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On Markov–Duffin–Schaeffer Inequalities with a Majorant. II

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We are continuing our studies on the so-called Markov inequalities with a majorant. Inequalities of this type provide an upper bound for the uniform norm in $[-1,1]$ of the k-th derivative of an algebraic polynomial p of degree n when $|p|$ is bounded on $[-1,1]$ by a certain curved majorant μ. A conjecture is that the exact upper bound $M_{k,\mu}$ is attained by the k-th derivative of the so-called snake-polynomial ω_μ which oscillates most between $\pm\mu$, i.e., that

$$M_{k,\mu} = \|\omega_\mu^{(k)}\|\,,$$

but it turned out to be a rather difficult question.

In our previous paper [3] we proved that this is true in the case of symmetric majorant μ provided the snake-polynomial ω_μ has a positive Chebyshev expansion. In this paper, we show that that the conjecture is valid under the assumption that the snake-polynomial has a positive or sign alternating Chebyshev expansion, hence for non-symmetric majorants μ as well.

1. Introduction

Throughout, $\mathcal{P}_n$ will stand for the class of real-valued algebraic polynomials of degree not exceeding n.

This paper continues our studies in [3] and it is dealing with the problem of estimating $\|p^{(k)}\|$, the max-norm in $[-1,1]$ of the k-th derivative of a polynomial $p \in \mathcal{P}_n$ obeying the restriction

$$|p(x)| \le \mu(x), \qquad x \in [-1,1],$$

where μ is a non-negative majorant. We want to find for which majorants μ the supremum of $\|p^{(k)}\|$ is attained by the so-called snake-polynomial ω_μ which

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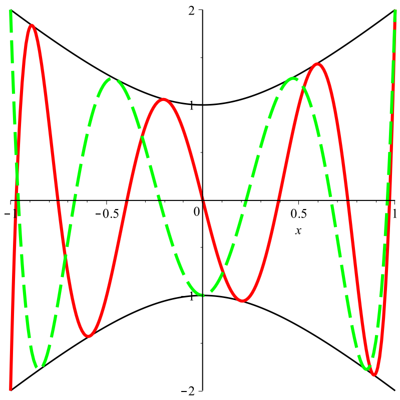

Figure 1. Markov inequality with a majorant μ: $|p| \le \mu$, $\|p^{(k)}\| \to \sup$

oscillates most between $\pm\mu$, namely by the polynomial $\omega_\mu \in \mathcal{P}_n$ that satisfies the following conditions

$$a) \quad |\omega_\mu(x)| \le \mu(x),\ x \in [-1,1]\,; \qquad b) \quad \omega_\mu(\tau_i^*) = (-1)^i \mu(\tau_i^*), \quad i = 0,\dots,n\,.$$

(This ω_μ is an analogue of the Chebyshev polynomial T_n for $\mu \equiv 1$, see Fig. 1.)

Actually, we are interested in those μ that provide the same supremum for $\|p^{(k)}\|$ under the weaker assumption

$$|p(x)| \le \mu(x), \qquad x \in \delta^* = (\tau_i^*)_{i=0}^n\,,$$

where δ^* is the set of oscillation points of ω_μ (see Fig. 2).

These two problems are generalizations of the classical results for $\mu \equiv 1$ of Markov [2] and Duffin-Schaeffer [1], respectively.

Problem 1.1 (Markov inequality with a majorant). Given $n, k \in \mathbb{N}$, $1 \le k \le n$, and a majorant $\mu \ge 0$, find

$$M_{k,\mu} := \sup\{\|p^{(k)}\| \,:\, p \in \mathcal{P}_n,\ |p(x)| \le \mu(x),\ x \in [-1,1]\}\,. \tag{1.1}$$

Problem 1.2 (Duffin–Schaeffer inequality with a majorant). Given $n, k \in \mathbb{N}$, $1 \le k \le n$, and a majorant $\mu \ge 0$, find

$$D^*_{k,\mu} := \sup\{\|p^{(k)}\| \,:\, p \in \mathcal{P}_n,\ |p(x)| \le \mu(x),\ x \in \delta^*\}\,. \tag{1.2}$$

In this setting, the results of Markov [2] and Duffin–Schaeffer [1] read:

$$\mu \equiv 1 \quad \Rightarrow \quad M_{k,\mu} = D^*_{k,\mu} = \|T_n^{(k)}\|\,, \quad 1 \le k \le n\,,$$

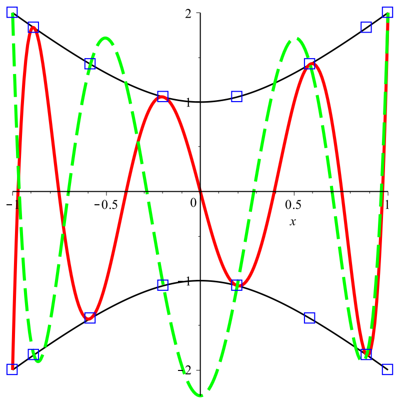

Figure 2. Duffin-Schaeffer inequality with a majorant μ: $|p|_{\delta^*} \le |\mu|_{\delta^*}$, $\|p^{(k)}\| \to \sup$

so, the question of interest is for which other majorants μ the snake-polynomial ω_μ is extremal to both Problems 1.1 and 1.2, i.e., when do we have the equalities

$$M_{k,\mu} \overset{?}{=} D^*_{k,\mu} \overset{?}{=} \|\omega_\mu^{(k)}\|\,. \tag{1.3}$$

Note that, for any majorant μ, we have $\|\omega_\mu^{(k)}\| \le M_{k,\mu} \le D^*_{k,\mu}$, so the question marks in (1.3) will be removed once we show that

$$D^*_{k,\mu} \le \|\omega_\mu^{(k)}\|\,. \tag{1.4}$$

Ideally, we would also like to know the exact numerical value of $\|\omega_\mu^{(k)}\|$ and that requires some kind of explicit expression for the snake-polynomial ω_μ. The latter is available for the class of majorants of the form

$$\mu(x) = \sqrt{R_s(x)}, \tag{1.5}$$

where R_s is a non-negative in $[-1,1]$ polynomial of degree s, so it is this class that we pay most of our attention to.

In our previous paper [3] we proved that inequality (1.4) is valid if $\widehat{\omega}_\mu := \omega_\mu^{(k-1)}$ belongs to the class Ω, which is defined by the following three conditions:

$$\widehat{\omega}_\mu \in \Omega: \quad \begin{array}{ll} 0) & \widehat{\omega}_\mu(x) = \prod_{i=1}^{\widehat{n}} (x - t_i), \quad t_i \in [-1,1]; \\ 1a) & \|\widehat{\omega}_\mu\|_{C[0,1]} = \widehat{\omega}_\mu(1); \quad 1b) \quad \|\widehat{\omega}_\mu\|_{C[-1,0]} = |\widehat{\omega}_\mu(-1)|; \\ 2) & \widehat{\omega}_\mu = \sum_{i=0}^{\widehat{n}} a_i T_i\,, \quad a_i \ge 0. \end{array} \tag{1.6}$$

Theorem 1.3 ([3]). *Let* $\omega_\mu^{(k-1)} \in \Omega$. *Then*

$$M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)\,.$$

Let us make some comments on the polynomial class Ω defined in (1.6).

For ω_μ, assumption (0) is redundant, as the snake-polynomial ω_μ of degree n has $n+1$ points of oscillations between $\pm\mu$, hence, all of its n zeros lie in the interval $[-1,1]$, thus the same is true for any of its derivatives. We wrote it down as we use this property repeatedly.

In the case of symmetric majorant μ, condition (1) becomes redundant too, as in this case the snake-polynomial ω_μ is either even or odd, hence all T_i in its Chebyshev expansion (2) are of the same parity, and that, coupled with the non-negativity of a_i in (2), implies (1a) and (1b). Therefore, for symmetric majorants μ, we have the following statement.

Theorem 1.4 ([3]). *Let* $\mu(x) = \mu(-x)$, *and let* ω_μ *be the corresponding snake-polynomial of degree* n. *If*

$$\omega_\mu^{(k_0-1)} = \sum_{i=0}^{\widehat{n}} a_i\, T_i\,, \quad a_i \ge 0,$$

then

$$M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)\,, \qquad k \ge k_0\,.$$

This theorem allowed us to establish in [3] Duffin-Schaeffer (and, thus, Markov) inequalities for various symmetric majorants μ of the form (1.5), see the next section for details.

