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On New Families of Radial Basis Functions*

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In this paper, we present two new families of conditionally positive definite radial basis functions which can be used in a basis of multivariate splines in $\mathbb{R}^d$, $d \in \mathbb{N}$. These families are the natural generalizations of well-known constructions of radial basis functions of the tension spline, the regularized spline, and the completely regularized spline.

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1. Introduction

An RBF-spline associated with a scattered mesh of distinct nodes $x_i \in \mathbb{R}^d$, $i = 1, \dots, N$, is defined as

$$\sigma(x) = \sum_{i=1}^{N} \lambda_i \phi(|x - x_i|) + p(x), \tag{1.1}$$

where $\phi \in C[0, \infty)$ is a univariate function, $|x - y|$ is the Euclidean distance between points $x, y \in \mathbb{R}^d$, $\lambda_i \in \mathbb{R}$ are arbitrary values, and $p(x)$ is a function belonging to the given finite-dimensional linear space $\mathcal{P}$ of continuous functions. Usually, $\mathcal{P}$ is the space $\mathbb{P}^d_{m-1}$ of polynomials of the degree less than $m \in \mathbb{Z}_+$ on $\mathbb{R}^d$ ($\mathbb{P}^d_{-1} = \{0\}$). The space $\mathcal{P}$ is called the *trend of spline*.

With given interpolation equations $\sigma(x_i) = z_i$, $i = 1, \dots, N$, we arrive to a system of N linear equations with $N + K$ unknowns, where $K = \dim \mathcal{P}$. To close the system, we assume that

$$\sum_{i=1}^{N} \lambda_i p(x_i) = 0 \qquad \forall\, p \in \mathcal{P} \tag{1.2}$$

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and arrive to the system of $N+K$ linear equations with $N+K$ unknowns.

To provide the unique solvability of the resulting system, we assume that the mesh of nodes $\{x_i\}$ is *nondegenerate with respect to* $\mathcal{P}$, i.e., the equations $p(x_i)=0$, $i=1,\dots,N$, have the sole trivial solution $p(x)=0$ on $\mathcal{P}$. Additionally, we suppose that the radial basis function $\phi(r)$ is *conditionally positive definite on* $\mathbb{R}^d$ *with respect to* $\mathcal{P}$, i.e., for any finite mesh of distinct points $y_i\in\mathbb{R}^d$ and coefficients $\mu_i\in\mathbb{R}$, $i=1,\dots,M$, $M\in\mathbb{N}$,

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\mu_i\mu_j\phi(|y_i-y_j|)>0 \quad \text{provided} \quad \sum_{i=1}^{M}\mu_i p(y_i)=0 \quad \forall\, p\in\mathcal{P},$$

where $\{\mu_i\in\mathbb{R},\ i=1,\dots,M\}$ is a nontrivial set (at least one entry differs from zero). For $\mathcal{P}=\mathbb{P}^d_{m-1}$, the function $\phi(r)$ is called the *conditionally positive definite of order* m *on* $\mathbb{R}^d$.

We denote as $\mathcal{R}^d_m$ the set of all conditionally positive definite radial basis functions of order m on $\mathbb{R}^d$. It is evident that $\mathcal{R}^{d_1}_{m_1}\subset\mathcal{R}^d_m$ if $m_1\le m$ and $d_1\ge d$.

Functions from $\mathcal{R}^\infty_m:=\bigcap_{d=1}^\infty\mathcal{R}^d_m$ are strongly connected with completely monotonic functions ($f\in C^\infty(0,\infty)$ is completely monotonic if $(-1)^k f^{(k)}\ge 0$ for every $k\in\mathbb{Z}_+$).

We denote as $\mathcal{M}_m$ the set of functions $f\in C^\infty(0,\infty)$ having completely monotonic mth derivative $(-1)^m f^{(m)}$. It is well known [9] that, if a function $f\in\mathcal{M}_m$ is bounded at zero and $f^{(m)}$ is not a constant, the function $f(r^2)$ belongs to $\mathcal{R}^\infty_m$. Conversely, $\phi\in\mathcal{R}^\infty_m\Rightarrow\phi(\sqrt{\cdot}\,)\in\mathcal{M}_m$ (see, e.g., [15]). It is also clear that, if $f\in\mathcal{M}_m$ is unbounded at zero and $f^{(m)}$ is not a constant, $f(r^2+c^2)$, $c\ne 0$, belongs to $\mathcal{R}^\infty_m$.

The strong connection between $\mathcal{R}^\infty_m$ and $\mathcal{M}_m$ allows us to investigate functions from $\mathcal{M}_m$ and then generate functions belonging to $\mathcal{R}^\infty_m$ using the following substitution:

$$f\in\mathcal{M}_m \implies \phi(r):=f(a^2(r^2+c^2))+p_{m-1}(r^2) \implies \phi\in\mathcal{R}^\infty_m, \tag{1.3}$$

where $a\in\mathbb{R}\setminus\{0\}$, $c\in\mathbb{R}$, and $p_{m-1}\in\mathbb{P}^1_{m-1}$. Here we suppose that $f^{(m)}$ is not a constant and $c\ne 0$ if the function f is unbounded at zero. We call the function $f\in\mathcal{M}_m$ to be *the generic function* for $\phi\in\mathcal{R}^\infty_m$ in this case.

The following family of functions is generic for many well-known radial basis functions (see, e.g., [13]):

$$f_\nu(t)=\begin{cases}\Gamma(-\nu)t^\nu, & \nu\in\mathbb{R}\setminus\mathbb{Z}_+,\\ (-1)^{\nu+1}t^\nu\ln t, & \nu\in\mathbb{Z}_+.\end{cases} \tag{1.4}$$

Namely,

- f_ν, $\nu\in\mathbb{R}\backslash\mathbb{Z}_+$, is the generic function for multiquadric $(-1)^{\lfloor\nu\rfloor+1}(r^2+c^2)^\nu$, $\nu>0$, and unverse multiquadric $(r^2+c^2)^\nu$, $\nu<0$;

- f_n, $n \in \mathbb{N}$, is the generic function for the radial basis function $(-1)^{n+1} r^{2n} \ln r$ of the thin-plate spline ($n = 1$) and polyharmonic spline ($n > 1$);
- f_ν, $\nu \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, generates the radial basis function $(-1)^{\lfloor \nu \rfloor + 1} r^{2\nu}$;
- f_n, $n \in \mathbb{Z}_+$, generates the radial basis function $(-1)^{n+1}(r^2+c^2)^n \ln(r^2+c^2)$ of DMM-spline [14].

