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Properties of Mappings Generated with Inequalities for Isotonic Linear Functionals

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We consider mappings generated by inequalities for isotonic linear functionals such as the inequalities of Chebyshev, Beckenbach-Dresher, Jensen-Mercer, Jensen, Hölder, Minkowski and their reversed versions. Properties like quasilinearity, boundedness and monotonicity are proved. Also, properties of the composite functional $x \mapsto h(v(x))\Phi\big(\frac{g(x)}{v(x)}\big)$ are mentioned, where g and v are functions associated with the mappings generated by the inequalities and Φ is a h-concave monotone function.

Keywords and Phrases: Isotonic linear functional, h-concave function, quasilinearity, Chebyshev's functional, Jensen's functional.

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1. Introduction and Preliminaries

It is known that some inequalities generate mappings based on the difference between their right- and left-hand side. Those mappings have some interesting properties such as quasilinearity, homogeneity, monotonicity, boundedness, etc. Here we mention only few papers with such motives: [2, 4, 7, 12, 15, 16, 22]. In these papers the considered inequalities and mappings involve sums and integrals. However, many classical inequalities allow the so-called functional version, where the sum or the integral is substituted by an isotonic linear functional. For some functional versions of the classical inequalities, the book [20] is a good beginning point of investigation, while some inequalities for isotonic linear functional will be mentioned or proved in this paper.

Meanwhile, Dragomir [8, 10, 9] gave series of results about quasilinearity of some composite functionals with applications to Jensen's, Hölder's, Minkowski's and Schwarz's functionals, and these results were generalized in [1, 18].

In this paper we give properties of mappings connected with inequalities for isotonic linear functional and apply results from [18] to them. In this introductory section we give definitions and describe some useful facts about the defined items. The second section is devoted to the Chebyshev functional, while in the third section we consider a functional which is generated by the generalized Beckenbach-Dresher inequality. In both sections the considered composite functionals have denominators which are superadditive functions and for their investigation we need general results from [18]. The last section is devoted to the properties of functionals which are related to the classical inequalities such as Jensen's, Hölder's, Minkowski's, Jensen-Mercer's and their reversed versions. The corresponding composite functionals have an additive function in the denominator and we apply a special case of the results given in [18].

Definition 1.1. Let E be a non-empty set and $\mathbf{L}$ be a class of real-valued functions on E having the properties:

L1. If $f, g \in L$, then $(af + bg) \in \mathbf{L}$ for all $a, b \in \mathbb{R}$;

L2. The function $\mathbf{1}$ belongs to $\mathbf{L}$ ($\mathbf{1}(t) = 1$ for $t \in E$).

A functional $A : \mathbf{L} \to \mathbf{R}$ is called an *isotonic linear functional* if the following assumptions are satisfied:

A1. $A(af + bg) = aA(f) + bA(g)$ for $f, g \in \mathbf{L}$, $a, b \in \mathbb{R}$;

A2. $f \in \mathbf{L}$, $f(t) \geq 0$ on E implies $A(f) \geq 0$.

Let C be a convex cone in the linear space X over $\mathbb{R}$ or $\mathbb{C}$. Let L be a real number, $L \neq 0$. A functional $f : C \to \mathbb{R}$ is called *L-superadditive* (resp., *L-subadditive)* on C if

$$f(x + y) \geq L(f(x) + f(y)) \quad \text{for any} \quad x, y \in C$$
$$(\text{resp., } f(x + y) \leq L(f(x) + f(y)) \quad \text{for any} \quad x, y \in C).$$

If $L = 1$, then the functional f is simply called *superadditive* (*subadditive*). In the latter case we term f as *quasilinear* functional.

A function $h : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(xy) \geq h(x)h(y) \qquad \text{for all} \quad x, y \in J.$$

If the reversed inequality is satisfied, then h is said to be a *submultiplicative* function. If the equality holds, then h is said to be a *multiplicative* function.

Let K be a real non-negative function. We will say that a functional f is *K-positive homogeneous* if

$$f(tx) = K(t)f(x) \qquad \text{for any} \quad t \geq 0 \quad \text{and} \quad x \in C.$$

The function K has to be multiplicative. Moreover, we have either $K \equiv 1$ or $K(0) = 0$. In particular, if $K(t) = t^k$, then we simply say that f is *positive homogeneous on C of order k*. If $k = 1$, we call it *positive homogeneous*.

Example 1.1. The function $x \mapsto x^s$ is:

1. superadditive and L-subadditive with $L = 2^{s-1} \geq 1$ for $s \in (1, \infty)$;
2. subadditive and L-superadditive with $L = 2^{s-1} \leq 1$ for $s \in (0, 1]$;
3. L-subadditive with $L = 2^{s-1} \leq \frac{1}{2}$ for $s < 0$.

In the sequel I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and h and f are non-negative functions defined on J and I, respectively.

Definition 1.2 ([21]). Let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an *h-convex* function if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y).$$

If the reversed inequality is satisfied, then f is called an *h-concave* function.

It is evident that this notion generalizes the concepts of classical convexity (for $h(t) = t$); of s-convexity in the second sense (for $h(t) = t^s$, $s \in (0,1)$), [3, 14]; of P-functions (for $h(t) = 1$), [19], and of Godunova-Levin functions (for $h(t) = t^{-1}$), [13].