However, for non-symmetric $\omega_\mu \in \Omega$ with a positive Chebyshev expansion, equality (1b) in (1.6) is often not valid for small k, and that did not allow us to bring our Duffin-Schaeffer-type results in [3] to a satisfactory level. For example, (1b) is not fulfilled in the case

$$\mu(x) = x+1, \qquad k=1,$$

although intuitively it is clear that the Duffin-Schaeffer inequality with such μ should be true, and we show that it is true, see Table 3 in the next section.

Here we show that, as we conjectured in [3], inequality (1.4) is valid under condition (1.6(2)) only, hence, the statement of Theorem 1.4 is true for non-symmetric majorants μ as well.

Theorem 1.5. *Given a majorant* $\mu \ge 0$, *let* ω_μ *be the corresponding snake-polynomial of degree* n. *If*

$$\omega_\mu^{(k_0-1)} = \sum_{i=0}^{\widehat{n}} a_i\, T_i\,, \quad a_i \ge 0,$$

then

$$M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)\,, \qquad k \ge k_0\,.$$

A short proof of this theorem is given in Section 3. It is based on a new idea which allows us to "linearize" the problem and reduce it to the following property of the Chebyshev polynomial T_n.

Proposition 1.6. *For a fixed $t \in [-1,1]$, define a polynomial $\tau_n(\cdot,t)$ as follows:*

$$\tau_n(x,t) := \frac{1-xt}{x-t}\,(T_n(x) - T_n(t))\,. \tag{1.7}$$

Then

$$\max_{x,t\in[-1,1]} |\tau_n'(x,t)| = T_n'(1). \tag{1.8}$$

The simple explicit form (1.7) of the polynomials $\tau_n(\cdot,t)$ enables us to draw the graphs of $\tau_n'(\cdot,t)$ using symbolic computations and thus to check inequality (1.8) numerically for rather large degrees n. Figure 3 shows that $\tau_n'(x,t)$, as a function of two variables, has $n-3$ local extrema, each of them equals approximately half the value of the global one, namely

$$\max_{|x|\le\cos\frac{\pi}{n}}\ \max_{|t|\le 1} |\tau_n'(x,t)| \approx \frac{1}{2} T_n'(1)\,.$$

Those extrema are very close to the extrema of $\frac{1}{2}(1-x^2)T_n''(x) + xT_n'(x)$

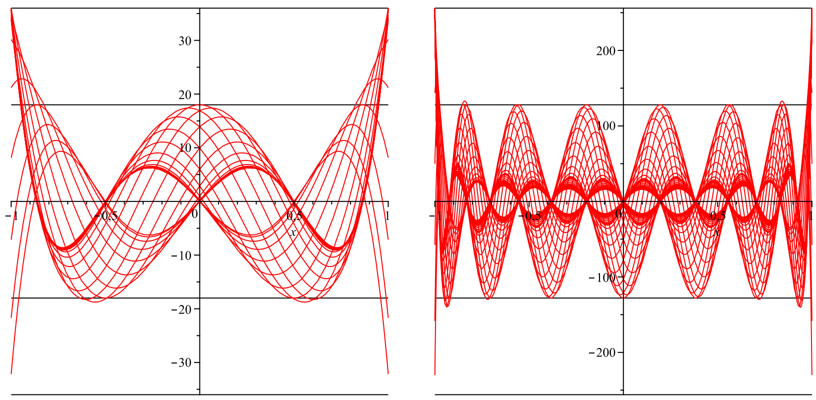

Figure 3. Graphs of $\tau_n'(\cdot,t)$ for $n=6$ (left) and $n=16$ (right)

although they are not the same. The rigorous proof of (1.8) turned out to be relatively long, and it would be interesting to find shorter arguments.

Organisation of the paper. In Section 2 we list a set of the majorants $\mu(x) = \sqrt{R_s(x)}$ to which our Theorem 1.5 is applicable, thus establishing Markov-Duffin-Schaeffer inequalities for those μ. Section 3 contains a short proof of Theorem 1.5 that uses Proposition 1.6 as its main ingredient. A proof of Proposition 1.6 is given then in Sections 4–8. Finally, in Section 9 we show that for the majorant $\mu_m(x) = (1-x^2)^{m/2}$, the snake-polynomial ω_μ is *not* extremal for the Duffin-Schaeffer inequality if $k \le m$.

2. Markov-Duffin-Schaeffer Inequalities for Various Majorants

1) Before our studies in [3], Markov- or Duffin-Schaeffer-type inequalities were obtained for the following majorants μ and derivatives k:

Table 1: Markov-type inequalities: $M_{k,\mu} = \omega_\mu^{(k)}(1)$

1°	$\sqrt{ax^2+bx+1},\ b\geq 0$	$k=1$	[7]	2°	$(1+x)^{\ell/2}(1-x^2)^{m/2}$	$k\geq m+\frac{\ell}{2}$	[4]
3°	$\sqrt{1+(a^2-1)x^2}$	all k	[7]	4°	$\sqrt{\prod_{i=1}^m(1+c_i^2x^2)}$	$k=1$	[8]

Table 2: Duffin-Schaeffer-type inequalities: $M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)$

2_1^*	$\sqrt{1-x^2}$	$k\geq 2$	[5]	2_2^*	$1-x^2$	$k\geq 3$	[6]

The next theorem combines results from our previous paper [3] with new results obtained here based on Theorem 1.5. In particular, it shows that, in cases 1^* and 4^*, Markov-type inequalities with $M_{k,\mu} = \omega_\mu^{(k)}(1)$ are valid also for $k \geq 2$, and in case 2^* they are valid for $k \geq m+1$ independently of ℓ. Moreover, in all our cases we have the stronger Duffin-Schaeffer inequalities.

Theorem 2.1. *Let μ be one of the majorant given in Table 3. Then, with the corresponding k_0, the (k_0-1)-st derivative of its snake-polynomial ω_μ satisfies*

$$\omega_\mu^{(k_0-1)} = \sum_i a_i T_i, \qquad a_i \geq 0\,, \tag{2.1}$$

hence, by Theorem 1.5,

$$M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)\,, \qquad k \geq k_0\,. \tag{2.2}$$

Table 3: Duffin-Schaeffer-type inequalities: $M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)$

1^*	$\sqrt{ax^2+bx+1},\ b\geq 0,$ $a\geq 0$ / $a<0$	$k\geq 1$ / $k\geq 2$	new	2^*	$(1+x)^{\ell/2}(1-x^2)^{m/2}$	$k>m$	new
3^*	$\sqrt{1+(a^2-1)x^2}$	$k\geq 2$	[3]	4^*	$\sqrt{\prod_{i=1}^m(1+c_i^2x^2)}$	$k\geq 1$	[3]
5^*	any $\sqrt{R_m(x^2)}$	$k>m$	[3]	6^*	any $\mu(x)=\mu(-x)$	$k>\lfloor\frac{n}{2}\rfloor$	[3]
7^*	$\sqrt{(1+c^2x^2)(1+(a^2-1)x^2)}$	$k\geq 2$	[3]	8^*	$\sqrt{1-a^2x^2+a^2x^4}$	$k\geq 1$	new

Proof. The proof of (2.1) for particular majorants consists of sometimes tedious calculations.

a) The cases 3^*-7^*, with symmetric majorants μ, are taken from [3] where we already proved (2.1) and then derived (2.2) from Theorem 1.3.

b) Here, we added one more symmetric case 8^* as an example of the majorant which is not monotonically increasing on $[0,1]$, but which is still providing Duffin-Schaeffer inequality for all $k \ge 1$. One can check that its snake-polynomial has the form

$$\omega_\mu(x) = \frac{1+b}{2}\,T_{n+2}(x) + \frac{1-b}{2}\,T_{n-2}(x), \qquad b = \sqrt{1-(\tfrac{a}{2})^2}\,.$$

c) In the non-symmetric case 1^*, we proved (2.1) for $k \ge 1$ if $a \ge 0$ and for $k \ge 2$ if $a < 0$ already in [3]. However, with Theorem 1.3 in [3] we were able to get (2.2) only for $k \ge 3$ whereas Theorem 1.5 covers the cases $k = 1,\,2,\ a \ge 0$ and $k = 2,\ a < 0$ as well.

