CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2013
(K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 235-251
Prof. Marin Drinov Academic Publishing House, Sofia, 2014

Approximation by Band-limited Scaling and Wavelet Expansions*

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The well-known sampling theorem (which is often called Kotel'nikov's formula) is very useful for engineers. Mathematicians are also interested in study of this formula and its generalizations in different aspects. From the point of view of wavelet theory, the sampling theorem is just an illustration for the Shannon MRA. Indeed, the function $\varphi(x) = \frac{\sin \pi x}{\pi x}$ is a scaling function for this MRA, and a function f from the sampling space V_j can be expanded as $f = \sum_{n\in\mathbb{Z}}\langle f, \varphi_{jn}\rangle\varphi_{jn}$, where $\varphi_{jn}(x) = 2^{j/2}\varphi(2^j x+n)$, which coincides with the Kotel'nikov's formula. We study operators $Q_j f = \sum_{n\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{jn}\rangle\varphi_{jn}$ and their approximation properties for some classes of band-limited functions φ and a wide class of tempered distributions $\widetilde{\varphi}$. In particular, for a class of differential operators L, we consider $\widetilde{\varphi}$ such that $Q_j f = \sum_{n\in\mathbb{Z}} Lf(2^{-j}\cdot)(-n)\varphi_{jn}$. The corresponding wavelet frame-type expansions are found. Replacing $\langle f, \widetilde{\varphi}_{jk}\rangle$ by $\langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle$, we extend these results for an essentially larger class of functions f.

Keywords and Phrases: sampling theorem, band-limited functions, tempered distributions, scaling expansions, wavelets, approximation order, modulus of continuity.

Mathematics Subject Classification 2010: 42C40, 41A25, 42A16.

1. Introduction

The well-known sampling theorem (Kotel'nikov's or Shannon's formula) states that

$$f(x) = \sum_{n\in\mathbb{Z}} f(2^{-j}n)\,\frac{\sin \pi(2^j x - n)}{\pi(2^j x - n)} \tag{1}$$

*Research supported by Grant 12-01-00216 of RFBR and state-financed project 9.38.62.2012 of SPbGU.

for any function $f \in L_2(\mathbb{R})$ whose Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. This formula is very useful for engineers. It was just Kotel'nikov [23] and Shannon [33] who started to apply this formula for signal processing, respectively in 1933 and 1949. Up to now, an overwhelming diversity of digital signal processing applications and devices are based on it and more than successfully use it. However, mathematicians knew this formula much earlier, actually, it can be found in the papers by Ogura [26] (1920), Whittaker [39] (1915), Borel [6] (1897), and even Cauchy [13] (1840).

Nowadays (1) is also an important and interesting formula for mathematicians. Butzer with co-authors recently published several papers [9, 10, 11], where they analyze sampling theorem, its applications and development. In particular, the equivalence of sampling theorem to some other classical formulas was established for some classes of band-limited functions. Also in [5, 12] they studied a generalization of sampling decomposition replacing the sinc-function $\operatorname{sinc}(x) := \frac{\sin \pi x}{\pi x}$ by certain linear combinations of B-splines. Another generalization of (1) was suggested by Trynin [35, 36].

Equality (1) holds only for functions $f \in L_2(\mathbb{R})$ whose Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. However the right hand side of (1) (the sampling expansion of f) has meaning for every continuous f with a good enough decay. The problem of approximation of f by its sampling expansions as $j \to +\infty$ was studied by many mathematicians. We mention only some of such results. Brown [7] proved that for every $x \in \mathbb{R}$

$$\Big| f(x) - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \operatorname{sinc}(2^j x + n) \Big| \le C \int\limits_{|\xi| > 2^{j-1}} |\widehat{f}(\xi)| d\xi, \qquad (2)$$

whenever the Fourier transform of f is summable on $\mathbb{R}$. It is known that the pointwise approximation by sampling expansions does not hold for arbitrary continuous f, even compactly supported. Moreover, Trynin [37] proved that there exists a continuous function vanishing outside of $(0, \pi)$ such that its deviation from the sampling expansion diverges at every point $x \in (0, \pi)$. Approximation by sampling expansions in L_p-norm was also actively studied. The best results were obtained by Bardaro, Butzer, Higgins, Stens and Vinti in [4] and [8]. They proved that

$$\Delta_p := \Big\| f - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \operatorname{sinc}(2^j \cdot + k) \Big\|_p \le C \tau_r(f, 2^{-j})_p,$$

where $\tau_r(f, \delta)_p$ is an $\mathbb{R}$-analog of the averaged modulus of smoothness introduced by Sendov and Popov [30] for finite intervals. Unfortunately, their $\tau_r(f, \delta)_p$ is not as good as the modulus introduced by Sendov and Popov, because it may be infinite for certain L_p-functions. However, they proved that $\tau_r(f, \delta)_p \underset{\delta \to 0}{\longrightarrow} 0$ if $f \in C \cap \Lambda_p$, where Λ_p consists of functions f such that

$$\sum_k |f(x_k)|^p (x_k - x_{k-1}) < \infty$$

for some class of admissible partition $\{x_k\}_k$ of $\mathbb{R}$. Also they proved that the Sobolev spaces W_p^n, $n \in \mathbb{N}$, are subspaces of Λ_p, and that for every $f \in W_p^n$

$$\Delta_p \le \frac{C\omega(f^{(n)}, 2^{-j})_p}{2^{-jn}}, \tag{3}$$

where $\omega(\ )_p$ is the modulus of continuity in L_p. Approximation by linear summation methods of sampling expansion was studied by Butzer, Schmeisser and Stens in [12] and [34], Kivinukk and Tamberg [21, 22].