Example 1.2. It is known ([21]), that the function $f(x) = x^\lambda$ is s-convex in the second sense if

$$\lambda \in (-\infty, 0] \cup [1, \infty),\ s \leq 1 \qquad \text{or} \qquad \lambda \in (0,1),\ s \leq \lambda.$$

The function $f(x) = x^\lambda$ is s-concave in the second sense if

$$\lambda \in (0,1),\ s \geq 1 \qquad \text{or} \qquad \lambda > 1,\ s \geq \lambda.$$

Examples of functions Φ, non-decreasing on $[0, \infty)$ and h-concave, where $h(x) = x^s$, $s > 1$ (but not concave) are for instance: $\Phi(x) = \arctan(x^s)$, $\Phi(x) = \tanh(x^s)$, $\Phi(x) = x^s/(1+x^s)$. The function

$$f(x) = \begin{cases} x^s, & x \in [0,1], \\ x, & x \in (1,b], \end{cases}$$

is non-decreasing, convex on (0,1], starshaped on $[0,b]$, $b \geq 1$, and s-concave in the second sense on $[0,b]$, ([18]).

Let us mention also Lemma 2.1 from [18]:

Lemma 1.1. *Let $x, y \in C$ and $f : C \to \mathbb{R}$ be a non-negative, L-superadditive and K-positive homogeneous functional on C. If $M \geq m > 0$ are such that $x - my, My - x \in C$, then*

$$\frac{1}{L} K(M) f(y) \geq f(x) \geq LK(m) f(y).$$

Composite functionals associated with monotone h-concave functions. In [18] the following theorem about quasilinearity of a composite functional η_Φ defined on a convex cone is proved. Here, a convex cone C is a subset of a linear space over F ($F = \mathbb{R}$ or $\mathbb{C}$) with the property: if $x, y \in C$ and $\alpha > 0$, then $x + y \in C$ and $\alpha x \in \mathbb{C}$.

Theorem 1.1. *Let h be a non-negative function which is k_1-positive homogeneous. Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ be an L-superadditive functional on C.*

(i) *If h is submultiplicative, $g : C \to [0, \infty)$ is an L-superadditive functional on C and $\Phi : [0, \infty) \to [0, \infty)$ is h-concave and non-decreasing, then the functional $\eta_\Phi : C \to \mathbb{R}$ defined by*

$$\eta_\Phi(x) := h(v(x))\Phi\Big(\frac{g(x)}{v(x)}\Big)$$

is $k_1(L)$-superadditive on C.

(ii) *If h is supermultiplicative, g is L-subadditive, Φ is h-convex and non-decreasing with $\Phi(0) = 0$, then η_Φ is $k_1(L)$-subadditive.*

A simple consequence of L-superadditivity and K-positive homogeneity of η_Φ is the boundedness property which is given in the following corollary, [18].

Corollary 1.1. *Let h be a non-negative submultiplicative function which is k_1-positive homogeneous. Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ be L-superadditive and k_2-positive homogeneous on C. Let $x, y \in C$ and assume that there exist $M \geq m \geq 0$ such that $x - my$ and $My - x \in C$. Let $K(t) = k_1(k_2(t))$.*

If $g : C \to [0, \infty)$ is an L-superadditive and k_2-positive homogeneous functional on C and $\Phi : [0, \infty) \to [0, \infty)$ is h-concave and non-decreasing, then

$$\frac{1}{k_1(L)}K(M)\eta_\Phi(y) \geq \eta_\Phi(x) \geq k_1(L)K(m)\eta_\Phi(y).$$

If we consider an additive function v, then with a proof similar to that of Theorem 1.1 we obtain the following proposition.

Proposition 1.1. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ be an additive functional on C.*

(i) *If h is non-negative submultiplicative, $g : C \to [0, \infty)$ is a superadditive functional on C and $\Phi : [0, \infty) \to [0, \infty)$ is h-concave and non-decreasing, then the functional $\eta_\Phi : C \to \mathbb{R}$ defined by*

$$\eta_\Phi(x) := h(v(x))\Phi\Big(\frac{g(x)}{v(x)}\Big)$$

is superadditive on C.

(ii) *If h is non-negative supermultiplicative, g is superadditive, Φ is h-convex and non-decreasing, then η_Φ is subadditive.*

Corollary 1.1 under the additional assumptions that v is additive and $L = 1$, $k_1(t) = k_2(t) = t$, becomes the following:

Corollary 1.2. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ be additive and positive homogeneous on C. Let $x, y \in C$ and assume that there exist $M \geq m \geq 0$ such that $x - my, My - x \in C$.*

If $g : C \to [0, \infty)$ is superadditive and $\Phi : [0, \infty) \to [0, \infty)$ is h-concave and non-decreasing with submultiplicative and positive homogeneous h, then

$$M\eta_\Phi(y) \geq \eta_\Phi(x) \geq m\eta_\Phi(y).$$

Furthermore, if $M = 1$, then the property of monotonicity holds

$$\eta_\Phi(y) \geq \eta_\Phi(x).$$

2. Functionals Associated with Chebyshev's Inequality

The classical Chebyshev inequality for integrals ([20, p. 197]) states that if f and g are similarly ordered real functions on $[a, b]$ and p is a non-negative weight function, then

$$\int_a^b p(x)\,dx \int_a^b p(x)f(x)g(x)\,dx \geq \int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx$$

provided that the integrals exist. If f and g are oppositely ordered, then the reversed inequality is valid. Let us recall that f and g are said to be similarly ordered if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \qquad \text{for all } \; x, y \in [a, b].$$

When the reversed inequality is satisfied, then f and g are called oppositely ordered.