d) The second non-symmetric case 2^* is new, but proving (2.1) in this case is relatively easy. For example, in the simplest situation when both m and ℓ are even, say, $m = 2m_1$ and $\ell = 2\ell_1$, we have

$$\omega_\mu(x) = (1+x)^{\ell_1}(x^2-1)^{m_1}T_n(x)\,,$$

and since $x^s T_n(x)$ has a positive Chebyshev expansion, we obtain

$$\omega_\mu(x) = (x^2-1)^{m_1}\sum_i a_i T_i(x)\,, \qquad a_i \ge 0\,.$$

We proved in [3] that $[(x^2-1)^{m_1}T_i(x)]^{(2m_1)}$ has a positive Chebyshev expansion as well, hence (2.1) is true with $k_0 = 2m_1 + 1 = m + 1$. □

2) There are two particular cases of a majorant μ and a derivative k for which Markov-type inequalities have been proved, but they cannot be extended to Duffin-Schaeffer-type within our method, as in those case $\omega_\mu^{(k-1)}$ does not have a positive Chebyshev expansion.

Table 4: Markov- but not Duffin-Schaeffer-type inequalities: $M_{k,\mu}=\omega_\mu^{(k)}(1)$, $D^*_{k,\mu}=?$

$1^\diamond$	$\sqrt{ax^2+bx+1},\ a<0,\ b\ge 0$	$k=1$	$2^\diamond$	$(1-x^2)^{m/2}$	$k=m$

In this respect, a natural question is whether this situation is due to imperfectness of our method, or whether it is because the equality $M_{k,\mu} = D^*_{k,\mu}$ is no longer valid. An indication that the latter is likely to be the case was given by the result of Rahman-Schmeisser [5] for the majorant $\mu_1(x) := \sqrt{1-x^2}$. Namely, they showed that

$$\mu_1(x) = \sqrt{1-x^2}, \quad k=1 \quad \Rightarrow \quad 2n = \omega'_{\mu_1}(1) = M_{1,\mu_1} < D^*_{1,\mu_1} = \mathcal{O}(n \ln n)\,.$$

Here, we show that, in case $2^\diamond$, i.e., for $\mu_m := (1-x^2)^{m/2}$ with any $m \in \mathbb{N}$, similar inequalities between Markov and Duffin-Schaeffer constants hold for all $k \le m$.

Theorem 2.2. *We have*

$$\mu_m(x) = (1-x^2)^{m/2}, \quad k \le m \quad \Rightarrow \quad \mathcal{O}(n^k) = M_{k,\mu_m} < D^*_{k,\mu_m} = \mathcal{O}(n^k \ln n)\,.$$

As to the remaining case $1^\diamond$, we believe that if $\mu(1) > 0$, i.e., except for the degenerate case $\mu(x) = \sqrt{1-x^2}$, we will have Duffin-Schaeffer inequality at least for large n:

$$\mu(x) = \sqrt{ax^2+bx+1}, \; a<0, \, b \ge 0, \qquad \Rightarrow \quad M_{1,\mu} = D_{1,\mu} = \omega'_\mu(1), \quad \forall n \ge n_\mu\,,$$

where n_μ depends on $\mu(1)$ (say, $n_\mu > \frac{1}{\mu(1)}$).

Remark 2.3. Obviously, $\omega_\mu(-x)$ is a snake-polynomial for the majorant $\widetilde{\mu}(x) = \mu(-x)$; moreover, if ω_μ has a positive (or negative) Chebyshev expansion, then $\omega_{\widetilde{\mu}}$ has a sign alternating Chebyshev expansion and vice versa. Hence, the assumption for a positive Chebyshev expansion in Theorems 1.5 and 2.1 can be replaced by the assumption for a sign alternating Chebyshev expansion, in which case we have $M_{k,\mu} = D^*_{k,\mu} = |\omega^{(k)}_\mu(-1)|$. We therefore have the following supplement to Table 3:

Table 3′: Duffin-Schaeffer-type inequalities: $M_{k,\mu} = D^*_{k,\mu} = |\omega^{(k)}_\mu(-1)|$

1′	$\sqrt{ax^2+bx+1},\ b<0,$ $a \ge 0$ / $a<0$	$k \ge 1$ / $k \ge 2$	new
2′	$(1-x)^{\ell/2}(1-x^2)^{m/2}$	$k>m$	new

3. Proof of Theorem 1.5

In [3], we used the following intermediate estimate as an upper bound for $D^*_{k,\mu}$.

Proposition 3.1 ([3]). *Given a majorant μ, let ω_μ be its snake-polynomial, let $\widehat{\omega}_\mu(x) := \omega^{(k-1)}_\mu(x)$, and let*

$$\phi_{\widehat{\omega}}(x,t_i) := \frac{1-xt_i}{x-t_i}\,\widehat{\omega}_\mu(x), \qquad \textit{where } t_i \textit{ are the zeros of } \widehat{\omega}_\mu. \tag{3.1}$$

Then

$$D^*_{k,\mu} \le \max\Big\{\|\widehat{\omega}'_\mu\|, \max_{x,t_i\in[-1,1]} |\phi'_{\widehat{\omega}}(x,t_i)|\Big\}. \tag{3.2}$$

We showed then in [3] that if $\widehat{\omega}_\mu$ belongs to the class Ω defined in (1.6), then $|\phi'_{\widehat{\omega}}(x,t_i)| \le \widehat{\omega}'_\mu(1) = \omega_\mu^{(k)}(1)$, and that led to Theorem 1.3.

Here, we prove a similar estimate that uses a continuous (with respect to t) analogue of (3.1).

Proposition 3.2. *Given a majorant μ, let ω_μ be its snake-polynomial, let $\widehat{\omega}_\mu = \omega_\mu^{(k-1)}$, and let*

$$\tau_{\widehat{\omega}}(x,t) := \frac{1-xt}{x-t}\big(\widehat{\omega}_\mu(x) - \widehat{\omega}_\mu(t)\big), \qquad t \in [-1,1]. \tag{3.3}$$

Then

$$D^*_{k,\mu} \le \max\Big\{\|\widehat{\omega}'_\mu\|, \max_{x,t\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t)|\Big\}. \tag{3.4}$$

Proof. Comparing definitions (3.1) and (3.3), we see that, since $\widehat{\omega}_\mu(t_i) = 0$, we have

$$\tau_{\widehat{\omega}}(x,t_i) = \frac{1-xt_i}{x-t_i}(\widehat{\omega}_\mu(x) - \widehat{\omega}_\mu(t_i)) = \frac{1-xt_i}{x-t_i}\,\widehat{\omega}_\mu(x) = \phi_{\widehat{\omega}}(x,t_i)\,.$$

Therefore,

$$\max_{x,t_i\in[-1,1]} |\phi'_{\widehat{\omega}}(x,t_i)| = \max_{x,t_i\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t_i)| \le \max_{x,t\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t)|\,,$$

and (3.4) follows from (3.2). □

Proof of Theorem 1.5. We want to show that if $\widehat{\omega}_\mu := \omega_\mu^{(k-1)}$ has a positive Chebyshev expansion, i.e.,

$$\widehat{\omega}_\mu = \sum_{i=0}^{\widehat{n}} a_i\,T_i, \qquad a_i \ge 0\,, \tag{3.5}$$

then

$$D^*_{k,\mu} \le \omega_\mu^{(k)}(1)\,.$$

By (3.4), we are done if we prove that

$$\max_{x,t\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t)| \le \widehat{\omega}'_\mu(1) \quad \Big(= \omega_\mu^{(k)}(1)\Big)\,.$$