From the point of view of wavelet theory, (1) is not a theorem, it is just an illustration for the Shannon MRA. Indeed, the function $\varphi(x) = \mathrm{sinc}(x)$ is a scaling function for this MRA, and a function f belongs to the sample space V_j if and only if its Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. So, such a function f can be expanded as $f = \sum_{n\in\mathbb{Z}} \langle f, \varphi_{jn}\rangle \varphi_{jn}$, which coincides with (1). Also, since $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA, any $f \in L_2(\mathbb{R})$ can be represented as

$$f = \lim_{j\to+\infty} \sum_{n\in\mathbb{Z}} \langle f, \varphi_{jn}\rangle \varphi_{jn}. \tag{4}$$

Moreover, (4) has an arbitrary large approximation order. This happens because the function $\varphi(x) = \frac{\sin \pi x}{\pi x}$ is band-limited, a similar property cannot be valid for other natural classes of φ, in particular, for compactly supported φ. Some generalizations of this fact are proved in the present paper.

We consider band-limited functions φ with continuous or discontinuous Fourier transform and study the corresponding scaling expansions (or quasi-projection operators) $\sum_{n\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jn}\rangle \varphi_{jn}$, where $\widetilde{\varphi}$ is, generally speaking, a tempered distribution. In particular, $\widetilde{\varphi}$ may be from the same class of band-limited functions.

The operators $Q_j f = \sum_{n\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jn}\rangle \varphi_{jn}$ appear very often in the papers concerned with wavelets. Probably one of the first appearing was in the well-known paper [14] by Cohen, Daubechies and Feaueav, where a method for the construction of biorthogonal wavelet bases was developed. In this case the functions φ, $\widetilde{\varphi}$ are in L_2, refinable, and their integer translations are biorthogonal. A method for the construction dual wavelet frames was developed in [28, 29] by Ron and Shen, where these operators also play an important role. In this case the integer translations of φ, $\widetilde{\varphi}$ should not be biorthogonal, but the functions are still in L_2 and refinable. Convergence and approximation properties of Q_j, with compactly supported $\varphi, \widetilde{\varphi}$ were actively studied by many authors (see [1, 2, 3, 18, 24, 19, 20, 27] and the references therein). Polynomial reproducibility plays a vital role in these results. The most general results for L_p-convergence were obtained by Jia in [20] who proved that

$$\|f - Q_j f\|_p \le C\omega_k(f, 2^{-j})_p \qquad \forall f \in L_p(\mathbb{R}),$$

under the assumptions: φ and $\widetilde{\varphi}$ are compactly supported, $\varphi \in L_p$, $\widetilde{\varphi} \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and Q_0 reproduces polynomials of degree $k-1$. The method based on the polynomial reproducibility is not appropriate for slowly decaying functions, such as functions φ with discontinuous Fourier transform. Another approach was employed by Jetter and Zhou [16, 17], and developed in [27], where Fourier transform technique was applied. Our results are obtained by using the latter method which allows to work with a wide class of band-limited functions φ and with a wide class of tempered distributions $\widetilde{\varphi}$. For appropriate band-limited or compactly supported functions $\widetilde{\varphi}$, the estimate $\|f - Q_j f\|_p \le C\omega_r(f, 2^{-j})_{L_p}$, where ω_r denotes the r-th modulus of continuity, is obtained for arbitrary $r \in \mathbb{N}$. For tempered distributions $\widetilde{\varphi}$, we prove that $Q_j f$ tends to f, $f \in S$, in L_p-norm, $p \ge 2$, with an arbitrary large approximation order. In particular, for the differential operators $Lf := \sum_{l=0}^{m} \alpha_l f^{(l)}$, we consider $\widetilde{\varphi}$ such that $\{\langle f, \widetilde{\varphi}_{jk}\rangle\}_{k\in\mathbb{Z}}$ interpolates $2^{-j/2}Lf(2^{-j}\cdot)$ at the integer points. Replacing $\langle f, \widetilde{\varphi}_{jk}\rangle$ by $\langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle$ we extend these results for an essentially larger class of functions f.