A version of the Chebyshev inequality for isotonic linear functionals states:

Let A be an isotonic linear functional on L. If f and g are similarly ordered functions and p is a non-negative weight function, then

$$A(p)A(pfg) \geq A(pf)A(pg). \tag{2.1}$$

If f and g are oppositely ordered, then the reversed inequality is valid.

This result was proved in [11]. The proof of the Chebyshev inequality for isotonic functional is based on the inequality

$$p(x)p(y)(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

Performing multiplication and applying the functional A first with respect to x and then with respect to y, we get (2.1).

Let us consider the Chebyshev functional T defined on the convex cone $C_T(A,f,g) = \{w \in L : w \geq 0,\ wf, wg, wfg \in L\}$:

$$T(w) = A(w)A(wfg) - A(wf)A(wg).$$

It is obvious that T is positive homogeneous of order 2, T is non-negative for similarly ordered functions f and g and T is non-positive for oppositely ordered functions f and g.

Theorem 2.1. (i) *If f and g are similarly ordered functions, then T is a superadditive functional. If f and g are oppositely ordered, then T is a subadditive functional.*

(ii) *If $w, v \in C_T(A,f,g)$ and $M \geq m > 0$ are such that $w - mv, Mv - w \in C_T(A,f,g)$, then for similarly ordered functions f and g*

$$m^2\, T(v) \leq T(w) \leq M^2\, T(v).$$

In particular, if $M = 1$, then we get monotonicity:

$$T(v) \geq T(w).$$

If f and g are oppositely ordered functions, then the reversed inequalities hold.

Proof. (i) If f and g are similarly ordered, then we have

$$(w(x)v(y) + v(x)w(y))(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

Performing multiplication and applying functional A first with respect to x and then with respect to y we get

$$A(v)A(wfg) + A(w)A(vfg) \geq A(vg)A(wf) + A(wg)A(vf). \tag{2.2}$$

For the expression $T(w+v) - T(w) - T(v)$ we have

$$\begin{aligned}
T(w+v) - T(w) - T(v)& \\
&= A(w+v)A(wfg + vfg) - A(wf + vf)A(wg + vg) \\
&\quad - A(w)A(wfg) + A(wf)A(wg) - A(v)A(vfg) - A(vf)A(vg) \\
&= A(v)A(wfg) + A(w)A(vfg) - A(vg)A(wf) + A(wg)A(vf).
\end{aligned}$$

By (2.2), the last expression is non-negative, hence $T(w+v) \geq T(w) + T(v)$.

(ii) If f and g are similarly ordered, then T is superadditive, non-negative and 2-positive homogeneous. Using Lemma 1.1, we get the assertion of (ii). The other cases are proved similarly. $\square$

Theorem 2.2. (i) *Let h be a non-negative, k_1-positive homogeneous sub-multiplicative function, and $\Phi : [0,\infty) \to [0,\infty)$ be h-concave and non-decreasing. Let f and g be similarly ordered.*

Then the functional η_T defined on $C_T(A,f,g)$ by

$$\eta_T(w) = h(A^2(w))\Phi\Big(\frac{T(w)}{A^2(w)}\Big).$$

is superadditive.

(ii) *Furthermore, if $w, v \in C_T(A,f,g)$ and $M \geq m > 0$ are such that $w - mv, Mv - w \in C_T(A,f,g)$, then*

$$k_1(M^2)h(A^2(v))\Phi\Big(\frac{T(v)}{A^2(v)}\Big) \geq h(A^2(w))\Phi\Big(\frac{T(w)}{A^2(w)}\Big)$$
$$\geq k_1(m^2)h(A^2(v))\Phi\Big(\frac{T(v)}{A^2(v)}\Big).$$

(iii) *If f and g are oppositely ordered, then the above statements (i) and (ii) are valid with T replaced with $-T$.*

Proof. Since $x \mapsto x^2$ is a superadditive function, we have that $w \mapsto A^2(w)$ is also superadditive. It is also positive homogeneous of order 2. The functional T is also non-negative, superadditive and positive homogeneous of order 2. Substituting in η_Φ in Theorem 1.1 $v(w) = A^2(w)$ and $g(w) = T(w)$, we obtain that $\eta_\Phi = \eta_T$ and it is superadditive. Claim (ii) is in fact Corollary 1.1 for the functional η_T. □

3. Functional Associated with the Beckenbach-Dresher Inequality

In this section we consider applications of Theorem 1.1 when the function v is superadditive. Let A and B be isotonic linear functionals on L. Let us define a convex cone $C_G(A,B,f,g)$, where

$$C_G(A,B,f,g) = \{w \in L : w, f, g \geq 0,\ wg, wf \in L,\ B(wg) > 0\}.$$

Theorem 3.1. (i) *Let $u \in \mathbb{R}^+$, $p, q \in (0,1]$ and $k \geq \max\{1, u\}$. Then the functional G defined by*

$$G(w) = \frac{A^{u/p}(wf)}{B^{(u-k)/q}(wg)}$$

is superadditive on $C_G(A,B,f,g)$.