We have

$$\begin{aligned}
\tau_{\widehat{\omega}}(x,t) \quad &:= \quad \frac{1-xt}{x-t}(\widehat{\omega}_\mu(x) - \widehat{\omega}_\mu(t)) = \frac{1-xt}{x-t}\sum_{i=1}^{\widehat{n}} a_i\Big(T_i(x) - T_i(t)\Big) \\
&= \quad \sum_{i=1}^{\widehat{n}} a_i \frac{1-xt}{x-t}\Big(T_i(x) - T_i(t)\Big) = \sum_{i=1}^{\widehat{n}} a_i \tau_i(x,t)\,,
\end{aligned}$$

where
$$\tau_i(x,t) := \frac{1-xt}{x-t}\big(T_i(x) - T_i(t)\big).$$
Respectively,
$$|\tau'_{\widehat\omega}(x,t)| \le \sum_{i=1}^{\widehat n} |a_i|\cdot|\tau'_i(x,t)| \overset{(a)}{=} \sum_{i=1}^{\widehat n} a_i\,|\tau'_i(x,t)| \overset{(b)}{\le} \sum_{i=1}^{\widehat n} a_i\,T'_i(1) \overset{(c)}{=} \widehat\omega'_\mu(1).$$
In the last display, equality (a) is due to assumption $a_i \ge 0$ in (3.5), equality (c) also follows from (3.5), and inequality (b) is the matter of Proposition 1.6 (which we are going to prove in the rest of the paper). □

4. Auxiliary Results

For a polynomial
$$\omega(x) = c\prod_{i=1}^{n}(x-t_i)\,, \quad -1 \le t_n \le \cdots \le t_1 \le 1, \quad c>0,$$
with all its zeros in the interval $[-1,1]$ (and counted in the reverse order), set
$$\phi(x,t_i) := \frac{1-xt_i}{x-t_i}\,\omega(x)\,, \qquad i=1,\dots,n. \tag{4.1}$$
For each i, we would like to estimate the norm $\|\phi'(\cdot,t_i)\|_{C[-1,1]}$, i.e., the maximum value of $|\phi(\cdot,t_i)|$, and the latter is attained either at the end-points $x=\pm1$, or at the points x where $\phi''(x,t_i)=0$.

In [3] we introduced two functions,
$$\psi_1(x,t) := \frac12(1-xt)\,\omega''(x) - t\,\omega'(x)\,, \tag{4.2}$$
$$\psi_2(x,t) := \frac12(1-x^2)\,\omega''(x) + \frac{x-t}{1-xt}\,\omega'(x) - \frac{x(1-t^2)}{(x-t)(1-xt)}\,\omega(x)\,. \tag{4.3}$$

In [3, Section 4] we obtained the following results.

Claim 4.1 ([3]). *We have*
$$|\phi'(\pm1,t_i)| \le |\omega'(\pm1)|\,.$$

Claim 4.2 ([3]). *For each i, both $\psi_{1,2}(\cdot,t_i)$ interpolate $\phi'(\cdot,t_i)$ at the points of its local extrema,*
$$\phi''(x,t_i)=0 \quad\Rightarrow\quad \phi'(x,t_i) = \psi_{1,2}(x,t_i)\,, \tag{4.4}$$
therefore
$$\|\phi'(\cdot,t_i)\|_* \le \|\psi_{1,2}(\cdot,t_i)\|\,,$$
where $\|f(\cdot)\|_*$ stands for the maximal critical value of f on $[-1,1]$.

Claim 4.3 ([3]). *With some specific functions* $f_\nu(\omega,\cdot)$, $1\le\nu\le 4$, *we have*

$$1)\quad |\psi_1(x,t_i)| \le \max_{\nu=1,2,3}|f_\nu(x)|, \qquad 0\le x\le 1, \qquad -1\le \tfrac{x-t_i}{1-xt_i}\le \tfrac{1}{2}\,;$$

$$2)\quad |\psi_2(x,t_i)| \le \max_{\nu=1,2}|f_\nu(x)|, \qquad t_1\le x\le 1; \qquad \tfrac{1}{2}\le \tfrac{x-t_i}{1-xt_i}\le 1;$$

and, under the additional assumption that $|\omega(x)|\le\omega(1)$ *for* $x\in[0,1]$,

$$3)\quad |\psi_2(x,t_i)| \le \max_{\nu=1,2,4}|f_\nu(x)|, \qquad 0\le x\le t_1, \qquad \tfrac{1}{2}\le \tfrac{x-t_i}{1-xt_i}\le 1.$$

Claim 4.4 ([3]). *Let*

$$\omega=\sum_{i=0}^{n} a_i\,T_i, \qquad a_i\ge 0\,,$$

Then

$$\max_{1\le\nu\le 4}|f_\nu(\omega,x)|\le\omega'(1)\,.$$

The next theorem follows immediately from Claims 4.1 – 4.4:

Theorem 4.5 ([3, Theorem 3.1]). *Let* $\omega\in\Omega$ *(see (1.6)), i.e., it satisfies the following three conditions*

$$0)\qquad \omega(x)=c\prod_{i=1}^{n}(x-t_i)\,, \qquad t_i\in[-1,1];$$

$$1a)\qquad \|\omega\|_{C[0,1]}=\omega(1), \qquad\qquad 1b)\quad \|\omega\|_{C[-1,0]}=|\omega(-1)|;$$

$$2)\qquad \omega=\sum_{i=0}^{n}a_iT_i, \quad a_i\ge 0\,.$$

Then

$$\max_{x,t_i\in[-1,1]}|\phi'(x,t_i)|\le\omega'(1)\,.$$

This theorem coupled with Proposition 3.1 gives Theorem 1.3, which was the main result in [3]. However, the main purpose of quoting here Claims 4.1 – 4.4 is to apply them to the particular polynomial $\omega(x)=c_0+T_n(x)$.

Firstly, we make a refinement of Claim 4.4, which is just a more accurate statement of what we proved in [3].

Claim 4.6. *Let*

$$\omega=c_0+\sum_{i=1}^{n}a_i\,T_i, \qquad a_i\ge 0\,,$$

Then

$$\max_{1\le\nu\le 4}|f_\nu(\omega,x)|\le\omega'(1)\,.$$

Proof. The functions $f_\nu(\omega;\cdot)$ are of the form

$$|f_\nu(\omega,x)| = |a_\nu(x)\omega''(x) + b_\nu(x)\omega'(x)| + c_\nu\|\omega'\|\,,$$

i.e., they depend on ω' rather than on ω, hence they are independent of the free term of the polynomial ω. □

Now, we formulate the statement that we will use in the next sections. It is a straightforward corollary of Claims 4.1-4.3 and Claim 4.6.

Proposition 4.7. *Let*

$$\omega(x) = c_0 + T_n(x) = c\prod_{i=1}^{n}(x-t_i), \quad |c_0| \le 1, \quad 1 \ge t_1 \ge \cdots \ge t_n \ge -1,$$

and let a pair of points (x, t_i) satisfy any of the following conditions:

$$\begin{aligned}
&1)\quad 0 \le x \le 1, && -1 \le \tfrac{x-t_i}{1-xt_i} \le \tfrac{1}{2}\,; \\
&2)\quad t_1 \le x \le 1; && \tfrac{1}{2} \le \tfrac{x-t_i}{1-xt_i} \le 1; \\
&3)\quad 0 \le x \le t_1, && \tfrac{1}{2} \le \tfrac{x-t_i}{1-xt_i} \le 1 \quad \textit{and} \quad |\omega(x)| \le \omega(1).
\end{aligned} \tag{4.5}$$

Then

$$\phi''(x,t_i) = 0 \quad \Rightarrow \quad |\phi'(x,t_i)| \le \omega'(1)\,. \tag{4.6}$$

5. Proof of Proposition 1.6

Here, we will prove Proposition 1.6, namely that the polynomial

$$\tau(x,t) := \tau_n(x,t) := \frac{1-xt}{x-t}\left(T_n(x) - T_n(t)\right), \tag{5.1}$$

considered as a polynomial in x (of degree n), admits the estimate

$$|\tau'(x,t)| \le T_n'(1)\,, \qquad x,t \in [-1,1]\,, \qquad n \in \mathbb{N}\,. \tag{5.2}$$

We prove it similarly to the techniques we used in [3] by considering, for a fixed t, the points x of local extrema of $\tau'(x,t)$ and the end-points $x = \pm 1$, and showing that at those points $|\tau'(x,t)| \le T_n'(1)$.