It is clear that

$$f_\nu \in \mathcal{M}_{(\lfloor \nu \rfloor + 1)_+}, \tag{1.5}$$

where $(k)_+ = \max\{k, 0\}$.

The paper is organized as follows.

In Section 2, we define two auxiliary families of functions — h_ν and $\tilde{h}_\nu$, $\nu \in \mathbb{R}$. The function h_ν generates Whittle–Matérn radial basis function [8] used with spline approximation in the Sobolev space $H^k(\mathbb{R}^d)$ when $\nu = k - d/2$. We prove that $h_\nu \in \mathcal{M}_0$, therefore, radial basis functions generated with it can be used for spline approximation in $\mathbb{R}^d$ for any $d \in \mathbb{N}$ and the parameter ν does not depend on d. The family of functions $\tilde{h}_\nu$ derives from the family f_ν (see (1.4)) and differs from it with slightly modified coefficients.

In Section 3, we present a two-parametric family of functions $h_{\nu,n}$, $\nu \in \mathbb{R}$, $n \in \mathbb{Z}_+$:

$$h_{\nu,0}(t) = \tilde{h}_\nu(t) - h_\nu(t); \qquad h_{\nu,n}(t) = \frac{\tilde{h}_{\nu+n}(t)}{n!\,2^n} - h_{\nu,n-1}(t), \quad n = 1, 2, \ldots$$

and prove that $h_{\nu,n} \in \mathcal{M}_{(\lfloor \nu \rfloor + n + 1)_+}$. We also investigate the boundedness of functions $h_{\nu,n}$ at zero. This family of functions was introduced and investigated by the author in [12].

In Section 4, we define two auxiliary families of functions — g_ν and $\tilde{g}_\nu$, $\nu \in \mathbb{R}$. The family g_ν is constructed with the help of incomplete Gamma-function and the family $\tilde{g}_\nu$ is again similar to f_ν.

In Section 5, we present another new two-parametric family of functions $g_{\nu,n}$, $\nu \in \mathbb{R}$, $n \in \mathbb{Z}_+$:

$$g_{\nu,0}(t) = \tilde{g}_\nu(t) - g_\nu(t); \qquad g_{\nu,n}(t) = g_{\nu+1,n-1}(t) - g_{\nu,n-1}(t), \quad n = 1, 2, \ldots$$

and prove that $g_{\nu,n} \in \mathcal{M}_{(\lfloor \nu \rfloor + n + 1)_+}$ and $g_{\nu,n}$ is bounded at zero for all $\nu \in \mathbb{R}$, $n \in \mathbb{Z}_+$.

In Section 6, we give calculation formulas for studied functions in special cases of integer and half-integer values of ν and show that the functions $h_{\nu,n}$ generalize the well-known constructions of tension spline and regularized spline [10] and the functions $g_{\nu,n}$ generalize the construction of completely regularized spline [11].

Finally, in Section 7, we discuss properties of radial basis functions generated with the functions of the new families and compare them with well-known radial basis functions.

2. The Auxiliary Families of Functions h_ν and $\tilde{h}_\nu$

We start with introducing the family of functions

$$h_\nu(t) = t^{\nu/2} K_\nu(\sqrt{t}), \qquad \nu \in \mathbb{R}, \quad t \in (0,\infty). \tag{2.1}$$

Here $K_\nu(x)$ is the modified Bessel function of the second kind of order ν (see, e.g., [1, Sect. 9.6]). The function $K_\nu(x)$ is continuous on $(0,\infty)$, unbounded at zero, positive, monotonic, and exponentially decaying as $x \to \infty$. Due to the symmetry $K_\nu(x) = K_{-\nu}(x)$, we establish

$$h_\nu(t) = t^\nu h_{-\nu}(t) \qquad \forall\, \nu \in \mathbb{R}, \quad t \in (0,\infty). \tag{2.2}$$

Theorem 1. *The function h_ν is differentiated by the formula*

$$h'_\nu(t) = -\frac{h_{\nu-1}(t)}{2} \qquad \forall\, \nu \in \mathbb{R}, \quad t \in (0,\infty). \tag{2.3}$$

Proof. We use the integral representation of the function $K_\nu(x)$ derived from [1, Sect. 9.6.25]:

$$K_\nu(x) = \frac{\Gamma(\nu+1/2)(2x)^\nu}{\Gamma(1/2)} \int_0^\infty \frac{\cos u}{(u^2+x^2)^{\nu+1/2}}\,du, \quad \nu+1/2>0, \;\; x>0. \tag{2.4}$$

Given $\nu > 0$, we derive from (2.4):

$$h_\nu(t) = \frac{\Gamma(\nu+1/2)}{\Gamma(1/2)}(2t)^\nu \int_0^\infty \frac{\cos u}{(u^2+t)^{\nu+1/2}}\,du.$$

Hence

$$\begin{aligned} h'_\nu(t) &= \frac{\Gamma(\nu+1/2)}{\Gamma(1/2)}\, 2\nu(2t)^{\nu-1} \int_0^\infty \frac{\cos u}{(u^2+t)^{\nu+1/2}}\,du \\ &\quad - \frac{\Gamma(\nu+3/2)}{\Gamma(1/2)}\,(2t)^\nu \int_0^\infty \frac{\cos u}{(u^2+t)^{\nu+3/2}}\,du \\ &= \nu t^{\nu/2-1} K_\nu(\sqrt{t}) - \frac{t^{(\nu-1)/2}}{2}\, K_{\nu+1}(\sqrt{t}) \\ &= \frac{t^{(\nu-1)/2}}{2}\Big(\frac{2\nu}{\sqrt{t}} K_\nu(\sqrt{t}) - K_{\nu+1}(\sqrt{t})\Big) \\ &= -\frac{t^{(\nu-1)/2}}{2}\, K_{\nu-1}(\sqrt{t}) = -\frac{h_{\nu-1}(t)}{2}. \end{aligned} \tag{2.5}$$