(ii) *If $w, v \in C_G(A,B,f,g)$ and $M \geq m > 0$ are such that $w - mv, Mv - w \in C_G(A,B,f,g)$, then*

$$m^{u/p+(k-u)/q}\, G(v) \leq G(w) \leq M^{u/p+(k-u)/q}\, G(v).$$

In particular, if $M = 1$, then

$$G(v) \geq G(w).$$

Proof. Let us define

$$\Phi(t) = t^u, \quad h(t) = t^k, \qquad k \geq \max\{1, u\},$$
$$v(w) = B^{1/q}(wg), \quad g_1(w) = A^{1/p}(wf), \qquad p, q \in (0,1].$$

Then h is multiplicative, Φ is non-decreasing h-concave (see Example 1.2), and v and g_1 are superadditive (see Example 1.1). The functional η_Φ from Theorem 1.1 is of the form

$$\eta_\Phi(w) = h(v(w))\Phi\Big(\frac{g_1(w)}{v(w)}\Big) = \frac{A^{u/p}(wf)}{B^{(u-k)/q}(wg)},$$

i.e. $\eta_\Phi = G$. Then the assumptions of Theorem 1.1(i) are satisfied, and thereby G is superadditive. The monotonicity and the boundedness in part (ii) are proved using Lemma 1.1 with $f = \eta_\Phi = G$ and $K(t) = t^{u/p+(k-u)/q}$. □

Remark 3.1. With $k = 1$, $u, p, q \in (0,1]$, $f \to f^p$, $g \to g^q$, Theorem 3.1 asserts that $G(w) = \frac{A^{u/p}(wf^p)}{B^{(u-1)/q}(wg^q)}$ is superadditive. This is a result obtained in [22]. Furthermore, in this case G is connected with the Beckenbach-Dresher inequality. Namely, if we put $A(f) = B(f) = \int f d\varphi$, $u = \frac{p}{p-q}$, $w = 1$, then the inequality which describes superadditivity of G is exactly the reversed Beckenbach-Dresher inequality. More about this inequality can be found in [22] and the references therein.

Remark 3.2. Depending on how $k \neq 0$, u and 0 are situated with respect to each other, we may consider 6 cases and get results for superadditivity and subaditivity of G. (In fact, the cases are more, as we have also dependence on p and q.) The idea is to use the inequalities

$$\min\{1, 2^{l-1}\}(a^l + b^l) \leq (a+b)^l \leq \max\{1, 2^{l-1}\}(a^l + b^l), \qquad l \geq 0,$$

and

$$(a+b)^l \leq 2^{l-1}(a^l + b^l), \qquad l \leq 0,$$

for $l = k$, $l = \frac{1}{p}$, $l = \frac{1}{q}$. We use also Hölder's inequality

$$(a+b)^s (c+d)^{1-s} \geq a^s c^{1-s} + b^s d^{1-s} \qquad \text{for} \quad s = \frac{u}{k}, \quad s \in [0,1],$$

and the reversed Hölder inequality for $s = \frac{u}{k}$ in the cases 1) $\frac{u}{k} < 0$, $1 - \frac{u}{k} > 0$, and 2) $\frac{u}{k} > 0$, $1 - \frac{u}{k} < 0$.

Theorem 3.2. *Let $k \neq 0$.*

(i) *If p, $q > 0$ and $0 \leq u \leq k$, then G is L_1-superadditive with*

$$L_1 = \min\{1, 2^{(1/p-1)u}\} \cdot \min\{1, 2^{(1/q-1)(k-u)}\} \cdot \min\{1, 2^{k-1}\}.$$

(ii) *If p, $q > 0$ and either $0 < k \leq u$ or $u \leq 0 < k$, then G is L_2-subadditive with*

$$L_2 = \max\{1, 2^{(1/p-1)u}\} \cdot \max\{1, 2^{(1/q-1)(k-u)}\} \cdot \max\{1, 2^{k-1}\}.$$

(iii) *If p, $q > 0$ and $k \leq u \leq 0$ (hence, $0 \leq \frac{u}{k} \leq 1$), then G is L_3-subadditive with*

$$L_3 = \max\{1, 2^{(1/p-1)u}\} \cdot \max\{1, 2^{(1/q-1)(k-u)}\}\, 2^{k-1}.$$

(iv) *If $p > 0$, $q < 0$, and $u < 0 < k$, then G is L_4-subadditive with*

$$L_4 = \max\{1, 2^{(1/p-1)u}\} \cdot 2^{(1/q-1)(k-u)} \max\{1, 2^{k-1}\}.$$

(v) *If $p < 0$, $q > 0$, and $0 < k \leq u$, then G is L_5-subadditive with*

$$L_5 = 2^{(1/p-1)u} \max\{1, 2^{(1/q-1)(k-u)}\} \cdot \max\{1, 2^{k-1}\}.$$

Proof. (i) Since the function $x \mapsto x^{1/p}$ is $\min\{1, 2^{1/p-1}\}$-superadditive, the function $x \mapsto x^{1/q}$ is $\min\{1, 2^{1/q-1}\}$-superadditive, and the function $x \mapsto x^k$ is $\min\{1, 2^{k-1}\}$-superadditive, we have for instance

$$\big(A(wf) + A(vf)\big)^{1/p} \geq \min\{1, 2^{1/p-1}\}\big(A(wf)^{1/p} + A(vf)^{1/p}\big),$$