Lemma 5.1. *If $x = \pm 1$, then $|\tau'(x,t)| \le T_n'(1)$.*

Proof. This inequality follows from the straightforward calculations:

$$\tau'(1,t) = T_n'(1) - \frac{1+t}{1-t}\left(T_n(1) - T_n(t)\right).$$

The last term is non-negative, hence $\tau'(1,t) \le T_n'(1)$. Also, since $1+t \le 2$ and $\frac{T_n(1)-T_n(t)}{1-t} \le T_n'(1)$, it does not exceed $2T_n'(1)$, hence $\tau'(1,t) \ge -T_n'(1)$. □

It remains to consider the local maxima of $|\tau'(\cdot,t)|$, i.e., the points (x,t) where $\tau''(x,t)=0$. Note that local maxima of the polynomial $\tau_n'(\cdot,t)$ exist only if $\tau_n(\cdot,t)$ is of degree $n \ge 3$; moreover, since $\tau(x,t) = \pm\tau(-x,-t)$, it is sufficient to prove the inequality (1.8) only on the half of the square $[-1,1]\times[-1,1]$. So, we have to deal only with the case

$$\mathcal{D}: \quad x \in [0,1], \quad t \in [-1,1]; \qquad n \ge 3.$$

We split the domain $\mathcal{D}$ into two main subdomains: $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where

$$\begin{aligned}
\mathcal{D}_1: &\quad x \in [0,1], \qquad t \in [-1,1], \qquad -1 \le \tfrac{x-t}{1-xt} \le \tfrac{1}{2}\,;\\
\mathcal{D}_2: &\quad x \in [0,1], \qquad t \in [-1,1], \qquad \tfrac{1}{2} \le \tfrac{x-t}{1-xt} \le 1\,;
\end{aligned}$$

with a further subdivision of $\mathcal{D}_2$: $\mathcal{D}_2 = \mathcal{D}_2^{(1)} \cup \mathcal{D}_2^{(2)} \cup \mathcal{D}_2^{(3)}$, where

$$\begin{aligned}
\mathcal{D}_2^{(1)}: &\quad x \in [0,1], \qquad t \in [\cos\tfrac{3\pi}{2n}, 1], \qquad \tfrac{1}{2} \le \tfrac{x-t}{1-xt} \le 1;\\
\mathcal{D}_2^{(2)}: &\quad x \in [0, \cos\tfrac{\pi}{n}], \quad t \in [-1, \cos\tfrac{3\pi}{2n}], \quad \tfrac{1}{2} \le \tfrac{x-t}{1-xt} \le 1;\\
\mathcal{D}_2^{(3)}: &\quad x \in [\cos\tfrac{\pi}{n}, 1], \quad t \in [-1, \cos\tfrac{3\pi}{2n}], \quad \tfrac{1}{2} \le \tfrac{x-t}{1-xt} \le 1.
\end{aligned}$$

Now, Proposition 1.6 follows from the following statement.

Proposition 5.2. *Let $n \ge 3$, and $\tau(x,t) := \tau_n(x,t)$ be defined by* (5.1).

a) If $(x,t) \in \mathcal{D}_1 \cup \mathcal{D}_2^{(1)}$ and $\tau''(x,t)=0$, then $|\tau'(x,t)| \le T_n'(1)$.

b) If $(x,t) \in \mathcal{D}_2^{(2)}$ and $\tau''(x,t)=0$, then $|\tau'(x,t)| \le T_n'(1)$.

c) If $(x,t) \in \mathcal{D}_2^{(3)}$, then $\tau''(x,t) \ne 0$.

Proofs of parts (a)-(c) are given in the next sections. Parts (b) and (c) are relatively simple and their proofs are independent of our results in [3]. For (a), we could not find similarly simple arguments, and chose to use our results from [3], namely Proposition 4.7, instead.

6. Proof of Proposition 5.2.a

The next statement is an adjustment of Proposition 4.7 to our needs.

Proposition 6.1. *For a fixed $t \in [-1,1]$, let t_1 be the rightmost zero of the polynomial*

$$\omega_*(\cdot) = T_n(\cdot) - T_n(t)\,,$$

and let a pair of points (x,t) *satisfy any of the following conditions:*

$$\begin{aligned} &1') \quad 0 \le x \le 1, && -1 \le \tfrac{x-t}{1-xt} \le \tfrac{1}{2}\,; \\ &2') \quad t_1 \le x \le 1; && \tfrac{1}{2} \le \tfrac{x-t}{1-xt} \le 1; \\ &3') \quad 0 \le x \le t_1, && \tfrac{1}{2} \le \tfrac{x-t}{1-xt} \le 1 \quad \text{and} \quad T_n(t) \le 0\,. \end{aligned} \tag{6.1}$$

Then

$$\tau''(x,t) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le T_n'(1)\,. \tag{6.2}$$

Proof. For a fixed $t \in [-1,1]$, the polynomial $\omega_*(\cdot) = T_n(\cdot) - T_n(t)$ has n zeros inside $[-1,1]$ counting possible multiplicities, i.e. $\omega_*(x) = c\prod(x-t_i)$, and $x = t$ is one of them, i.e., $t = t_i$ for some i. Therefore, conditions $(1')$-$(3')$ for (x,t) in (6.1) are equivalent to the conditions (1)-(3) for (x,t_i) in (4.5), in particular, the inequality $|\omega_*(x)| < \omega_*(1)$ in 4.5(3) follows from $T_n(t) \le 0$. Hence, the implication (4.6) for ϕ_* is valid. But, since $t = t_i$, we have

$$\tau(x,t) = \frac{1-xt}{x-t}\,(T_n(x) - T_n(t)) = \frac{1-xt_i}{x-t_i}\,\omega_*(x) = \phi_*(x,t_i),$$

so (6.2) is identical to (4.6). □

Lemma 6.2. *Let* $(x,t) \in \mathcal{D}_1 = \{x \in [0,1], t \in [-1,1], -1 \le \frac{x-t}{1-xt} \le \frac{1}{2}]\}$. *Then*

$$\tau''(x,t) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le T_n'(1)\,.$$

Proof. Condition $(x,t) \in \mathcal{D}_1$ is identical to condition $(1')$ in Proposition 6.1, hence the conclusion. □

Lemma 6.3. *Let* $(x,t) \in \mathcal{D}_2^{(1)} = \{x \in [0,1], t \in [\cos\frac{3\pi}{2n}, 1], \frac{1}{2} \le \frac{x-t}{1-xt} \le 1]\}$. *Then*

$$\tau''(x,t) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le T_n'(1)\,.$$

Proof. We split $\mathcal{D}_2^{(1)}$ into two further subsets:

$$2a) \quad t \in [\cos\tfrac{3\pi}{2n}, \cos\tfrac{\pi}{2n}]\,, \qquad 2b) \quad t \in [\cos\tfrac{\pi}{2n}, 1]\,.$$

2a) For $t \in [\cos\frac{3\pi}{2n}, \cos\frac{\pi}{2n}]$ we have $T_n(t) \le 0$, so we apply Proposition 6.1 where we use condition $(3')$ if $x < t_1$, and condition $(2')$ otherwise.