In the penultimate identity in (2.5), we use the well-known recurrence formula

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x} K_\nu(x). \tag{2.6}$$

Given $\nu \le 0$, we deduce from (2.4) and the identity $K_\nu(x) = K_{-\nu}(x)$:

$$h_\nu(t) = t^{\nu/2} K_{-\nu}(\sqrt{t}) = \frac{\Gamma(-\nu+1/2)}{\Gamma(1/2)}\, 2^{-\nu} \int_0^\infty \frac{\cos u}{(u^2+t)^{-\nu+1/2}}\,du.$$

Therefore,

$$h'_\nu(t) = -\frac{\Gamma(-\nu+3/2)}{\Gamma(1/2)}\, 2^{-\nu} \int_0^\infty \frac{\cos u}{(u^2+t)^{-\nu+3/2}}\, du = -\frac{h_{\nu-1}(t)}{2}.$$

The theorem is proved. □

Corollary 1. *$h_\nu \in \mathcal{M}_0$ for every $\nu \in \mathbb{R}$.*

Now we introduce the family of functions

$$\tilde{h}_\nu(t) = \begin{cases} \dfrac{\Gamma(-\nu)t^\nu}{2^{\nu+1}}, & \nu \in \mathbb{R} \setminus \mathbb{Z}_+, \\ (-1)^{\nu+1} \dfrac{t^\nu[\ln(t/4) - \psi(1) - \psi(\nu+1)]}{\nu!\, 2^{\nu+1}}, & \nu \in \mathbb{Z}_+. \end{cases} \tag{2.7}$$

Here $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$ and $\gamma = 0.577215\ldots$ is the Euler constant.

One can easy establish that functions $\tilde{h}_\nu$ are differentiated similarly to h_ν:

$$\tilde{h}'_\nu(t) = -\frac{\tilde{h}_{\nu-1}(t)}{2} \qquad \forall\, \nu \in \mathbb{R},\ \ t \in (0,\infty). \tag{2.8}$$

We will further use the power series for h_n, $n \in \mathbb{Z}_+$, derived from [1, Sect. 9.6.10, 9.6.11]:

$$\begin{aligned} h_n(t) &= 2^{n-1} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (-t/4)^k \\ &\quad + (-1)^{n+1} \frac{t^n}{2^{n+1}} \sum_{k=0}^{\infty} [\ln(t/4) - \psi(k+1) - \psi(n+k+1)] \frac{(t/4)^k}{k!\,(n+k)!}, \end{aligned} \tag{2.9}$$

Evidently, $\tilde{h}_n(t)$ coincides with nth term of (2.9) (the first term from the last sum).

Taking into account the limit form $K_\nu(x) \sim \Gamma(\nu)/2 \cdot (x/2)^{-\nu}$ for small x and $\nu > 0$ [1, Sect. 9.6.9], the identity $K_\nu(x) = K_{-\nu}(x)$, and (2.9) for $n = 0$, we obtain the limit form for h_ν near to zero:

$$h_\nu(t) \sim \begin{cases} \dfrac{\Gamma(-\nu)t^\nu}{2^{\nu+1}}, & \nu < 0, \\ \tilde{h}_0(t), & \nu = 0, \\ \Gamma(\nu)2^{\nu-1}, & \nu > 0. \end{cases} \tag{2.10}$$

Comparing (2.10) with (2.7), we conclude that $\tilde{h}_\nu$ coincides with the main singular part of h_ν for $\nu \le 0$.

3. The Family of Functions $h_{\nu,n}$

Now we define the recurrence formula for $h_{\nu,n}$:

$$h_{\nu,0}(t) = \tilde{h}_\nu(t) - h_\nu(t); \qquad h_{\nu,n}(t) = \frac{\tilde{h}_{\nu+n}(t)}{n!\,2^n} - h_{\nu,n-1}(t), \quad n = 1, 2, \ldots \tag{3.1}$$

In other words,

$$h_{\nu,n}(t) = (-1)^{n+1}\Big(h_\nu(t) - \sum_{k=0}^{n}(-1)^k \frac{\tilde{h}_{\nu+k}(t)}{k!\,2^k}\Big). \tag{3.2}$$

We will use in the proof below the following estimates for a completely monotonic function $f \in C^\infty(0,\infty)$ whose derivatives of orders $n = 0, \ldots, m$ are bonded at zero:

$$(-1)^n\Big(f(t) - \sum_{k=0}^{n}\frac{t^k}{k!}f^{(k)}(0)\Big) \le 0 \qquad \text{for} \quad t \ge 0, \ n \in \{0, \ldots, m\}. \tag{3.3}$$

Really, since f is completely monotonic, its derivative is nonpositive on $(0,\infty)$, hence $f(t) \le f(0)$. Further, f'' is nonnegative, hence $f(t) \ge f(0) + tf'(0)$, and so on.

Theorem 2. $h_{\nu,n} \in \mathcal{M}_{(\lfloor\nu\rfloor+n+1)_+}$ *for every* $\nu \in \mathbb{R}$, $n \in \mathbb{Z}_+$.

Proof. From (2.3) and (2.8) it follows that

$$h'_{\nu,n}(t) = -\frac{h_{\nu-1,n}(t)}{2} \qquad \forall\, \nu \in \mathbb{R}, \ n \in \mathbb{Z}_+, \ t \in (0,\infty). \tag{3.4}$$

Given $\nu + n < 0$, we apply (2.2), (2.10), and (2.7) to (3.2) and derive:

$$\begin{aligned} h_{\nu,n}(t) &= (-1)^{n+1}t^\nu\Big(h_{-\nu}(t) - \sum_{k=0}^{n}\frac{(-1)^k t^k}{k!\,2^k}\cdot\frac{\Gamma(-\nu-k)}{2^{\nu+k+1}}\Big) \\ &= (-1)^{n+1}t^\nu\Big(h_{-\nu}(t) - \sum_{k=0}^{n}\frac{t^k}{k!}\cdot\frac{(-1)^k h_{-\nu-k}(0)}{2^k}\Big) \\ &= (-1)^{n+1}t^\nu\Big(h_{-\nu}(t) - \sum_{k=0}^{n}\frac{t^k}{k!}h_{-\nu}^{(k)}(0)\Big). \end{aligned} \tag{3.5}$$

Since $-\nu-n > 0$, the derivatives of $h_{-\nu}$ till order n are bounded at zero. Hence, taking into account (3.3), we conclude from (3.5) that $h_{\nu,n}$ is nonnegative for $\nu + n < 0$. Therefore, from (3.4) we derive $h_{\nu,n} \in \mathcal{M}_0$.