and hence

$$\begin{aligned}
G(w+v) &= \big(A(wf) + A(vf)\big)^{u/p}\big(B(wg) + B(vg)\big)^{(k-u)/q} \\
&\geq \min\{1, 2^{(1/p-1)u}\} \cdot \min\{1, 2^{(1/q-1)(k-u)}\} \\
&\qquad \times \Big[\big(A(wf)^{1/p} + A(vf)^{1/p}\big)^{u/k}\big(B(wg)^{1/q} + B(vg)^{1/q}\big)^{1-u/k}\Big]^k \\
&\geq \min\{1, 2^{(1/p-1)u}\} \cdot \min\{1, 2^{(1/q-1)(k-u)}\} \\
&\qquad \times \Big[A(wf)^{u/(kp)} B(wg)^{(k-u)/(kq)} + A(vf)^{u/(kp)} B(vg)^{(k-u)/(kq)}\Big]^k \\
&\geq L_1\big[A(wf)^{u/p} B(wg)^{(k-u)/q} + A(vf)^{u/p} B(vg)^{(k-u)/q}\big] \\
&= L_1\big(G(w) + G(v)\big),
\end{aligned}$$

where for the intermediate estimation we have used Hölder's inequality.

(ii) Let us consider the case p, $q > 0$, $0 \leq k \leq u$, the other case can be proved analogously. The function $x \mapsto x^{1/p}$ is $\max\{1, 2^{1/p-1}\}$-subadditive, the function $x \mapsto x^{1/q}$ is $\min\{1, 2^{1/q-1}\}$-superadditive and the function

$x \mapsto x^k$ is $\min\{1, 2^{k-1}\}$-superaddditive. Using the reversed Hölder inequality with negative exponent $1 - \frac{u}{k}$ and positive exponent $\frac{u}{k}$, we get the result as follows:

$$\begin{aligned}
G(w+v) &= \Big[\big(A(wf)+A(vf)\big)^{u/(kp)}\big(B(wg)+B(vg)\big)^{(k-u)/(kq)}\Big]^k \\
&\le \Big[\max\{1, 2^{(1/p\,-1)u/k}\}\big(A(wf)^{1/p}+A(vf)^{1/p}\big)^{u/k} \\
&\qquad \times \max\{1, 2^{(1/q\,-1)(1-u/k)}\}\big(B(wg)^{1/q}+B(vg)^{1/q}\big)^{1-u/k}\Big]^k \\
&\le \max\{1, 2^{(1/p\,-1)u}\}\cdot\max\{1, 2^{(1/q\,-1)(k-u)}\} \\
&\qquad \times \big[A(wf)^{u/(kp)}B(wg)^{(k-u)/(kq)}+A(vf)^{u/(kp)}B(vg)^{(k-u)/(kq)}\big]^k \\
&\le L_2\big[A(wf)^{u/p}B(wg)^{(k-u)/q}+A(vf)^{u/p}B(vg)^{(k-u)/q}\big] \\
&= L_2\big(G(w)+G(v)\big).
\end{aligned}$$

(iii) The function $x \mapsto x^{1/p}$ is $\max\{1, 2^{1/p\,-1}\}$-subadditive, the function $x \mapsto x^{1/q}$ is $\max\{1, 2^{1/q\,-1}\}$-subadditive, and the function $x \mapsto x^k$ is 2^{k-1}-subadditive. We use Hölder's inequality with positive exponents $\frac{u}{k}$ and $1 - \frac{u}{k}$ to deduce subadditivity of G as follows:

$$\begin{aligned}
G(w+v) &= \Big[\big(A(wf)+A(vf)\big)^{u/(kp)}\big(B(wg)+B(vg)\big)^{(k-u)/(kq)}\Big]^k \\
&\le \Big[\max\{1, 2^{(1/p\,-1)u/k}\}\big(A(wf)^{1/p}+A(vf)^{1/p}\big)^{u/k} \\
&\qquad \times \max\{1, 2^{(1/q\,-1)(1-u/k)}\}\big(B(wg)^{1/q}+B(vg)^{1/q}\big)^{1-u/k}\Big]^k \\
&\le L_3\big[A(wf)^{u/(kp)}B(wg)^{(k-u)/(kq)}+A(vf)^{u/(kp)}B(vg)^{(k-u)/(kq)}\big]^k \\
&\le L_3\big[A(wf)^{u/p}B(wg)^{(k-u)/q}+A(vf)^{u/p}B(vg)^{(k-u)/q}\big]^k \\
&= L_3\big(G(w)+G(v)\big).
\end{aligned}$$

(iv) We have

$$\big(A(wf)+A(vf)\big)^{1/p} \ge \min\{1, 2^{1/p\,-1}\}\big(A(wf)^{1/p}+A(vf)^{1/p}\big),$$

and hence

$$\big(A(wf)+A(vf)\big)^{u/(kp)} \le \max\{1, 2^{(1/p\,-1)u/k}\}\big(A(wf)^{1/p}+A(vf)^{1/p}\big)^{u/k}.$$

On the other hand,

$$\big(B(wg)+B(vg)\big)^{(k-u)/(kq)} \le 2^{(1/q\,-1)(1-u/k)}\big(B(wg)^{1/q}+B(vg)^{1/q}\big)^{1-u/k}.$$