2b) For $t \in [\cos\frac{\pi}{2n}, 1]$, the Chebyshev polynomial $T_n(t)$ is increasing, hence t is the rightmost zero t_1 of the polynomial $\omega_*(x) = T_n(x) - T_n(t)$. Now, we use the inequality $\frac{1}{2} \le \frac{x-t}{1-xt} \le 1$ for $(x,t) \in \mathcal{D}_2^{(1)}$. Since $t = t_1$, we have

$$\frac{1}{2} \le \frac{x-t_1}{1-xt_1} \le 1 \quad \Rightarrow \quad t_1 \le x \le 1,$$

so we apply Proposition 6.1 with condition $(2')$. □

7. Proof of Proposition 5.2.b

Lemma 7.1. *Let* $(x,t)\in\mathcal{D}_2^{(2)}=\{x\in[0,\cos\frac{\pi}{n}], t\in[-1,\cos\frac{3\pi}{2n}], \frac{1}{2}\le\frac{x-t}{1-xt}\le 1\}$. *Then*

$$\tau''(x,t)=0 \quad\Rightarrow\quad |\tau'(x,t)|\le T_n'(1)\,.$$

Proof. We note that the assumption $t\in[-1,\cos\frac{3\pi}{2n}]$ is not used in the proof. With $\omega_*(x)=T_n(x)-T_n(t)$, we have $\tau(x,t)=\phi_*(x,t_i)$, hence by Claim 4.2,

$$\tau''(x,t)=0 \quad\Rightarrow\quad |\tau'(x,t)|=|\psi_2(x,t)|\,,$$

where

$$\psi_2(x,t):=\frac{1}{2}(1-x^2)\,\omega_*''(x)+\frac{x-t}{1-xt}\,\omega_*'(x)-\frac{x(1-t^2)}{(x-t)(1-xt)}\,\omega_*(x)\,. \qquad (7.1)$$

Let us prove that

$$\max_{(x,t)\in\mathcal{D}_2^{(2)}}|\psi_2(x,t)|\le T_n'(1). \qquad (7.2)$$

Making the substitution $\gamma:=\frac{x-t}{1-xt}$ into (7.1), so that $\gamma\in[\frac{1}{2},1]$, we obtain

$$\begin{aligned}\psi_2(x,t)&=\frac{1}{2}(1-x^2)\,\omega_*''(x)+\gamma\,\omega_*'(x)-\frac{1-\gamma^2}{\gamma}\,\frac{x}{1-x^2}\,\omega_*(x)\\&=:g_\gamma(x)-h_\gamma(x)\,,\end{aligned} \qquad (7.3)$$

where $g_\gamma(x)$ is the sum of the first two terms, and $h_\gamma(x)$ is the third one, so that

$$|\psi_2(x,t)|\le|g_\gamma(x)|+|h_\gamma(x)|\,. \qquad (7.4)$$

Let us evaluate both g_γ and h_γ.

1) Since $\omega_*(x)=T_n(x)-T_n(t)$, we have

$$2g_\gamma(x)=(1-x^2)T_n''(x)+2\gamma T_n'(x)=(x+2\gamma)T_n'(x)-n^2T_n(x)\,,$$

so that, using Cauchy's inequality and the well-known identity for Chebyshev polynomials, we obtain

$$\begin{aligned}2|g_\gamma(x)| &= n\left|nT_n(x)-\frac{x+2\gamma}{n\sqrt{1-x^2}}\sqrt{1-x^2}T_n'(x)\right|\\ &\le n\Big(n^2T_n(x)^2+(1-x^2)T_n'(x)^2\Big)^{1/2}\left(1+\frac{(x+2\gamma)^2}{n^2(1-x^2)}\right)^{1/2}\\ &= n^2\left(1+\frac{(x+2\gamma)^2}{n^2(1-x^2)}\right)^{1/2},\end{aligned}$$

so that

$$|g_\gamma(x)|\le n^2\frac{1}{2}\left(1+\frac{(x+2\gamma)^2}{n^2(1-x^2)}\right)^{1/2}. \qquad (7.5)$$

2) For the function h_γ in (7.3), since $\omega_*(x) = T_n(x) - T_n(t)$ does not exceed 2 in the absolute value, we have the trivial estimate

$$|h_\gamma(x)| \le \frac{1-\gamma^2}{\gamma}\,\frac{2x}{1-x^2} \;= n^2\frac{1-\gamma^2}{\gamma}\,\frac{2x}{n^2(1-x^2)}\,. \qquad (7.6)$$

3) So, from (7.4), (7.5) and (7.6), we have

$$\max_{x,t\in\mathcal{D}_2^{(2)}} |\psi_2(x,t)| \le T_n'(1)\max_{x,\gamma} F(x,\gamma)\,,$$

where

$$F(x,\gamma) := \frac{1}{2}\left(1+\frac{(x+2\gamma)^2}{n^2(1-x^2)}\right)^{1/2} + \frac{1-\gamma^2}{\gamma}\,\frac{2x}{n^2(1-x^2)}\,,$$

and the maximum is taken over $\gamma \in [\frac{1}{2},1]$ and $x \in [0,x_n]$, where $x_n = \cos\frac{\pi}{n}$. Clearly, $F(x,\gamma) \le F(x_n,\gamma)$, so we are done with (7.2) once we prove that $F(x_n,\gamma) \le 1$. We have

$$\begin{aligned} F(x_n,\gamma) &= \frac{1}{2}\left(1+\frac{(\cos\frac{\pi}{n}+2\gamma)^2}{n^2\sin^2\frac{\pi}{n}}\right)^{1/2} + \frac{1-\gamma^2}{\gamma}\,\frac{2\cos\frac{\pi}{n}}{n^2\sin^2\frac{\pi}{n}} \\ &\le \frac{1}{2}\left(1+\frac{(1+2\gamma)^2}{4^2\sin^2\frac{\pi}{4}}\right)^{1/2} + \frac{1-\gamma^2}{\gamma}\,\frac{2\cdot 1}{4^2\sin^2\frac{\pi}{4}} \;=:\; G(\gamma), \qquad n \ge 4, \end{aligned}$$

where we have used that $\cos\frac{\pi}{n} < 1$ and the fact that the sequence $(n^2\sin^2\frac{\pi}{n})$ is increasing. Hence, $F(x_n,\gamma) \le 1$ for all $n \ge 3$ if

$$F(x_3,\gamma) \le 1, \qquad G(\gamma) \le 1\,, \qquad \gamma \in [\tfrac{1}{2},1]\,.$$

The latter is seen to be true on Figure 4. Formally, it is easy to show that

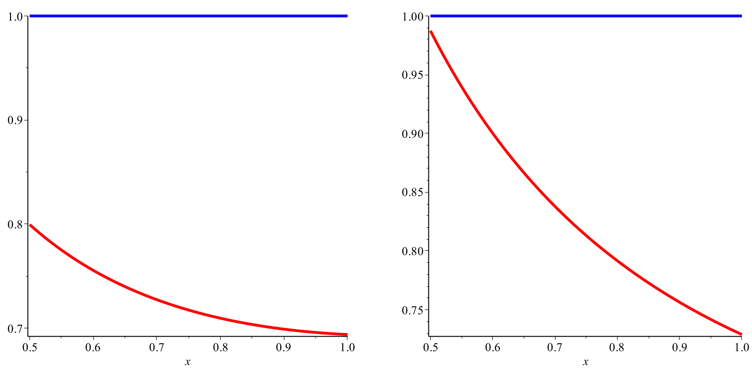

Figure 4. The graphs of $F(x_3,\gamma)$ (left) and $G(\gamma)$ (right), $\gamma \in [\frac{1}{2},1]$.

$$G'(\gamma) \le G'(1) < 0, \qquad F'(x_3,\gamma) \le F'(x_3,1) < 0, \qquad \gamma \in [\tfrac{1}{2},1],$$

i.e., both functions are decreasing on $[\frac{1}{2},1]$, and then verify that $G(\frac{1}{2}) < 1$ and $F(x_3,\frac{1}{2}) < 1$. □

8. Proof of Proposition 5.2.c

Lemma 8.1. *Let* $x \in \mathcal{D}_2^{(3)} = \{x \in [\cos\frac{\pi}{n},1], t \in [-1,\cos\frac{3\pi}{2n}], \frac{1}{2} \le \frac{x-t}{1-xt} \le 1\}$. *Then*

$$\tau''(x,t) \ne 0\,.$$

We prove this statement in several steps, and the restriction $\frac{1}{2} \le \frac{x-t}{1-xt} \le 1$ is irrelevant to the proof.

Lemma 8.2. *a) If* $t \in [-1,0]$, *then* $\tau''(x,t) \ne 0$ *for* $x \in [\cos\frac{\pi}{n},\infty)$.
b) If $t \in (0,1]$, *then* $\tau''(x,t)$ *has at most one zero in* $[\cos\frac{\pi}{n},\infty)$, *and* $\tau''(x,t)<0$ *for large* x.