The differential expansions $\sum_{k\in\mathbb{Z}} Lf(2^{-j}\cdot)(-k)\varphi_{jk}$ generalize the sampling expansions. Note that study of their approximation properties can be useful for engineering applications. Indeed, engineers do not deal with functions, they only have some discrete information about the function. If values of the function at equidistributed nodes are known, then sampling expansion is very good for the recovering the function. However, sometimes the values are known approximately. Assume that some device gives the average value of a function f on the interval $[2^{-j}k, 2^{-j}(k+h)]$ instead of $f(2^{-j}k)$, i.e., one knows the values

$$\frac{1}{2^{-j}h}\int\limits_{2^{-j}k}^{2^{-j}(k+h)} f(t)\,dt \approx \frac{1}{2^{-j}h}\int\limits_{0}^{2^{-j}h}\sum_{l=0}^{m}\frac{f^{(l)}(2^{-j}k)t^l}{l!}\,dt = \sum_{l=0}^{m}\frac{h^l}{(l+1)!}\frac{d^l f(2^{-j}\cdot)}{dx^l}(k).$$

But the latter sum is nothing as $Lf(2^{-j}\cdot)(k)$ with $\alpha_l = \frac{1}{(l+1)!}h^l$.

Finally, we discuss wavelet frame-type decompositions corresponding to the sampling and differential scaling expansions.

2. Notations

Let $\mathbb{N}$, $\mathbb{Z}_+$ $\mathbb{Z}$, $\mathbb{R}$, be the sets of positive integers, non-negative integers, integers and real numbers, respectively.

The Schwartz class of functions defined on $\mathbb{R}$ is denoted by S, and S' is the dual space of S, i.e. the space of tempered distributions. We will use the basic notion and facts from distribution theory which can be found, e.g., in [15] or [38]. If $f \in S$, $g \in S'$, then $\overline{\langle f, g\rangle} := \langle g, f\rangle := g(f)$. If $f \in S'$, then $\widehat{f}$ denotes its Fourier transform defined by $\langle \widehat{f}, \widehat{g}\rangle = \langle f, g\rangle$, $g \in S$.

If $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$, then $\langle f, g\rangle := \int_{\mathbb{R}} f\,\overline{g}$. If f is a function defined on $\mathbb{R}$, we set

$$f_{jk}(x) := 2^{j/2} f(2^j x + k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{R}.$$

If $f \in S'$, $j \in \mathbb{Z}$, $k \in \mathbb{R}$, we define f_{jk} by

$$\langle f_{jk}, g\rangle = \langle f, g_{-j,-2^{-j}k}\rangle \qquad \forall g \in S.$$

For convenience, sometimes we will write $2^{j/2}f(2^j x+k)$ instead of f_{jk} even for $f \in S'$.

If $\langle f, \widetilde{\varphi}_{jk}\rangle$ has meaning and the series $\sum_{k\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{jk}\rangle\varphi_{jk}$ converges in some sense, we set

$$Q_j(\varphi, \widetilde{\varphi}, f) = Q_j(f) := \sum_{k\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{jk}\rangle\varphi_{jk}.$$

We use W_p^n, $1 \le p < \infty$, $n \in \mathbb{N}$, to denote the Sobolev space on $\mathbb{R}$, i.e. the set of functions which derivatives up to order n are in $L_p(\mathbb{R})$, with usual Sobolev norm.

We use ∇_t to denote the difference operator given by $\nabla_t f = f(\cdot) - f(\cdot - t)$. The n-th modulus of continuity of a function f in $L_p(\mathbb{R})$ is defined by

$$\omega_n(f,h)_{L_p} = \sup_{|t|\le h}\|\nabla_t^n f\|_p, \qquad h \ge 0.$$

If F is a 1-periodic function and $F \in L(0,1)$, then $\widehat{F}(k) = \int_0^1 F(x)e^{-2\pi ikx}\,dx$ is its k-th Fourier coefficient.

3. Scaling Approximation in L_2

In this section we study the operators $Q_j(\varphi, \widetilde{\varphi}, f)$ in $L_2(\mathbb{R})$ for wide classes of tempered distributions $\widetilde{\varphi}$ and band-limited functions φ.

Let $\varphi \in S'$, its Fourier transform $\widehat{\varphi}$ be defined on $\mathbb{R}$ and n times differentiable on $\mathbb{Z}$. One says that the Strang-Fix condition of order n holds for φ if

$$\frac{d^k\widehat{\varphi}}{d\xi^k}(l) = 0, \qquad k = 0, \dots, n,$$

for all $l \in \mathbb{Z}$, $l \ne 0$.