Using these estimates and the reversed Hölder inequality with negative exponent $\frac{u}{k}$ and positive exponent $1 - \frac{u}{k}$, we get

$$\begin{aligned}
G(w+v) &\le \max\{1, 2^{(1/p-1)u}\}\, 2^{(1/q-1)(k-u)} \\
&\qquad \times \Big[\big(A(wf)^{1/p} + A(vf)^{1/p}\big)^{u/k}\big(B(wg)^{1/q} + B(vg)^{1/q}\big)^{1-u/k}\Big]^k \\
&\le \max\{1, 2^{(1/p-1)u}\}\, 2^{(1/q-1)(k-u)} \\
&\qquad \times \big[A(wf)^{u/(kp)} B(wg)^{(k-u)/(kq)} + A(vf)^{u/(kp)} B(vg)^{(k-u)/(kq)}\big]^k \\
&\le L_4\big[A(wf)^{u/p} B(wg)^{(k-u)/q} + A(vf)^{u/p} B(vg)^{(k-u)/q}\big] \\
&= L_4\big(G(w) + G(v)\big).
\end{aligned}$$

(v) The proof of this claim is analogous to the proof of (iv). □

4. Mappings Generated with the Classical Inequalities and Their Reverses for Isotonic Functionals

Let us define several mappings which arise from the functional versions of classical inequalities such as Jensen's, Hölder's and Minkowski's inequality and their reversed inequalities. In the sequel we assume that L satisfies conditions **L1** and **L2** and A is an isotonic linear functional on L. We define below several convex cones. Assume that A is an isotonic linear functional on L, φ is a continuous function on an interval $I \subseteq \mathbb{R}$, $a, b \in I$, $a < b$, $f, g \in \mathbf{L}$, $f_0, g_0 \in \mathbb{R}^+$, $p \in \mathbb{R}$, $p \neq 0$ and $q = \frac{p-1}{p}$. Set

$$\begin{aligned}
&C_J(A, \varphi, g) = \big\{w \in \mathbf{L} : wg, w\varphi(g) \in \mathbf{L},\ w \ge 0,\ A(w) > 0, \tfrac{A(w\phi(g))}{A(w)} \in I\big\}, \\
&C_{RJ}(A, \varphi, g, g_0) = \big\{(w_0, w) \in \mathbb{R}^+ \times \mathbf{L} : w \ge 0,\ wg, w\varphi(g) \in \mathbf{L}, \\
&\qquad\qquad\qquad\qquad 0 < A(w) < w_0,\ \tfrac{w_0 g_0 - A(wg)}{w_0 - A(w)} \in I\big\}, \\
&C_{JM}(A, \varphi, g, a, b) = \big\{w \in \mathbf{L} : w \ge 0,\ A(w) > 0,\ wg, w\varphi(g) \in \mathbf{L},\ \tfrac{A(wg)}{A(w)} \in [a, b]\big\}, \\
&C_H(A, f, g, p) = \big\{w \in \mathbf{L} : w, f, g \ge 0,\ wf^p, wg^q, wfg \in \mathbf{L},\ (P_1) \text{ is satisfied}\big\}, \\
&C_{POP}(A, f, g, f_0, g_0, p) = \big\{(w_0, w) \in \mathbb{R}^+ \times \mathbf{L} : w, f, g \ge 0,\ wf^p, wg^q, wfg \in \mathbf{L}, \\
&\qquad\qquad w_0 f_0^p - A(wf^p) > 0,\ w_0 g_0^q - A(wg^q) > 0,\ (P_1) \text{ is satisfied}\big\}, \\
&C_M(A, f, g, p) = \big\{w \in \mathbf{L} : w, f, g \ge 0,\ wf^p, wg^q, w(f+g)^p \in \mathbf{L},\ (P_2) \text{ is satisfied}\big\}, \\
&C_{BEL}(A, f, g, f_0, g_0, p) = \big\{(w_0, w) \in \mathbb{R}^+ \times \mathbf{L} : w, f, g \ge 0, \\
&\qquad wf^p, wg^q, w(f+g)^p \in \mathbf{L},\ w_0 f_0^p > A(wf^p),\ w_0 g_0^p > A(wg^p),\ (P_2) \text{ is satisfied}\big\},
\end{aligned}$$

where properties (P_1) and (P_2) are defined as follows:

(P_1) If $0 < p < 1$, then $A(wg^q) > 0$; if $p < 0$, then $A(wf^p) > 0$.

(P_2) If $0 < p < 1$ or $p < 0$, then $A(wg^q) > 0$ and $A(wf^p) > 0$.

Definition 4.1. *The Jensen functional J is defined as*

$$J(w) = A(w\varphi(g)) - A(w)\varphi\Big(\frac{A(wg)}{A(w)}\Big) \qquad \text{where } w \in C_J(A, \varphi, g).$$

The reversed Jensen functional RJ is defined on cone $C_{RJ}(A, \varphi, g, g_0)$, $g_0 \in I$, by

$$RJ(w_0, w) = (w_0 - A(w))\varphi\Big(\frac{w_0 g_0 - A(wg)}{w_0 - A(w)}\Big) - w_0\varphi(g_0) + A(w\varphi(g)).$$

The Jensen-Mercer functional JM is defined on $C_{JM}(A, \varphi, g, a, b)$ by

$$JM(w) = A(w)(\varphi(a) + \varphi(b)) - A(w\varphi(g)) - A(w)\varphi\Big(a + b - \frac{A(wg)}{A(w)}\Big).$$

From Jensen's inequality for isotonic linear functional (often called Jessen's inequality), [20, p. 113], and from the Jensen-Mercer inequality for isotonic linear functionals ([5]) we deduce that J and JM are non-negative if the function φ is convex, and J and JM are non-positive if φ is concave. The same holds for $RJ(w)$ from the functional version of the reversed Jensen inequality, which is proved in [20, p. 124].