Proof. By definition,

$$\tau(x,t) = \frac{1-xt}{x-t}\,(T_n(x)-T_n(t))\,.$$

For a fixed $t \in [-1,1]$, the polynomial $\omega_*(\cdot) = T_n(\cdot) - T_n(t)$ has n zeros inside $[-1,1]$, say (t_i), one of them at $x = t$, so $t = t_{i_0}$ for some i_0. From definition, we see that the polynomial $\tau(\cdot,t)$ has the same zeros as $\omega_*(\cdot)$ except t_{i_0} which is replaced by $1/t_{i_0}$. So, if $(s_i)_{i=1}^n$ and $(t_i)_{i=1}^n$ are the zeros of $\tau(\cdot,t)$ and $\omega_*(\cdot,t)$ respectively, counted in the reverse order, then

$$1)\quad s_i \le t_i \le s_{i-1}, \quad \text{if} \quad t \le 0, \qquad 2)\quad s_{i+1} \le t_i \le s_i, \quad \text{if} \quad t > 0\,.$$

That means that zeros of $\tau(\cdot,t)$ and $\omega_*(\cdot)$ interlace, hence, by Markov's lemma, the same is true for the zeros of any of their derivatives. In particular, if $(s_i'')_{i=1}^{n-2}$ and $(t_i'')_{i=1}^{n-2}$ are the zeros of $\tau''(\cdot,t)$ and $\omega_*''(\cdot,t)$, respectively, counted in the reverse order, then

$$1'')\quad s_1'' < t_1'', \quad \text{if} \quad t \le 0, \qquad 2'')\quad s_2'' < t_1'' < s_1'', \quad \text{if} \quad t > 0\,.$$

Since $\omega_*'' = T_n''$, its rightmost zero t_1'' satisfies $t_1'' < \cos\frac{\pi}{n}$ as the latter is the rightmost zero of T_n'.

Hence, if $t \le 0$, then the rightmost zero s_1'' of $\tau''(\cdot,t)$ satisfies $s_1'' < \cos\frac{\pi}{n}$, and that proves claim *a*) of the lemma. On the other hand, if $t > 0$, then there is at most one zero of $\tau''(\cdot,t)$ on $[\cos\frac{\pi}{n},\infty)$, and that proves the first part of claim *b*) of the lemma. The second part of *b*) follows from the observation that, for $t > 0$, the polynomial $\tau(\cdot,t)$ has a negative leading coefficient, hence $\tau''(x,t) < 0$ for large x. □

Corollary 8.3. *If, for a fixed* $t \in [0,1]$, $\tau''(x,t) \ge 0$ *at* $x = 1$, *then* $\tau''(x,t) > 0$ *for all* $x \in [\cos\frac{\pi}{n}, 1)$.

Proof. By Lemma 8.2, there is at most one zero of $\tau''(\cdot,t)$ on $[\cos\frac{\pi}{n}, \infty)$, and $\tau''(x,t) < 0$ for large x. Hence, if $\tau''(x,t) \ge 0$ at $x = 1$, then $\tau''(\cdot,t)$ does not change its sign on $[\cos\frac{\pi}{n}, 1)$. □

Lemma 8.4. *If* $t \in [0, \cos\frac{3\pi}{2n}]$, *then* $\tau''(x,t) > 0$ *for* $x \in [\cos\frac{\pi}{n}, 1]$.

Proof. By Corollary 8.3, it suffices to prove that $\tau''(x,t) \ge 0$ at $x = 1$ provided $t \in [0, \cos\frac{3\pi}{2n}]$. By direct calculations, we have

$$\tau''(x,t) = \frac{1-xt}{x-t}\,T_n''(x) - 2\,\frac{1-t^2}{(x-t)^2}\,T_n'(x) + 2\,\frac{1-t^2}{(x-t)^3}\,(T_n(x) - T_n(t)),$$

so we need to prove that

$$\tau''(1,t) = \frac{n^2(n^2-1)}{3} - 2\,\frac{1+t}{1-t}\,n^2 + 2\,\frac{1+t}{(1-t)^2}\,(1 - T_n(t)) \ge 0, \qquad (8.1)$$

where we have used that $T_n(1) = 1$, $T_n'(1) = n^2$, and $T_n''(1) = \frac{n^2(n^2-1)}{3}$.

1) Since the last term in (8.1) is non-negative for $t \in [-1,1)$, this inequality will certainly be true if

$$\frac{n^2(n^2-1)}{3} - 2\,\frac{1+t}{1-t}\,n^2 \ge 0 \quad \Rightarrow \quad t \le \frac{n^2-7}{n^2+5}.$$

We have

$$\cos\frac{3\pi}{2n} < \frac{n^2-7}{n^2+5}, \quad 3 \le n \le 6, \quad \text{and} \quad \cos\frac{2\pi}{n} < \frac{n^2-7}{n^2+5} < \cos\frac{3\pi}{2n}, \quad n \ge 7.$$

That proves (8.1), and hence the lemma, for all $t \in [0, \cos\frac{3\pi}{2n}]$ if $3 \le n \le 6$, and for all $t \in [0, \cos\frac{2\pi}{n}]$ if $n \ge 7$.

2) So, it remains to prove that (8.1) is valid for $t \in [\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}]$ and $n \ge 7$. To this end, we consider the function

$$\begin{aligned} f(t) &:= (1-t)\tau''(1,t) \\ &= (1-t)\frac{n^2(n^2-1)}{3} - 2(1+t)n^2 + 2\,(1+t)\,\frac{1-T_n(t)}{1-t}. \end{aligned}$$

Clearly, $f(1) = 0$ and it is easy to see that $f(\cos\frac{2\pi}{n}) > 0$.

Let us prove next that f is convex on $I = [\cos\frac{2\pi}{n}, \infty)$. Indeed, the first two terms are linear in t whereas the last term consists of two factors, both convex, positive and increasing on I. The latter claim is obvious for the factor $1+t$, and it is also true for the factor $P_n(t) := \frac{1-T_n(t)}{1-t}$, since this P_n is a polynomial with a positive leading coefficient whose rightmost zero is the double zero at $t = \cos\frac{2\pi}{n}$.

Thus, f is convex on $[\cos\frac{2\pi}{n}, \infty)$, and it also satisfies $f(\cos\frac{2\pi}{n}) > 0$ and $f(1) = 0$. Therefore, if $f(t_*) > 0$ for some $t_* \in (\cos\frac{2\pi}{n}, 1)$, then $f(t) > 0$ for all $t \in [\cos\frac{2\pi}{n}, t_*]$. Hence, it suffices to show that $\tau''(1, t_*) > 0$ for $t_* = \cos\frac{3\pi}{2n}$. Putting this t_* into (8.1) and noting that $T_n(t_*) = 0$, we obtain

$$\tau''(1, t_*) = \frac{n^2(n^2-1)}{3} - 2n^2 u + \frac{2}{1+\cos\frac{3\pi}{2n}}\, u^2 \overset{?}{>} 0, \quad u := \cot^2\frac{3\pi}{4n}. \tag{8.2}$$

Inequality (8.2) will certainly be true if $\frac{n^2(n^2-1)}{3} - 2n^2 u + u^2 > 0$, and a sufficient condition for the latter is the inequality

$$\cot^2\frac{3\pi}{4n} = u < n^2\Big(1 - \sqrt{\frac{2}{3} + \frac{1}{3n^2}}\Big).$$

Since $\cot\alpha < \alpha^{-1}$ for $0 < \alpha < \frac{\pi}{2}$, this condition is fulfilled if

$$\Big(\frac{4}{3\pi}\Big)^2 < 1 - \sqrt{\frac{2}{3} + \frac{1}{3n^2}}\,,$$

and that is true for $n \ge 8$. For $n = 7$, one can verify (8.2) directly. □

9. Proof of Theorem 2.2

In this section, we prove that, for the majorant

$$\mu_m(x) = (1 - x^2)^{m/2}, \tag{9.1}$$

its snake-polynomial ω_μ is *not* extremal for the Duffin-Schaeffer inequality for $k \le m$, precisely that for the value

$$D^*_{k,\mu_m} := \sup_{|p(x)|_{\delta^*} \le |\mu_m(x)|_{\delta^*}} \|p^{(k)}\|$$

where $\delta^* = (\tau_i^*)$ is the set of points of oscillation of ω_{μ_m} between $\pm\mu_m$, we have

$$D^*_{k,\mu_m} > \|\omega_\mu^{(k)}\|, \qquad k \le m.$$

The snake-polynomial for μ_m in (9.1) is given by the formula

$$\omega_{\mu_m}(x) = \begin{cases} (x^2-1)^s\, T_n(x), & m = 2s, \\ \frac{1}{n}(x^2-1)^s\, T_n'(x), & m = 2s-1\,, \end{cases} \tag{9.2}$$

so its oscillation points are the sets

$$\delta_n^{(1)} := \Big(\cos\tfrac{\pi i}{n}\Big)_{i=0}^n, \qquad \delta_n^{(2)} := \Big(\cos\tfrac{\pi(i-1/2)}{n}\Big)_{i=1}^n,$$

at which $|T_n(x)| = 1$ and $|T_n'(x)| = \frac{n}{\sqrt{1-x^2}}$, respectively, with additional multiple points at $x = \pm 1$.