Theorem 1 ([32, Theorem 2]). *Let $f \in S$, $\varphi \in L_2(\mathbb{R})$, $\widehat{\varphi}$ be essentially bounded and compactly supported, $\widetilde{\varphi}$ be a tempered distribution which Fourier transform $\widehat{\widetilde{\varphi}}$ is a function on $\mathbb{R}$ essentially bounded on $[-\frac{1}{2}, \frac{1}{2}]$ and such that $\big|\widehat{\widetilde{\varphi}}(\xi)\big| \le C_{\widetilde{\varphi}}|\xi|^{N_0}$ for almost all $\xi \notin [-\frac{1}{2}, \frac{1}{2}]$, $N_0 \ge 0$. Then:*

(a) *for every $j \in \mathbb{Z}$, the series $\sum_{k\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{jk}\rangle\varphi_{jk}$ converges unconditionally in $L_2(\mathbb{R})$, in particular, $Q_j(\varphi, \widetilde{\varphi}, f) \in L_2(\mathbb{R})$;*

(b) *if $\widehat{\varphi}$ is continuous on $\mathbb{Z}$, $\widehat{\widetilde{\varphi}}$ is continuous at zero and $\overline{\widehat{\varphi}(0)}\widehat{\widetilde{\varphi}}(0) = 1$, then the Strang-Fix condition of order 0 for φ is necessary and sufficient for the convergence of $Q_j(\varphi, \widetilde{\varphi}, f)$ to f in L_2-norm as $j \to +\infty$;*

(c) *if there exist $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$ such that $\overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}}$ is n-times boundedly differentiable on $[-\delta, \delta]$, $\widehat{\varphi}$ is n-times boundedly differentiable on $[l-\delta, l+\delta]$ for all $l \in \mathbb{Z} \setminus \{0\}$, $(1 - \overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}})^{(k)}(0) = 0$, $k = 0, \dots, n-1$, and the Strang-Fix condition of order $n-1$ holds for φ, then for every $j > 0$*

$$\|f - Q_j(\varphi, \widetilde{\varphi}, f)\|_2 \le C 2^{-jn} \|f\|_{W_2^{n^*}},$$

where $n^* \ge \max\{n, N_0 + \frac{1}{2} + \epsilon\}$, $\epsilon > 0$, C does not depend on f and j;

(d) *if there exists $\delta \in (0, \frac{1}{2})$ such that $\overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}} = 1$ a.e. on $[-\delta, \delta]$, $\widehat{\varphi} = 0$ a.e. on $[l-\delta, l+\delta]$ for all $l \in \mathbb{Z} \setminus \{0\}$, then $f = Q_j(\varphi, \widetilde{\varphi}, f)$ whenever* $\operatorname{supp} \widehat{f} \subset$ *$[-2^j\delta, 2^j\delta]$.*

Note that any compactly supported $\widetilde{\varphi} \in S'$ satisfies the conditions of Theorem 1 because of the Paley-Wiener theorem for tempered distributions. Also any function $\widetilde{\varphi}$ in $L_2(\mathbb{R})$ with essentially bounded Fourier transform and any $\widetilde{\varphi}$ in $L(\mathbb{R})$ satisfies the conditions of Theorem 1.

Item (c) in Theorem 1 looks very similar to the main result of Jetter and Zhou [17] with only the following difference: the deviation of $f \in S$ from Q_j is estimated via W_2^n-norm of f in [17] (instead of $W_2^{n^*}$-norm in our theorem). However this is possible only under the additional assumptions, in particular, they assume some restriction on the order of growth of $\widehat{\widetilde{\varphi}}$ at infinity, which decreases the class of distributions $\widetilde{\varphi}$ for given φ. For example, if $\widetilde{\varphi} = \delta + a\delta'$, $a \ne 0$, and assumptions of (c) are satisfied with $n = 1$, then the results in [17] do not give an answer about approximation order and even about convergence, while it follows from (c) that convergence holds with approximation order 1 in this case. Also, in some situations, it is not easy to check if $\widehat{\widetilde{\varphi}}$ has a required order of growth. In particular, if one needs $\widetilde{\varphi}$ to be refinable, the standard way for the construction is to start with a mask $\widetilde{m}_0$ and to define $\widehat{\widetilde{\varphi}}$ as the infinite product $\prod_{j=1}^{\infty} \widetilde{m}_0(2^{-j}\xi)$ which is a distribution, generally speaking. It is not clear how to find the order of growth of such $\widehat{\widetilde{\varphi}}$.

For compactly supported $\varphi, \widetilde{\varphi}$, item (b) was also proved in [27], where the multidimensional case with matrix dilation was considered.

Under the assumptions of Theorem 1, the value $\langle f, \widetilde{\varphi}_{jn} \rangle$ has meaning only for functions f from S. If $\widetilde{\varphi} \in L_2(\mathbb{R})$ and $\sum_{k\in\mathbb{Z}} |\widehat{\widetilde{\varphi}}(\cdot + k)|^2$, then Q_j is defined on the whole $L_2(\mathbb{R})$. In this case

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + k)|^2 \le B, \qquad \sum_{k\in\mathbb{Z}} |\widehat{\widetilde{\varphi}}(\xi + k)|^2 \le \widetilde{B},$$

i.e., the systems $\{\varphi_{jk}\}_{k\in\mathbb{Z}}$, $\{\widetilde{\varphi}_{jk}\}_{k\in\mathbb{Z}}$ are Bessel with the constants $B, \widetilde{B}$ respectively, which implies

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi_{jk} \rangle|^2 \le B\|f\|^2, \qquad \sum_{k\in\mathbb{Z}} |\langle f, \widetilde{\varphi}_{jk} \rangle|^2 \le \widetilde{B}\|f\|^2,$$

for every $f \in L_2(\mathbb{R})$ and $j \in \mathbb{Z}$. This yields (a) and the uniform boundedness of the operators Q_j in $L_2(\mathbb{R})$:

$$\Big\| \sum_{k\in\Omega} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_2 \le B\widetilde{B}\|f\| \qquad \forall f \in L_2(\mathbb{R}).$$

So, it suffices to prove (b), (c) and (d) only for functions f from a dense subspace of $L_2(\mathbb{R})$. But (b), (c) and (d) are valid for $f \in S$ due to Theorem 1. Item (c) also follows from the results of Jetter and Zhou [17].