Given $\nu + n \ge 0$, $(-1)^{\lfloor\nu\rfloor+n+1}h_{\nu,n}^{(\lfloor\nu\rfloor+n+1)}(t) = 2^{-\lfloor\nu\rfloor-n-1}h_{\nu_1,n}(t)$, where $\nu_1 = \nu - \lfloor\nu\rfloor - n - 1 < -n$ and, hence, $h_{\nu_1,n} \in \mathcal{M}_0$. Therefore, $h_{\nu,n} \in \mathcal{M}_{\lfloor\nu\rfloor+n+1}$. The theorem is proved. □

Theorem 3. *The limit form for $h_{\nu,n}$ for small t is the following*:

$$h_{\nu,n}(t) \sim \begin{cases} \dfrac{\tilde{h}_{\nu+n+1}(t)}{(n+1)!\,2^{n+1}}, & \nu \le -n-1, \\ O(1), & -n-1 < \nu < 0, \\ o(1), & \nu = 0, \\ (-1)^{n+1}\Gamma(\nu)2^{\nu-1}, & \nu > 0. \end{cases} \tag{3.6}$$

Proof. We will prove the estimates from (3.6) in the order of cases and split the first case into two parts.

1. For $\nu + n < -1$, the function $h_{-\nu}^{(n+1)}(t) = (-1)^{n+1}h_{-\nu-n-1}(t)/2^{n+1}$ is bounded at zero. Hence, applying (3.5), (2.3), and (2.10), we obtain

$$\begin{aligned} h_{\nu,n}(t) &= (-1)^{n+1}t^{\nu}\Big(\frac{t^{n+1}h_{-\nu}^{(k)}(0)}{(n+1)!} + o(t^{n+1})\Big) \\ &= \frac{t^{\nu+n+1}h_{-\nu-n-1}(0)}{(n+1)!\,2^{n+1}} + o(t^{\nu+n+1}) \\ &= \frac{t^{\nu+n+1}\Gamma(-\nu-n-1)2^{-\nu-n-2}}{(n+1)!\,2^{n+1}} + o(t^{\nu+n+1}) \\ &= \frac{\tilde{h}_{\nu+n+1}(t)}{(n+1)!\,2^{n+1}} + o(t^{\nu+n+1}). \end{aligned}$$

2. For $\nu + n = -1$, we derive from (3.5):

$$\begin{aligned} h_{-n-1,n}(t) &= (-1)^{n+1}t^{-n-1}\Big(h_{n+1}(t) - \sum_{k=0}^{n}\frac{t^k}{k!}\cdot\frac{(-1)^k h_{n+1-k}(0)}{2^k}\Big) \\ &= (-1)^{n+1}t^{-n-1}\Big(h_{n+1}(t) - \sum_{k=0}^{n}\frac{t^k}{k!}\cdot\frac{(-1)^k(n-k)!\,2^{n-k}}{2^k}\Big) \\ &= (-1)^{n+1}t^{-n-1}\Big(h_{n+1}(t) - 2^n\sum_{k=0}^{n}\frac{(n-k)!}{k!}(-t/4)^k\Big). \end{aligned} \tag{3.7}$$

Substituting (2.9) for h_{n+1} into (3.7), we obtain

$$\begin{aligned} h_{-n-1,n}(t) &= -\frac{\ln(t/4) - \psi(1) - \psi(n+2)}{(n+1)!\,2^{n+2}} + O(t\ln t) \\ &= \frac{\tilde{h}_0(t) + (\psi(n+2)+\gamma)/2}{(n+1)!\,2^{n+1}} + O(t\ln t). \end{aligned}$$

3. From Cases 1 and 2 and (3.4), we conclude that

$$h_{\nu,n}(t) = \begin{cases} p_{\lfloor\nu\rfloor+n+1}(t) + O(t^{\nu+n+1}), & \nu \in (-1-n, 0)\setminus\mathbb{Z}, \\ p_{\nu+n}(t) + O(t^{\nu+n+1}\ln t), & \nu \in (-1-n, 0)\cap\mathbb{Z}, \end{cases}$$

where $p_k(t) \in \mathbb{P}^1_k$. Therefore, $h_{\nu,n}(t) = O(1)$ for $-1-n < \nu < 0$.

4. For $\nu = 0$, all terms in (3.2) except h_0 and $\tilde{h}_0$ vanish at zero. Therefore, it is enough to prove that $h_{0,0}(t) = \tilde{h}_0(t) - h_0(t) = o(1)$. From (2.9) for $n = 0$, we deduce

$$h_{0,0}(t) = \frac{t}{4}\Big(\frac{1}{2}\ln\frac{t}{4} + \gamma - 1\Big) + O(t^2 \ln t). \tag{3.8}$$

5. For $\nu > 0$, all terms in (3.2) except h_ν vanish at zero. Taking into account the limit value for $h_\nu(0)$ from (2.10) for $\nu > 0$, we receive the required limit value for $h_{\nu,n}(0)$. The theorem is proved. □

4. The Auxiliary Families of Functions g_ν and $\tilde{g}_\nu$

Now we consider the family of functions

$$g_\nu(t) = t^\nu \Gamma(-\nu, t), \qquad \nu \in \mathbb{R}. \tag{4.1}$$

Here $\Gamma(a,t) = \int_t^\infty e^{-x} x^{a-1}\,dx$ is the incomplete Gamma-function [1, Sect. 6.5.3].

Taking into account the differentiation formula [1, Sect. 6.5.26] and positiveness of $\Gamma(a,t)$ for $t > 0$, we conclude that

$$g'_\nu(t) = -g_{\nu-1}(t) \qquad \forall\, \nu \in \mathbb{R},\;\; t \in (0,\infty) \tag{4.2}$$

and $g_\nu \in \mathcal{M}_0$ for every $\nu \in \mathbb{R}$.

For $n \in \mathbb{Z}_+$, we obtain from [1, Sect. 5.1.12]:

$$g_n(t) = (-1)^{n+1}\frac{t^n[\ln t - \psi(n+1)]}{n!} - \sum_{\substack{k=0\\k\neq n}}^{\infty}{}' \frac{(-t)^k}{(k-n)\,k!}. \tag{4.3}$$

From [1, Sect. 6.5.3, 6.5.4], we derive

$$g_\nu(t) = \Gamma(-\nu)t^\nu - \Gamma(-\nu)\gamma^*(-\nu,t), \qquad \nu \in \mathbb{R}\setminus\mathbb{Z}_+, \tag{4.4}$$

where $\gamma^*(a,t)$ is a single valued analytic function of a and t possessing no finite singularities. It has two representation in series:

$$\gamma^*(a,t) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(a+k+1)} = \frac{1}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{(-t)^k}{(a+k)\,k!}. \tag{4.5}$$

In the second representation of γ^*, $(-a) \notin \mathbb{Z}_+$.