Now, we introduce the pointwise Duffin-Schaeffer function:

$$d_{k,\mu}^*(x) := \sup_{|p|_{\delta^*} \le |\mu_m|_{\delta^*}} |p^{(k)}(x)| = \begin{cases} \displaystyle\sup_{|q|_{\delta_n^{(1)}} \le |T_n|_{\delta_n^{(1)}}} |(x^2-1)^s q(x)]^{(k)}|, & m = 2s \\ \displaystyle\sup_{|q|_{\delta_n^{(2)}} \le \frac{1}{n}|T_n'|_{\delta_n^{(2)}}} |(x^2-1)^s q(x)]^{(k)}|, & m = 2s-1 \end{cases}$$

and note that

$$D_{k,\mu}^* = \|d_{k,\mu}^*(\cdot)\| \ge d_{k,\mu}^*(0).$$

Proposition 9.1. *We have*

$$D_{k,\mu_m}^* \ge \mathcal{O}(n^k \ln n).$$

Proof. We split the proof into two cases, for even and for odd m in (9.2), respectively.

Case 1 ($m = 2s$). Let us show that, for a fixed $k \in \mathbb{N}$, and for all large $n \not\equiv k \pmod 2$, there is a polynomial q_1 of degree n such that

$$1)\quad |q_1(x)|_{\delta_n^{(1)}} \le 1, \qquad 2)\quad \big|[(x^2-1)^s q_1(x)]^{(k)}\big|_{|x=0} = \mathcal{O}(n^k \ln n).$$

1) Set

$$P(x) := (x^2-1)T_n'(x) = n\prod_{i=0}^n (x - t_i), \qquad (t_i)_{i=0}^n = (\cos\tfrac{\pi i}{n})_{i=0}^n = \delta_n^{(1)}, \quad (9.3)$$

and, having in mind that $t_{n-i} = -t_i$, define the polynomial of degree n

$$q_1(x) := \frac{1}{n^2}P(x)\sum_{i=1}^{(n-1)/2}\Big(\frac{1}{x-t_i} - \frac{1}{x+t_i}\Big) =: \frac{1}{n^2}P(x)U(x). \qquad (9.4)$$

This polynomial vanishes at those t_i that do not appear under the sum, i.e., at $t_0 = 1$, $t_n = -1$ and, for even n, at $t_{n/2} = 0$. At all other t_i it has the absolute value $|q(t_i)| = \frac{1}{n^2}|P'(t_i)| = 1$, by virtue of the equality $P'(x) = n^2 T_n(x) + xT_n'(x)$. Hence,

$$|q_1(t_i)| \le |T_n(t_i)| = 1, \qquad \forall t_i \in \delta_n^{(1)}.$$

2) We see from (9.4) that U is an even function, $P(x) = (x^2-1)T_n'(x)$ is either even or odd polynomial, and for their non-vanishing derivatives at $x=0$ we have

$$\begin{aligned} |P^{(r)}(0)| &= |T_n^{(r+1)}(0) - r(r-1)T_n^{(r-1)}(0)| = \mathcal{O}(n^{r+1}), \quad n \not\equiv r \,(\mathrm{mod}\, 2), \\ |U^{(r)}(0) &= 2\,r! \sum_{j=1}^{(n-1)/2} \frac{1}{(\sin\frac{\pi j}{n})^{r+1}} = \begin{cases} \mathcal{O}(n\ln n), & r=0, \\ \mathcal{O}(n^{r+1}), & r=2r_1 \ge 2. \end{cases} \end{aligned}$$

Respectively, in the Leibnitz formula for $q_1^{(k)}(0)$,

$$q_1^{(k)}(0) = \frac{1}{n^2}[P(x)U(x)]^{(k)}_{|x=0} = \frac{1}{n^2}\sum_{r=0}^{k}\binom{k}{r} P^{(k-r)}(0)U^{(r)}(0),$$

if $k \not\equiv n(\mathrm{mod}\, 2)$ then the term $P^{(k)}(0)U(0) = \mathcal{O}(n^{k+2}\ln n)$ dominates, hence

$$q_1^{(k)}(0) = \mathcal{O}(n^k \ln n) \quad \Rightarrow \quad [(x^2-1)^s q_1(x)]^{(k)}_{|x=0} = \mathcal{O}(n^k\ln n), \qquad k \not\equiv n(\mathrm{mod}\, 2).$$

Case 2 ($m = 2s-1$). Similarly, for a fixed k, and for all large $n \equiv k\,(\mathrm{mod}\, 2)$, the polynomial q_2 of degree $n-1$ defined as

$$q_2(x) := \frac{1}{n^2}T_n(x)\sum_{i=1}^{(n-1)/2}\Big(\frac{1}{x-t_i} - \frac{1}{x+t_i}\Big), \qquad (t_i)_{i=1}^n = (\cos\tfrac{\pi(i-1/2)}{n})_{i=1}^n = \delta_n^{(2)},$$

satisfies

$$1) \quad |q_2(x)|_{\delta_n^{(2)}} \le \frac{1}{n}|T_n'(x)|_{\delta_n^{(2)}}, \qquad 2) \quad |(x^2-1)^s q_2^{(k)}(x)|_{|x=0} = \mathcal{O}(n^k\ln n).$$

Proposition 9.1 is proved. □

Proposition 9.2. *Let* $\mu_m(x) = (1-x^2)^{m/2}$. *Then*

$$M_{k,\mu_m} := \sup_{|p(x)|\le|\mu_m(x)|} \|p^{(k)}\| = \mathcal{O}(n^k), \quad k \le m. \tag{9.5}$$

Proof. Pierre and Rahman [4] proved that

$$M_{k,\mu_m} = \max\Big(\|\omega_N^{(k)}\|, (\|\omega_{N-1}^{(k)}\|\Big), \qquad k \ge m, \tag{9.6}$$

where ω_N and ω_{N-1} are the snake-polynomial for μ_m of degree N and $N-1$, respectively. However, they did not investigate which norm is bigger and at what point $x \in [-1,1]$ it is attained. We proved in [3] that, for functions

$$f(x) := (x^2-1)^s T_n(x), \quad g(x) := \frac{1}{n}(x^2-1)^s T_n'(x)$$

we have

$$\|f^{(k)}\| = f^{(k)}(1), \quad k \ge 2s, \qquad \|g^{(k)}\| = g^{(k)}(1), \quad k \ge 2s-1.$$

Since f and g are exactly the snake-polynomials for $\mu_m(x) = (1-x^2)^{m/2}$ for $m = 2s$ and $m = 2s-1$, respectively, we can refine the result of Pierre and Rahman in (9.6) as

$$M_{k,\mu_m} = \omega_N^{(k)}(1) = \omega_\mu^{(k)}(1), \qquad k \ge m\,.$$

It is easy to find that $f^{(k)}(1) = \mathcal{O}(n^{2(k-s)})$ and $g^{(k)}(1) = \mathcal{O}(n^{2(k-s)+1})$, hence $\omega_\mu^{(k)}(1) = \mathcal{O}(n^{2k-m})$, in particular,

$$M_{m,\mu_m} = \omega_\mu^{(m)}(1) = \mathcal{O}(n^m)\,, \tag{9.7}$$

and that proves (9.5) for $k = m$. For $k < m$, we observe that

$$k < m \quad \Rightarrow \quad \mu_m \le \mu_k \quad \Rightarrow \quad M_{k,\mu_m} \le M_{k,\mu_k} \overset{(9.7)}{=} \mathcal{O}(n^k)\,,$$

and that completes the proof. □

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