4. Scaling Approximation in L_p

Denote by $\mathcal{BL}$ the class of functions φ defined by

$$\varphi(x) = \sum_{n=1}^{N} \int_{A_n}^{B_n} \theta_n(\xi) e^{2\pi i x\xi}\, d\xi,$$

where θ_n is absolutely continuous on $[A_n, B_n]$ and $\omega(\theta_n', h)_{L_1[A_n,B_n]} = O(h^\alpha)$, $\alpha > 0$. By the Paley-Wiener theorem, φ is an entire function, and it is not difficult to see that

$$\varphi(x) = \sum_{n=1}^{N} \frac{\theta(B_n)e^{2\pi i B_n x} - \theta(A_n)e^{2\pi i A_n x}}{2\pi i x} + O\Big(\frac{1}{|x|^{1+\alpha}}\Big), \qquad x \to \infty,$$

in particular, $\varphi \in L_p(\mathbb{R})$ for any $p > 1$.

Theorem 2 ([32, Corollaries 5 and 6]). *Let $1 < p < \infty$, $f \in L_p(\mathbb{R})$, $\varphi \in \mathcal{BL}$; $\widetilde{\varphi} \in \mathcal{BL}$ or $\widetilde{\varphi} \in L_q(\mathbb{R})$ and $\widetilde{\varphi}$ be compactly supported. Then the series $\sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}$ converges unconditionally in $L_p(\mathbb{R})$ and*

$$\Big\| \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_p \le K\|f\|_p, \tag{5}$$

where K does not depend on f and j.

Note that (5) follows from the classical Plancherel-Pólya theorem in the case $\varphi = \widetilde{\varphi} = \mathrm{sinc}$. The proof of Theorem 2 in [32] does not use complex methods, it is based on the L_p-boundedness of the maximal function of Hilbert transform.

Theorem 3 ([32, Theorem 7]). *Let $1 < p < \infty$, $f \in L_p(\mathbb{R})$, functions $\varphi, \widetilde{\varphi}$ be as in Theorem 2. If there exists $\delta \in (0, \frac{1}{2})$ such that $\overline{\widehat{\widetilde{\varphi}}}\widehat{\varphi} = 1$ a.e. on $[-\delta, \delta]$, $\widehat{\varphi} = 0$ a.e. on $[l - \delta, l + \delta]$ for all $l \in \mathbb{Z} \setminus \{0\}$, then for every $r \in \mathbb{N}$*

$$\Big\| f - \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_p \le C\omega_r(f, 2^{-j})_{L_p},$$

where C does not depend on f and j.

Now we are interested in results similar to Theorem 3 for the case $\widetilde{\varphi} \in S'$. Unfortunately, we have no analogs of Theorem 2 for this case. Instead, we have Theorem 4 which is based on the following statement.

Lemma 1 ([32, Lemma 8]). *Let $f \in S$, $\varphi \in \mathcal{BL}$, $\widetilde{\varphi}$ and N_0 be as in Theorem 1, $2 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $n^* > N_0 + \frac{1}{p}$. Then the series $\sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}$ converges unconditionally in $L_p(\mathbb{R})$ and*

$$\Big\|\sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}\Big\|_p \le K\bigg(\|\widehat{f}\|_q + 2^{-jn^*}\Big(\int\limits_{-\infty}^{\infty} |(2\pi\xi)|^{qn^*} |\widehat{f}(\xi)|^q d\xi\Big)^{1/q}\bigg),$$

where K does not depend on f and j.

Further in this section we give an insignificant improvement of the results presented in [32, Section 3].

Theorem 4. *Let $f \in S$, $\varphi \in \mathcal{BL}$, $\widetilde{\varphi}$ and N_0 be as in Theorem 1, $2 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $n^* > N_0 + \frac{1}{p}$. If there exists $\delta \in (0, \frac{1}{2})$ such that $\overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}} = 1$ a.e. on $[-\delta, \delta]$, $\widehat{\varphi} = 0$ a.e. on $[l-\delta, l+\delta]$ for all $l \in \mathbb{Z} \setminus \{0\}$, then*

$$\Big\|f - \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}\Big\|_p \le \frac{C}{2^{jn^*}}\Big(\int\limits_{|\xi|>2^j\delta} |\xi|^{qn^*} |\widehat{f}(\xi)|^q d\xi\Big)^{1/q} \le \frac{C'\|f\|_{W_1^{r+1}}}{2^{j(r+1/p)}}, \quad (6)$$

where $r \in \mathbb{N}$, $r \ge n^*$, C and C' do not depend on f and j.