Denote

$$\tilde{g}_\nu(t) = \begin{cases} \Gamma(-\nu)t^\nu, & \nu \in \mathbb{R}\setminus\mathbb{Z}_+,\\[4pt] (-1)^{\nu+1}\dfrac{t^\nu[\ln t - \psi(\nu+1)]}{\nu!}, & \nu \in \mathbb{Z}_+. \end{cases} \tag{4.6}$$

It follows from (4.3) and (4.4) that $\tilde{g}_\nu(t) \sim g_\nu(t)$ for $\nu \le 0$ and small $t > 0$. In other words, $\tilde{g}_\nu$ coincides with the main singular part of g_ν for $\nu \le 0$.

One can easy prove the differentiation formula for $\tilde{g}_\nu$:

$$\tilde{g}'_\nu(t) = -\tilde{g}_{\nu-1}(t) \qquad \forall\, \nu \in \mathbb{R},\ \ t \in (0,\infty). \tag{4.7}$$

5. The Family of Functions $g_{\nu,n}$

Finally, we define the family of functions $g_{\nu,n}$, $\nu \in \mathbb{R}$, $n \in \mathbb{Z}_+$:

$$\begin{aligned} g_{\nu,0}(t) &= \tilde{g}_\nu(t) - g_\nu(t); \\ g_{\nu,n}(t) &= g_{\nu+1,n-1}(t) - g_{\nu,n-1}(t), \qquad n = 1, 2, \ldots \end{aligned} \tag{5.1}$$

Theorem 4. $g_{\nu,n} \in \mathcal{M}_{(\lfloor\nu\rfloor+n+1)_+}$ *for every* $\nu \in \mathbb{R}$, $n \in \mathbb{Z}_+$.

Proof. From (4.2) and (4.7) it follows that

$$g'_{\nu,n}(t) = -g_{\nu-1,n}(t) \qquad \forall\, \nu \in \mathbb{R},\ \ n \in \mathbb{Z}_+,\ \ t \in (0,\infty). \tag{5.2}$$

For $n = 0$, from (4.4) and (4.5) we obtain

$$g_{\nu,0}(t) = \Gamma(-\nu)\gamma^*(-\nu,t) = e^{-t}\sum_{k=0}^{\infty} \frac{\Gamma(-\nu)t^k}{\Gamma(-\nu+k+1)}, \qquad \nu \in \mathbb{R}\setminus\mathbb{Z}_+. \tag{5.3}$$

Clearly, $g_{\nu,0}(t) \ge 0$ for $\nu < 0$ and $t > 0$. Taking into account the differentiation formula (5.2), we conclude that $g_{\nu,0} \in \mathcal{M}_0$ for $\nu < 0$. Hence, $g_{\nu,0} \in \mathcal{M}_{\lfloor\nu\rfloor+1}$ for $\nu \ge 0$.

For $n > 0$, the statement of the theorem follows from the formula

$$g_{\nu,n}(t) = e^{-t}\sum_{k=0}^{\infty} \frac{\Gamma(-\nu-n)(k+n)!}{\Gamma(-\nu+k+1)k!}\, t^k, \qquad \nu + n \in \mathbb{R}\setminus\mathbb{Z}_+, \tag{5.4}$$

valid for $\nu + n < 0$. This formula can be proved using induction by n. The theorem is proved. □

Now we will establish limit values for the functions $g_{\nu,n}$ at zero. We start from the case $n = 0$. Using (4.3) for $\nu \in \mathbb{Z}_+$ and the second representation of function γ^* from (4.5) substituted into the identity $g_{\nu,0}(t) = \Gamma(-\nu)\gamma^*(-\nu,t)$, we obtain a general formula for calculation $g_{\nu,0}$ for any ν:

$$g_{\nu,0}(t) = {\sum_{\substack{k=0\\k\ne\nu}}^{\infty}}' \frac{(-t)^k}{(k-\nu)\,k!}, \qquad \nu \in \mathbb{R}. \tag{5.5}$$

Here the term with index $k = \nu$ is excluded from the sum if $\nu \in \mathbb{Z}_+$. For $\nu \notin \mathbb{Z}_+$, summation is done without any exclusion.

From (5.5), we derive for small t:

$$g_{\nu,0}(t) \sim \begin{cases} -\nu^{-1}, & \nu \in \mathbb{R} \setminus \{0\}, \\ -t, & \nu = 0. \end{cases} \tag{5.6}$$

Therefore, using recurrence formula (5.1), we conclude that $g_{\nu,n}$ is bounded at zero for every $\nu \in \mathbb{R}$, $n \in \mathbb{Z}_+$. From (5.2) it also follows that all the derivatives of the function $g_{\nu,n}$ are bounded at zero.

6. Special Cases

In practice, two special cases of calculation formulas for considered functions have an interest — when the value of the parameter ν is an integer or a half-integer. In these cases, the considered functions can be efficiently calculated.

6.1. Calculation Formulas for h_ν and $h_{\nu,n}$

To calculate $h_n(t)$, $n \in \mathbb{Z}_+$, we start from $h_0(t) = K_0(\sqrt{t})$ and $h_1(t) = \sqrt{t}K_1(\sqrt{t})$. For greater n, we use the recurrence formula

$$h_{n+1}(t) = th_{n-1}(t) + 2nh_n(t) \tag{6.1}$$

deduced from (2.6).

Calculation formulas for $h_{n+1/2}(t)$, $n \in \mathbb{Z}_+$, are derived from [2, Sect. 7.2.6]:

$$h_{n+1/2}(t) = \sqrt{\pi/2}\, e^{-\sqrt{t}} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,2^k}\, t^{(n-k)/2}. \tag{6.2}$$

The case of a negative ν is reduced to the case of a positive value of the parameter by applying the identity $h_\nu(t) = t^\nu h_{-\nu}(t)$. For example,

$$h_{-1/2}(t) = t^{-1/2} h_{1/2}(t) = \sqrt{\frac{\pi}{2t}} e^{-\sqrt{t}}.$$

To evaluate the function $h_{\nu,n}(t)$ for $t > 0$, we use (3.2) and do the optimization by taking out of the summation the common part of functions $\tilde{h}_{\nu+k}$. The sum in this case reduces to evaluation of a polynomial. In the case of $\nu+k \in \mathbb{Z}_+$, the constants added to $\ln(t/4)$ could be omitted.

