by Corollary 2 we have $p \notin \overline{E_{II}^+}$, which implies $p \notin \overline{E^+}$.

Based on *Cases 1–3*, we obtain $\overline{E_{II}^-} \cap \overline{E_{II}^+} = \emptyset$. Proposition 4 is proved. □

Finally, the proof of Theorem 1 is completed by using Corollary 1 and Proposition 4 in view of Property (P_3). □

3. Graphical Illustrations

Here we present the graphs of some typical polynomials from E_Δ.

Figure 1 depicts the graph of a strictly concave extreme polynomial p with parameters $x_0 = 0.43$ and $y_0 = 0.35$ (see I). This polynomial attains its norm at the points $X_0 = (x_0, y_0)$, O, A, and B. Actually, $p(X_0) = 1$, while $p(O) = p(A) = p(B) = -1$. The peak of p is denoted by $Z_0 = (X_0, 1)$.

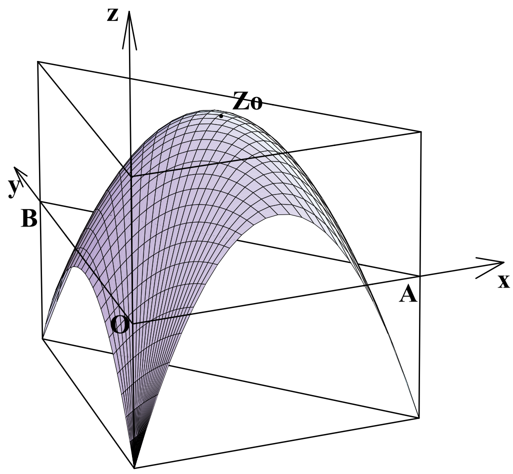

Figure 1. A strictly definite extreme polynomial from B_Δ.

Figure 2 illustrates an indefinite polynomial q of type II.1, with $\beta = 0.6$. As it is seen, $\mathcal{M}(q) = [OA] \cup \{X_3\}$, where $X_3 \in [AB]$. We have $q\big|_{[OA]} \equiv 1$ and $q(X_3) = -1$.

Finally, a polynomial r of type II.2 is shown in Figure 3. It has parameters $\alpha = 0.4$, $\beta = -0.3$, and $\gamma = 0.2$. In this case $\mathcal{M}(r)$ consists of three points:

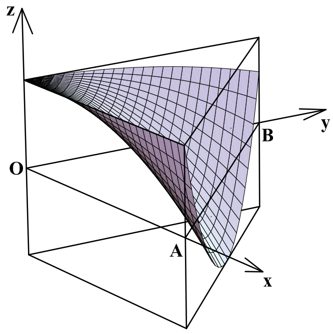

Figure 2. An indefinite extreme polynomial of type II.1.

$X_1 \in [OA]$, $X_2 \in [OB]$, and $X_3 \in [AB]$. The signature of r is positive since $r(X_1) = r(X_2) = 1$ and $r(X_3) = -1$.

Note that graphs of semidefinite extreme polynomials are given in [10].

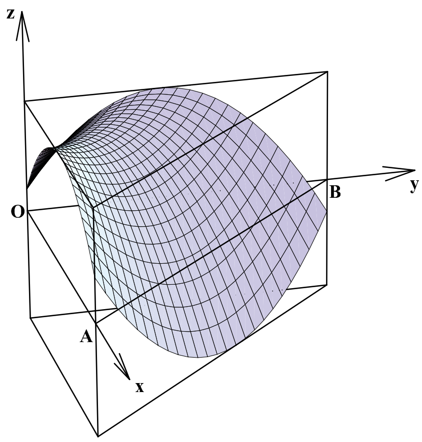

Figure 3. An indefinite extreme polynomial of type II.2.

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