Proof. For any interval $[a, b]$, the function $\widehat{f}$ can be approximated in $L_q[a, b]$ by infinitely smooth functions supported on $[a, b]$. So, given j, one can find a function $\widehat{F}_j \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} \widehat{F}_j \subset [-2^j\delta, 2^j\delta]$ and

$$\Big(\int\limits_{-2^j\delta}^{2^j\delta} |\widehat{f}(\xi) - \widehat{F}_j(\xi)|^q d\xi\Big)^{1/q} \le 2^{-jn^*}\Big(\int\limits_{|\xi|>2^j\delta} |\xi|^{qn^*} |\widehat{f}(\xi)|^q d\xi\Big)^{1/q}. \quad (7)$$

This yields

$$\Big(\int\limits_{-2^j\delta}^{2^j\delta} |\xi|^{qn^*} |\widehat{f}(\xi) - \widehat{F}_j(\xi)|^q d\xi\Big)^{1/q} \le \Big(\int\limits_{|\xi|>2^j\delta} |\xi|^{qn^*} |\widehat{f}(\xi)|^q d\xi\Big)^{1/q}. \quad (8)$$

Let F_j be a function which Fourier transform is $\widehat{F}_j$. Evidently $F_j \in S$ and, by Theorem 1, item (d),

$$F_j = \sum_{k\in\mathbb{Z}} \langle F_j, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}.$$

Using this and Lemma 1, taking into account that, due to the Hausdorff-Young inequality $\|f - F_j\|_p \le C_1 \|\widehat{f} - \widehat{F}_j\|_q$, we have

$$\Big\|f - \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}\Big\|_p \le \|f - F_j\|_p + \Big\|\sum_{k\in\mathbb{Z}} \langle (f - F_j)\widetilde{\varphi}_{jk}\rangle \varphi_{jk}\Big\|_p$$

$$\le C_2 \Bigg(\Big(\int\limits_{-\infty}^{\infty} |\widehat{f}(\xi) - \widehat{F}_j(\xi)|^q d\xi\Big)^{1/q} + 2^{-jn^*}\Big(\int\limits_{-\infty}^{\infty} |(2\pi\xi)|^{qn^*} |\widehat{f}(\xi) - \widehat{F}_j(\xi)|^q d\xi\Big)^{1/q}\Bigg).$$

Combining this with (7), (8) and taking into account that $\operatorname{supp} \widehat{F}_j \subset [-2^j\delta, 2^j\delta]$ and

$$\Big(\int\limits_{|\xi|>2^j\delta} |\widehat{f}(\xi)|^q d\xi\Big)^{1/q} \le \delta^{-n^*} 2^{-jn^*} \Big(\int\limits_{|\xi|>2^j\delta} |\xi|^{qn^*} |\widehat{f}(\xi)|^q d\xi\Big)^{1/q},$$

we obtain the left inequality in (6).

Integrating by parts $r+1$ times, we have

$$|\widehat{f}(\xi)| = \Big|\int\limits_{-\infty}^{\infty} f(t) e^{-2\pi i t\xi} dt\Big| = \Big|\frac{1}{(2\pi i\xi)^{r+1}} \int\limits_{-\infty}^{\infty} f^{(r+1)}(t) e^{-2\pi i t\xi} dt\Big| \le \frac{\|f\|_{W_1^{r+1}}}{(2\pi\xi)^{r+1}}.$$

It follows that

$$2^{-jn^*}\Big(\int\limits_{|\xi|>2^j\delta} |\xi|^{qn^*} |\widehat{f}(\xi)|^q d\xi\Big)^{1/q} \le C_3 2^{-jn^*} \frac{\|f\|_{W_1^{r+1}}}{2^{j(r+1-n^*-1/q)}} = C_3 \frac{\|f\|_{W_1^{r+1}}}{2^{j(r+1/p)}},$$

which completes the proof. $\square$

Consider the special case $\widetilde{\varphi} = \delta$. If $f \in S$, then $\langle f, \widetilde{\varphi}_{jk}\rangle = \langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle = 2^{-j/2} f(-2^{-j}k)$. The latter equality may be extended to a larger class of functions f. Assume that $f = \widehat{g}$,

$$\int\limits_{-\infty}^{\infty} (1 + |\xi|^{n^*})^q |g(\xi)|^q \, d\xi < \infty, \qquad 1 < q \le 2, \quad n^* > \frac{1}{p}. \tag{9}$$

It is easy to see that $g \in L_1(\mathbb{R})$ and $g \in L_q(\mathbb{R})$ in this case, which yields the continuity of f and $f \in L_p(\mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$. Hence, we have

$$2^{-j/2} f(-2^{-j}k) = 2^{-j/2} \int\limits_{-\infty}^{\infty} g(\xi) e^{2\pi i\xi 2^{-j}k} d\xi = \langle g^-, \widehat{\widetilde{\varphi}_{j,k}}\rangle, \quad g^-(\xi) = g(-\xi).$$