Also, in [20] one can find Hölder's and Minkowski's inequality for isotonic functionals and their reversed inequalities, which is called Popoviciu's and Bellman's inequality, respectively, [20, pp. 113–114 and 124–125]. Inspired by these inequalities, we define the following functionals.

Definition 4.2. *The Hölder functional H is defined by*

$$H(w) = A^{1/p}(wf^p)A^{1/q}(wg^q) - A(wfg), \qquad w \in C_H(A, f, g, p).$$

The Popoviciu functional POP is defined by

$$POP(w_0, w) = w_0 f_0 g_0 - A(wfg) - \big(w_0 f_0^p - A(wf^p)\big)^{1/p}\big(w_0 g_0^q - A(wg^q)\big)^{1/q},$$

where $(w_0, w) \in C_{POP}(A, f, g, f_0, g_0, p)$.

The Minkowski functional M is defined as

$$M(w) = \big[A^{1/p}(wf^p) + A^{1/p}(wg^p)\big]^p - A(w(f+g)^p), \qquad w \in C_M(A, f, g, p).$$

The Bellman functional BEL is defined by

$$BEL(w_0, w) = w_0(f_0 + g_0)^p - A(w(f+g)^p) \\ - \big[\big(w_0 f_0^p - A(wf^p)\big)^{1/p} + \big(w_0 g_0^p - A(wg^p)\big)^{1/p}\big]^p,$$

where $(w_0, w) \in C_{BEL}(A, f, g, f_0, g_0, p)$.

If $p > 1$, then $H(w) \geq 0$ and $POP(w) \geq 0$, while if $0 < p < 1$ or $p < 0$, then $H(w) \leq 0$ and $POP(w) \leq 0$, [20, pp. 113 and 125]. If $p \geq 1$ or $p < 0$, then $M(w) \geq 0$ and $BEL(w_0, w) \geq 0$, while if $0 < p < 1$, then $M(w) \leq 0$ and $BEL(w_0, w) \leq 0$, [20, pp. 114 and 125–126]. Let us mention that J, JM, RJ, H, M, POP and BEL are positive homogeneous functionals.

In the following theorem we give results about quasilinearity of the above-defined mappings.

Theorem 4.1. (a) *If φ is convex, then functionals J and JM are superadditive and RJ is subadditive. If φ is concave, then functionas J and JM are subadditive and RJ is superadditive.*

(b) *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then H is superadditive and POP is subadditive. If $0 < p < 1$ or $p < 0$, then H is subadditive and POP is superadditive.*

(c) *If $p > 1$ or $p < 0$, then M is superadditive and BEL is subadditive. If $0 < p < 1$, then M is subadditive and BEL is superadditive.*

Proof. (a) The proof for J is given in [17]. Let us prove the claim for JM. Let φ be convex. A simple calculation yields:

$$\begin{aligned} JM(w+v)-JM(w)-JM(v) &= A(w)\varphi\Big(a+b-\frac{A(wg)}{A(w)}\Big)+A(v)\varphi\Big(a+b-\frac{A(vg)}{A(v)}\Big) \\ &\quad - A(w+v)\varphi\Big(a+b-\frac{A((w+v)g)}{A(w+v)}\Big) \geq 0. \end{aligned}$$

The last inequality follows from the classical Jensen inequality

$$(p_1+p_2)\varphi\Big(\frac{p_1}{p_1+p_2}\,x_1 + \frac{p_2}{p_1+p_2}\,x_2\Big) \leq p_1\varphi(x_1) + p_2\varphi(x_2),$$

applied with

$$p_1 = A(w), \quad p_2 = A(v), \quad x_1 = a+b-\frac{A(wg)}{A(w)}, \quad x_2 = a+b-\frac{A(vg)}{A(v)}.$$

(b) We shall prove only the first case because the proof of the second one is similar. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} &POP(w_0+v_0, w+v) - POP(w_0, w) - POP(v_0, v) \\ &\quad = (w_0+v_0)f_0g_0 - A((w+v)fg) \\ &\qquad - \big((w_0+v_0)f_0^p - A((w+v)f^p)\big)^{1/p}\big((w_0+v_0)g_0^q - A((w+v)g^q)\big)^{1/q} \\ &\qquad - w_0f_0g_0 - A(wfg) + \big(w_0f_0^p - A(wf^p)\big)^{1/p}\big(w_0g_0^q - A(wg^q)\big)^{1/q} \\ &\qquad - v_0f_0g_0 - A(vfg) + \big(v_0f_0^p - A(vf^p)\big)^{1/p}\big(v_0g_0^q - A(vg^q)\big)^{1/q} \\ &\quad = \big(w_0f_0^p - A(wf^p)\big)^{1/p}\big(w_0g_0^q - A(wg^q)\big)^{1/q} \\ &\qquad + \big(v_0f_0^p - A(vf^p)\big)^{1/p}\big(v_0g_0^q - A(vg^q)\big)^{1/q} \\ &\qquad - \big((w_0+v_0)f_0^p - A((w+v)f^p)\big)^{1/p}\big((w_0+v_0)g_0^q - A((w+v)g^q)\big)^{1/q} \\ &\quad \leq 0, \end{aligned}$$

where for the last inequality we have substituted $a = w_0 f_0^p - A(wf^p)$, $b = v_0 f_0^p - A(vf^p)$, $c = w_0 g_0^q - A(wg^q)$ and $d = v_0 g_0^q - A(vg^q)$ in the classical Hölder inequality

$$a^{1/p}c^{1/q} + b^{1/p}d^{1/q} \leq (a+b)^{1/p}(c+d)^{1/q}.$$

(c) We again do only one case, namely the superadditivity of BEL, the other cases follow in the same way. Similar to what was done in (b) we see that

$$\begin{aligned} &BEL(w_0+v_0, w+v) - BEL(w_0, w) - BEL(v_0, v) \\ &\qquad = -[(a+b)^{1/p} + (c+d)^{1/p}]^p + [a^{1/p} + c^{1/p}]^p + [b^{1/p} + d^{1/p}]^p \end{aligned}$$