The function $h_{0,0}(t) = -0.5\ln(t/4) - \gamma - K_0(\sqrt{t})$ should be evaluated in a special way when $t < \epsilon$, where ϵ is the relative accuracy of calculation on computer. We use in this case the formula

$$h_{0,0}(t) \approx \frac{t}{4}\Big(\frac{1}{2}\ln\frac{t}{4} + \gamma - 1\Big)$$

derived from (3.8).

The function $h_{-1/2,0}(t) = \sqrt{\pi/(2t)}\bigl(1 - e^{-\sqrt{t}}\bigr)$ is also evaluated in a special way for $t < \epsilon$:

$$h_{-1/2,0}(t) \approx \sqrt{\frac{\pi}{2}}\Bigl(1 - \frac{\sqrt{t}}{2}\Bigr).$$

6.2. Calculation Formulas for g_ν and $g_{\nu,n}$

The recurrence formula

$$e^{-t} - tg_{\nu-1}(t) = \nu g_\nu(t), \qquad \nu \in \mathbb{R}, \tag{6.3}$$

is easy established from (4.1) by applying integration by parts. Hence, functions g_ν for integer and half-integer values of ν can be easy calculated if we could evaluate g_{-1}, $g_{-1/2}$, and g_0. For $\nu \in \mathbb{Z}$, we use formulas [1, Sect. 5.1.45, 5.1.46] and derive to well-known exponential integral functions:

$$g_n(t) = \begin{cases} \alpha_{-n-1}(t), & n < -1, \\ E_{n+1}(t), & n > -1, \end{cases} \qquad g_{-1}(t) = \alpha_0(t) = E_0(t) = \frac{e^{-t}}{t}. \tag{6.4}$$

For $\nu = -1/2$, we deduce from [1, Sect. 6.5.17, 7.1.5]:

$$g_{-1/2}(t) = \sqrt{\pi}\,\frac{1 - \operatorname{erf}\sqrt{t}}{\sqrt{t}}. \tag{6.5}$$

From (6.3) and (5.1) we obtain the recurrence formula

$$e^{-t} + tg_{\nu-1,0}(t) = \begin{cases} -\nu g_{\nu,0}(t), & \nu \in \mathbb{R} \setminus \mathbb{Z}_+, \\ 1, & \nu = 0, \\ -\nu g_{\nu,0}(t) + \dfrac{(-t)^\nu}{\nu!}, & \nu \in \mathbb{N}, \end{cases} \tag{6.6}$$

which allows us to evaluate $g_{\nu,0}(t)$ in the case of an integer or a half-integer value of ν by starting from $g_{-1/2,0}(t)$ or $g_{0,0}(t)$. From (6.4), (6.5), and (4.6), we deduce

$$g_{-1/2,0}(t) = \sqrt{\pi}\,\frac{\operatorname{erf}\sqrt{t}}{\sqrt{t}}, \qquad g_{0,0}(t) = -\ln t - \gamma - E_1(t).$$

For $t < \epsilon$, we use the limit forms from (5.6) and assign

$$g_{-1/2,0}(t) = 2, \qquad g_{0,0}(t) = -t.$$

Functions $g_{\nu,n}(t)$, $n > 0$, are easy evaluated by the recurrence formula (5.1).

6.3. Comparison with Radial Rasis Functions Introduced by Mitáš and Mitášová

In Table 1, we show examples of radial basis functions generated with particular functions from families h_ν, $h_{\nu,n}$, g_ν, and $g_{\nu,n}$. The coefficient c is given in the cases when a generic function is unbounded at zero.

Table 1. Examples of radial basis functions

Ex.	Generic function	Radial basis function	Order
1	$h_{-1/2}(t) = \sqrt{\pi/(2t)}\, e^{-\sqrt{t}}$	$\sqrt{\pi/2}\,(r^2+c^2)^{-1/2} e^{-(r^2+c^2)^{1/2}}$	0
2	$h_0(t) = K_0(\sqrt{t})$	$K_0((r^2+c^2)^{1/2})$	0
3	$h_{1/2}(t) = \sqrt{\pi/2}\, e^{-\sqrt{t}}$	$\sqrt{\pi/2}\, e^{-r}$	0
4	$h_{3/2}(t) = \sqrt{\pi/2}\, e^{-\sqrt{t}}(\sqrt{t}+1)$	$\sqrt{\pi/2}\, e^{-r}(r+1)$	0
5	$h_{-1/2,0}(t) = \sqrt{\pi/(2t)}\,(1-e^{-\sqrt{t}})$	$\sqrt{\pi/2}\,\frac{1}{\varphi r}(1-e^{-\varphi r})$	0
6	$h_{0,0}(t) = -\frac{1}{2}\ln\frac{t}{4} - \gamma - K_0(\sqrt{t})$	$-\ln\frac{\varphi r}{2} - \gamma - K_0(\varphi r)$	1
7	$h_{1/2,0}(t) = -\sqrt{\pi/2}\,(\sqrt{t}+e^{-\sqrt{t}})$	$-\sqrt{\pi/2}\,(\varphi r + e^{-\varphi r})$	1
8	$h_{-1/2,1}(t) = \sqrt{\pi/(2t)}\,(-t/2-1+e^{-\sqrt{t}})$	$\sqrt{\pi/2}\,\frac{\tau}{r} \quad -\frac{r^2}{2\tau^2} - 1 + e^{-r/\tau}$	1
9	$h_{0,1}(t) = \frac{t}{8} \quad \ln\frac{t}{4} + 2\gamma - 1 \quad +$ $\frac{1}{2}\ln\frac{t}{4} + \gamma + K_0(\sqrt{t})$	$\frac{r^2}{4\tau^2} \quad \ln\frac{r}{2\tau} + \gamma - \frac{1}{2} \quad +$ $\ln\frac{r}{2\tau} + \gamma + K_0 \quad \frac{r}{\tau}$	2
10	$h_{1/2,1}(t) = \sqrt{\pi/2} \quad \frac{\sqrt{t^3}}{6} + \sqrt{t} + e^{-\sqrt{t}}$	$\sqrt{\pi/2} \quad \frac{r^3}{6\tau^3} + \frac{r}{\tau} + e^{-r/\tau}$	2
11	$g_{-1}(t) = \frac{e^{-t}}{t}$	$\frac{e^{-(r^2+c^2)}}{r^2+c^2}$	0
12	$g_{-1/2}(t) = \sqrt{\pi}\frac{1-\operatorname{erf}\sqrt{t}}{\sqrt{t}}$	$\sqrt{\pi}\frac{1-\operatorname{erf}\sqrt{r^2+c^2}}{\sqrt{r^2+c^2}}$	0
13	$g_0(t) = E_1(t)$	$E_1(r^2+c^2)$	0
14	$g_{1/2}(t) = 2 \quad e^{-t} - \sqrt{\pi t} \quad 1 - \operatorname{erf}\sqrt{t}$	$2 \quad e^{-r^2} - \sqrt{\pi}r \quad 1 - \operatorname{erf} r$	0
15	$g_{-1/2,0}(t) = \sqrt{\pi}\,\frac{\operatorname{erf}\sqrt{t}}{\sqrt{t}}$	$\sqrt{\pi}\,\frac{\operatorname{erf}(\varphi r/2)}{\varphi r/2}$	0
16	$g_{0,0}(t) = -\ln t - \gamma - E_1(t)$	$-\ln \quad (\varphi r/2)^2 \quad - \gamma - E_1 \quad (\varphi r/2)^2$	1
17	$g_{1/2,0}(t) = -2 \quad \sqrt{\pi t}\operatorname{erf}\sqrt{t} + e^{-t}$	$-2 \quad \sqrt{\pi}\frac{\varphi r}{2}\operatorname{erf}(\varphi r/2) + e^{-(\varphi r/2)^2}$	1