Note that if $f \in W_1^n$, $n \in \mathbb{N}$, $f^{(n)} \in Lip_{L_1}\alpha$, $0 \le \alpha < 1$ ($\alpha > 0$ for $n = 1$), then $|\widehat{f}(\xi)| \le \frac{C}{1+|\xi|^{n+\alpha}}$. So, both f and $\widehat{f}$ are in $L(\mathbb{R})$, which yields that the

Fourier inversion formula holds almost everywhere. On the other hand, both the functions $f(x)$ and $\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi x}\,d\xi$ are continuous on $\mathbb{R}$. It follows that the functions coincide everywhere on $\mathbb{R}$. So, $f = \widehat{g}$, where $g(\xi) = \widehat{f}(-\xi)$, and it is easy to see that (9) is fulfilled whenever $n^* < n - 1 + \frac{1}{p} + \alpha$.

Now one can repeat the proofs of Lemma 1 and Theorem 4 for the expansion $\sum_{k\in\mathbb{Z}}\langle g^-, \widehat{\widetilde{\varphi}_{jk}}\rangle\varphi_{jk}$, which leads to the following statement.

Theorem 5. *Let $2 \le p < \infty$, $n^* > \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$, g is a function satisfying (9), $f = \widehat{g}$, $\varphi \in \mathcal{BL}$. If there exists $\delta \in (0, \frac{1}{2})$ such that $\widehat{\varphi} = 1$ a.e. on $[-\delta, \delta]$, $\widehat{\varphi} = 0$ a.e. on $[l-\delta, l+\delta]$ for all $l \in \mathbb{Z}\setminus\{0\}$, then*

$$\Big\|f - 2^{-j/2}\sum_{k\in\mathbb{Z}} f(-2^{-j}k)\varphi_{jk}\Big\|_p \le \frac{C}{2^{jn^*}}\Big(\int\limits_{|\xi|>2^j\delta} |\xi|^{qn^*}|g(\xi)|^q d\xi\Big)^{1/q},$$

where C does not depend on f and j. In particular, if $f \in W_1^n$, $n \in \mathbb{N}$, $f^{(n)} \in Lip_{L_1}\alpha$, $0 \le \alpha < 1$ ($\alpha > 0$ for $n = 1$), then

$$\Big\|f - 2^{-j/2}\sum_{k\in\mathbb{Z}} f(-2^{-j}k)\varphi_{jk}\Big\|_p \le \frac{C'}{2^{j(n-1+\alpha+1/p)}}, \tag{10}$$

where C' does not depend on j.

Let us compare this theorem for the special case $\varphi = \mathrm{sinc}$ with the results obtained by the Butzer's team in [4], [8], where inequality (3) was proved for $f \in W_p^n$, $n \in \mathbb{N}$. So, the minimal requirement for the convergence (in terms of Sobolev spaces) is $f \in W_p^1$, and the minimal approximation order following from (3) is $o(2^{-j})$. Theorem 5 gives approximation order $O\big(2^{-j(\alpha+1/p)}\big)$ whenever $f \in W_1^1$, $f' \in Lip_{L_1}\alpha$, $\alpha > 0$. Observe that the latter class of functions is not a subset of W_p^1 for $\alpha \le \frac{1}{q}$. If $\alpha > \frac{1}{q}$, then our estimate (10) follows also from (3). On the other hand, the space W_1^1 is a subset of Λ_p. So, the L_p-convergence of the sampling expansion of $f \in W_1^1$ follows from [4], [8], however these results do not give the order of approximation. It remains unclear for us if the class of functions defined by (9) is a subset of Λ_p. Finally note that (3) is proved for any $p > 1$, while our technique works only for the case $p \ge 2$.

Consider next a differential operator $Lf := \sum_{l=0}^m \overline{\alpha_l} f^{(l)}$. If $\widetilde{\varphi} = (-1)^l\delta^{(l)}$, $l = 0, \dots, m$, $f = \widehat{g}$ and g satisfies (9) with $n^* > l + \frac{1}{p}$, then f is l times differentiable, $f^{(l)}$ is continuous, and

$$\begin{aligned} f^{(l)}(-2^{-j}k) &= \int\limits_{-\infty}^{\infty} g(\xi)(2\pi i\xi)^l e^{2\pi i\xi 2^{-j}k}\,d\xi \\ &= 2^{jl}(-1)^l \int\limits_{-\infty}^{\infty} g(\xi)\overline{\widehat{\delta^{(l)}}(2^{-j}\xi)e^{2\pi i\xi 2^{-j}k}}\,d\xi \\ &= 2^{j(l+1/2)}\langle g^-, \widehat{\widetilde{\varphi}_{jk}}\rangle, \qquad g^-(\xi) = g(-\xi). \end{aligned}$$

Hence, if $\widetilde{\varphi} = \sum_{l=0}^m \alpha_l(-1)^l\delta^{(l)}$, then the sequence $\{\langle g^-, \widehat{\widetilde{\varphi}_{jk}}\rangle\}_{k\in\mathbb{Z}}$ interpolates $2^{-j/2}Lf(2^{-j}\cdot)$ at the integer points. Again one can repeat the proofs of Lemma 1 and Theorem 4 for the expansion $\sum_{k\in\mathbb{Z}}\langle g^-, \widehat{\widetilde{\varphi}_{jk}}\rangle\varphi_{jk}$, which leads to the following statement.