If $0 < p < 1$ using classical Minkowski inequality we get superadditivity of BEL. □

As a consequence of quasilinearity we obtain boundedness and monotonicity properties. For the sake of simplicity, we say that a functional F satisfies property B_{inc} on cone C if

$$m\,F(x) \leq F(y) \leq M\,F(x),$$

where $x, y \in C$ is such that $y - mx, Mx - y \in C$. The functionl F is said to satisfy property B_{dec} if the reversed inequalities hold.

It is easy to see that if F is a non-negative, superadditive and positive homogeneous functional, then F satisfies B_{inc}, while if F is non-positive, subadditive and positive homogeneous, then F satisfies B_{dec} (see the proof of Theorem 2.1). Indeed, we have the following corollary.

Corollary 4.1. (i) *Property B_{inc} is satisfied when F is one of the following functionals:*

J or JM for convex φ; RJ when φ is concave; H for $p > 1$; M for $p < 0$ or $p > 1$.

(ii) *Property B_{dec} is satisfied when F is one of the following functionals:*

J or JM for concave φ; RJ when φ is convex; H for $p < 0$ or $p \in (0,1)$; M for $p \in (0,1)$.

Moreover, if $m = 1$, then the above boundedness property becomes monotonicity property.

Remark 4.1. Some of results given in Theorem 4.1 and Corollary 4.1 are known, especially when A is a sum or an integral. Of course, the most investigated map is Jensen's functional. Properties like quasilinearity, monotonicity and boundedness of Jensen's functional for sums are given in [7, 12], quasilinearity and monotonicity of Jensen's functional for isotonic functional (or Jessen's functional) can be found in [17], while boundedness property with applications is investigated in [6]. Properties of the discrete Jensen-Mercer functional are given in [16]. Properties of Minkowski's functional for integrals are given in [15], while some results about Hölder's and Minkowski's functionals for sums in normed spaces are given in [10]. To the best of our knowledge, properties of the reversed Jensen functional, Popoviciu's and Bellman's functional do not appear in literature in any form: discrete, integral or in the language of isotonic functionals.

Let us now apply our results from the first section to Jensen's functional.

Theorem 4.2. (i) *Let h be a non-negative submultiplicative function, and $\Phi : [0, \infty) \to [0, \infty)$ be h-concave and non-decreasing. If φ is a convex function, then the functional η_J defined on $C_J(A, \varphi, g)$ by*

$$\eta_J(w) = h(A(w))\Phi\Big(\frac{A(w\varphi(g))}{A(w)} - \varphi\Big(\frac{A(wg)}{A(w)}\Big)\Big)$$

is superadditive.

(ii) *If h is non-negative supermultiplicative, φ is convex, Φ is h-convex and non-decreasing, then η_J is subadditive.*

(iii) *Let assumptions of (i) be satisfied. If h is positive homogeneous, $w, v \in C_J(A, \varphi, g)$ and $M \geq m > 0$ is such that $w - mv, Mv - w \in C_J(A, \varphi, g)$, then*

$$m\eta_J(v) \leq \eta_J(w) \leq M\eta_J(v).$$

Furthermore, if $M = 1$, then

$$0 \leq \eta_J(w) \leq \eta_J(v).$$

Proof. Put in the definition of the functional η_Φ from Proposition 1.1

$$x \to w, \qquad v(x) \to A(w), \qquad g(x) \to J(w).$$

Then v is additive and positive homogeneous, and if φ is convex, then g is superadditive and by Proposition 1.1 we obtain that η_J is superadditive. Another statements follows from the results of Corollary 1.2. $\square$

Let us finally mention that the investigation of the functionals

$$\eta_{JM}(w) = h(A(w))\Phi\Big(\varphi(a) + \varphi(b) - \varphi\Big(a + b - \frac{A(wg)}{A(w)}\Big) - \frac{A(w\varphi(g))}{A(w)}\Big),$$

$$\eta_H(w) = h(A(w))\Phi\Big(\frac{1}{A(w)}\big(A^{1/p}(wf^p)A^{1/q}(wg^q) - A(wfg)\big)\Big),$$

$$\eta_M(w) = h(A(w))\Phi\Big(\frac{1}{A(w)}\big(\big[A^{1/p}(wf^p) + A^{1/p}(wg^p)\big]^p - A(w(f+g)^p)\big)\Big),$$

$$\eta_{RJ}(w_0, w) = h(w_0 - A(w))\Phi\Big(\varphi\Big(\frac{w_0 g_0 - A(wg)}{w_0 - A(w)}\Big) - \frac{w_0\varphi(g_0) - A(w\varphi(g))}{w_0 - A(w)}\Big),$$

yields results of a very similar nature to those given in Theorem 4.2.

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