We use the substitutions $t \mapsto (\varphi r)^2$ in Examples 5–7, $t \mapsto (r/\tau)^2$ in Examples 8–10, and $t \mapsto (\varphi r/2)^2$ in Examples 15–17 with some positive values of φ and τ.

Comparing radial basis functions from Examples 6–9 with [10] and taking into account the substitution (1.3), we conclude that $h_{1-d/2,0}$ (Examples 6, 7) generates the radial basis function of the tension spline on $\mathbb{R}^d$, $d = 1, 2$, and $h_{1-d/2,1}$ (Examples 8, 9) generates the radial basis function of the regularized spline on $\mathbb{R}^d$, $d = 2, 3$. Let us note a small difference between radial basis functions from Examples 8, 9 and the corresponding formulas from [10]:

- The order of conditional positive definiteness of the function in Example 8 is equal to 1, but in [10] a trend space consisting of linear functions is required, i.e., the order is equal to 2.
- There is an errata in formula (55) of the radial basis function of the regularized spline in $\mathbb{R}^2$ — π in the argument of ln should be replaced with τ.

Comparing radial basis functions from Examples 15, 16 with [11], we also conclude that $g_{1-d/2,0}$ generates the radial basis function of the completely regularized spline on $\mathbb{R}^d$, $d = 2, 3$.

7. Discussion

Now we return back to the interpolation by RBF-spline constructed with the help of a radial basis function ϕ, which is conditionally positive definite with respect to a trend space $\mathcal{P}$. It is well-known [7], that the interpolating RBF-spline minimizes in this case a specific seminorm $\Phi(\cdot)$ in the proper Hilbert space $\mathcal{N}_\Phi(\Omega)$ of functions on $\Omega \subset \mathbb{R}^d$, where Ω is a domain containing mesh nodes. This space of functions is called the *native space* and it is uniquely identified by the triple $\langle \phi, \mathcal{P}, \Omega \rangle$. The seminorm can be presented in the form $\Phi(u) = \|Tu\|_Y$, where T is a bounded linear operator acting from $\mathcal{N}_\Phi(\Omega)$ to a Hilbert space Y, and the function ϕ generates the reproducing kernel $G(x, y) = \phi(|x - y|)$ of the space $\mathcal{N}_\Phi(\Omega)$ equipped with the seminorm $\Phi(\cdot)$ [3]. The reproducing kernel G is also the Green's function of the operator T^*T (see, e.g., [6]).

In the case of $\mathcal{P} = \mathbb{P}^d_{m-1}$, $\Omega = \mathbb{R}^d$, and $\phi \in \mathcal{R}^d_m$, the native space and minimized seminorm can be associated with the triple $\langle \phi, m, d \rangle$. In Table 2 we show particular cases of seminorms minimized by RBF-splines. We use here the notation

$$\|D^k u\|^2_Y := \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|D^\alpha u\|^2_Y, \qquad k \in \mathbb{Z}_+,$$

where $D^\alpha u$ is the partial derivative of u of the multiindex α. See [5] for the definition of the function space $\tilde{H}^s(\mathbb{R}^d)$.

Table 2. Examples of variational functionals minimized by RBF-splines

Ex.	$\Phi(u)^2$	$\phi(r)$	Restrictions
1	$\|D^m u\|_{L_2}^2$	$f_{m-d/2}(r^2)$	$m - d/2 > 0$
2	$\|D^m u\|_{\tilde{H}^s}^2$	$f_{m+s-d/2}(r^2)$	$m > m + s - d/2 > 0$
3	$\|u\|_{L_2}^2 + \|D^m u\|_{L_2}^2$	$h_{m-d/2}(r^2)$	$m - d/2 > 0$
4	$\varphi^2\|D^1 u\|_{L_2}^2 + \|D^2 u\|_{L_2}^2$	$h_{1-d/2,0}((\varphi r)^2)$	$2 - d/2 > 0$
5	$\|D^2 u\|_{L_2}^2 + \tau^2\|D^3 u\|_{L_2}^2$	$h_{1-d/2,1}((r/\tau)^2)$	$3 - d/2 > 0$
6	$\sum_{k=1}^{\infty} \frac{\|D^k u\|_{L_2}^2}{\varphi^{2k}(k-1)!}$	$g_{1-d/2,0}((\varphi r/2)^2)$	—

Radial basis functions of the tension and regularized splines can be also obtained with the approach described in [4]. After appropriate scaling, the functional from Example 4 can be written as

$$\|D^1 u\|_{L_2}^2 + \|D^2 u\|_{L_2}^2 = (Lu, u)_{L_2}, \quad \text{where} \quad L = (-\Delta)(-\Delta + I).$$

Following [4], we conclude that the radial basis function of the tension spline is the difference of the radial basis function $\tilde{h}_{1-d/2}(r^2)$ corresponding to the operator $(-\Delta)$ and the radial basis function $h_{1-d/2}(r^2)$ corresponding to the operator $(-\Delta + I)$. As a result, we obtain $h_{1-d/2,0}(r^2)$.

The radial basis function of the regularized spline is obtained as a linear combination of 3 functions, because the respective differential operator has the form $L = (-\Delta)(-\Delta)(-\Delta + I)$. Therefore, the linear combination will include 2 functions $\tilde{h}_\nu(r^2)$ and one function $h_\nu(r^2)$ with appropriate values of ν, and we obtain $h_{1-d/2,1}(r^2)$.

Taking into account these considerations, we can conclude that $h_{1-d/2,k}(r^2)$ corresponds to minimization of the functional $\|D^{k+1}u\|_{L_2}^2 + \|D^{k+2}u\|_{L_2}^2$.

Finally, we will give some recommendations on selection an appropriate radial basis function from the wide range of possible variants. The first question we should answer is, what is the order of smoothness of functions from the native space corresponding to $\langle \phi, m, d \rangle$? This order is strongly connected with the order of smoothness of the function ϕ at zero or, in terms of its generic function f, with a value of n such that $|f^{(k)}(0)| < \infty$, $k = 0, \ldots, n$.

We introduce the Tailor series residual operator

$$\mathcal{T}_\nu f(t) = f(t) - \sum_{k \in \mathbb{Z}_+,\ k < \nu} \frac{f^{(k)}(0)}{k!}\, t^k.$$

Note that $\mathcal{T}_\nu f = f$ for $\nu \leq 0$. Note also that $\mathcal{T}_\nu \tilde{h}_\nu = \tilde{h}_\nu$.

From (2.10) for $\nu \le 0$ and the similarity of the differentiation rules (2.3) and (2.8), we conclude that $\mathcal{T}_\nu h_\nu(t) \sim \tilde{h}_\nu(t)$ for small t. Therefore, a h_ν-spline constructed with $h_\nu(r^2)$ behaves in a small neighborhood of interpolation nodes as $\tilde{h}_\nu$-spline and the orders of convergence of these two splines in the domain of nodes condensation are the same. Similarly, from (3.6) for $\nu \le -n-1$ and (2.8), (3.4), we conclude that $\mathcal{T}_{\nu+n+1} h_{\nu,n}(t) \sim \tilde{h}_{\nu+n+1}(t)/((n+1)!\,2^{n+1})$ for small t. Therefore, the orders of convergence of $h_{\nu,n}$-spline and $\tilde{h}_{\nu+n+1}$-spline are the same. Applying operator $\mathcal{T}_\nu$ to g_ν, we also conclude that g_ν-spline has the similar behavior in a small neighborhood of interpolation nodes as the $\tilde{g}_\nu$-spline.

Taking into account the similarity of functions f_ν, $\tilde{h}_\nu$, and $\tilde{g}_\nu$, we obtain that the functions $h_\nu(r^2)$, $g_\nu(r^2)$, $f_\nu(r^2)$, and $h_{\nu-n-1,n}(r^2)$ produce splines with similar convergence properties. But the difference between these splines is essential outside of a neighborhood of interpolation nodes.

The main difference consists in the requirements on the degree of polynomials to be involved in the trend space $\mathcal{P}$. We say that $\mathcal{P}$ is the *polynomial trend of order m* if $\mathcal{P} = \mathbb{P}^d_{m-1}$. In the case of h_ν- and g_ν-splines, the minimal order of the polynomial trend is zero. For f_ν-spline it is $(\lfloor \nu \rfloor + 1)_+$ and for $h_{\nu-n-1,n}$-spline it is $(\lfloor \nu \rfloor)_+$. Although h_ν- and g_ν-splines don't require any trend, we can use a trend space with h_ν- and g_ν-splines. This allows us to construct hybrid interpolations, for example, by combining a high-order radial basis function $h_\nu(r^2)$ with a low-degree polynomial trend, say with linear functions. Of course, in h_ν- and g_ν-spline approximation, the scaling of radial basis functions is very important, contrary to f_ν-spline approximation where scaling has no affect on the interpolating spline constructed.

Both functions $h_\nu(r^2)$ and $g_\nu(r^2)$ decay exponentially fast as $r \to \infty$: $h_\nu(r^2) \approx r^{\nu-1/2}e^{-r}$ and $g_\nu(r^2) \approx e^{-r^2}/r^2$. Therefore, they can be used in the *adaptive greedy algorithm* (see, e.g., [15]) as an alternative to finitely supported radial basis functions.

Interpolation with the radial basis function $h_{\nu-n-1,n}((ar)^2)$ produces intermediate constructions between f_ν- and $f_{\nu-1}$-splines. As $a \to 0$, the $h_{\nu-n-1,n}$-spline tends to f_ν-spline, and, as $a \to \infty$, the $h_{\nu-n-1,n}$-spline tends to $f_{\nu-1}$-spline. Concerning to $g_{\nu,n}$-splines, we only can say that they should behave similarly to interpolation with infinitely-differentiable radial basis functions such as gaussian, multiquadric, or inverse multiquadric, or any other radial basis function generated by the formula $\phi(r) = f(r^2 + c^2)$ with nonzero value of c.

In conclusion, we would like to mention one essential possibility of RBF-spline approximation which remains out of scope of the theoretical study, but it is important in practice. Using RBF-spline approximation, we can involve additional information on approximated function such as first-order breaks or areas with big gradients. If we know that somewhere in the domain the approximated function has a peculiarity and this peculiarity could be described with a given function, we can subtract this function values from approximated data, then approximate the rest of data with RBF-spline and after that add the peculiarity function back to the result. But this method works if we exactly know the peculiarity function.

In the case when the peculiarity can be described as a linear combination of functions with unknown coefficients, we can directly add these functions into the basis of the trend space $\mathcal{P}$ and solve the spline-approximation problem with the extended trend. Of course, to obtain the close system of linear equations, we should also add corresponding equations to (1.2). We call a spline constructed with extended trend as *spline with external drift* by analogy with kriging with external drift used in geostatistics.

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