Theorem 6. *Let $2 \le p < \infty$, $m \in \mathbb{N}$, $n^* > m + \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$, g is a function satisfying (9), $f = \widehat{g}$, $\widetilde{\varphi} = \sum_{l=0}^m \alpha_l(-1)^l\delta^{(l)}$, $\varphi \in \mathcal{BL}$. If there exists $\delta \in (0, 1/2)$ such that $\overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}} = 1$ a.e. on $[-\delta, \delta]$, $\widehat{\varphi} = 0$ a.e. on $[l-\delta, l+\delta]$ for all $l \in \mathbb{Z}\setminus\{0\}$, then*

$$\Big\|f - 2^{-j/2}\sum_{k\in\mathbb{Z}} Lf(2^{-j}\cdot)(k)\varphi_{jk}\Big\|_p \le \frac{C}{2^{jn^*}}\Big(\int\limits_{|\xi|>2^j\delta}|\xi|^{qn^*}|g(\xi)|^q d\xi\Big)^{1/q},$$

where C does not depend on f and j. In particular, if $f \in W_1^n$, $n \ge m+1$, $f^{(n)} \in Lip_{L_1}\alpha$, $0 \le \alpha < 1$ ($\alpha > 0$ for $n = m+1$), then

$$\Big\|f - 2^{-j/2}\sum_{k\in\mathbb{Z}} Lf(2^{-j}\cdot)(k)\varphi_{jk}\Big\|_p \le \frac{C}{2^{j(n-1+\alpha+1/p)}}.$$

5. Uniform Convergence of Scaling Expansions

In this section we study the pointwise approximation by sampling and differential scaling expansions. As above we consider a differential operator $Lf := \sum_{l=0}^m \overline{\alpha_l} f^{(l)}$. Recall that if $\widetilde{\varphi} = \sum_{l=0}^m \alpha_l(-1)^l\delta^{(l)}$, then the sequence $\{\langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle\}_{k\in\mathbb{Z}}$ interpolates $2^{-j/2}Lf(2^{-j}\cdot)$ at the integer points whenever the Fourier transform of f decays fast enough.

Let $\widetilde{\mathcal{BL}}$ be a class of functions φ such that

$$\varphi(x) = \int\limits_{-\infty}^{\infty}\theta(\xi)e^{2\pi i x\xi}\,d\xi, \tag{11}$$

where θ is a compactly supported function of bounded variation.

The following statement generalizes Brown's inequality (2).

Theorem 7. *Let $m \in \mathbb{Z}_+$, $f = \widehat{g}$ and the functions $g(\xi)$, $\xi^m g(\xi)$ be summable on $\mathbb{R}$, $\varphi \in \widetilde{\mathcal{BL}}$, $\widetilde{\varphi} = \sum_{l=0}^m \alpha_l(-1)^l\delta^{(l)}$. If there exists $\delta \in (0, \frac{1}{2})$ such that $\overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}} = 1$ a.e. on $[-\delta, \delta]$, $\widehat{\varphi} = 0$ a.e. on $[l-\delta, l+\delta]$ for all $l \in \mathbb{Z}\setminus\{0\}$, then for every $x \in \mathbb{R}$ and $j \in \mathbb{Z}$ the series $\sum_{k\in\mathbb{Z}} Lf(2^{-j}\cdot)(k)\varphi_{jk}(x)$ converges, and*

$$\Big|f(x) - 2^{-j/2}\sum_{k\in\mathbb{Z}} Lf(2^{-j}\cdot)(k)\varphi_{jk}(x)\Big| \le C2^{-jm}\int\limits_{|\xi|>2^j\delta}|\xi^m g(\xi)|\,d\xi,$$

where C does not depend on f, j and x.

Proof. Let φ be given by (11), $x \in \mathbb{R}$. Set $\Theta(\xi) := \sum_{s\in\mathbb{Z}} \theta(\xi+s)e^{2\pi i x(\xi+s)}$. By the Poisson summation formula, Θ is a summable 1-periodic function, and its n-th Fourier coefficient is

$$\widehat{\Theta}(n) = \int_{-\infty}^{\infty} \theta(\xi)e^{2\pi i x\xi}e^{-2\pi i n\xi}\,d\xi = \widehat{\theta}(n-x) = \varphi(x-n).$$

Since Θ is a function of bounded variation, its Fourier series converges to the function almost everywhere, and the partial Fourier sums are uniformly bounded. Using this and Lebesgue's dominated convergence theorem, for every $l = 0, \dots, m$ we derive

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f^{(l)}(-n)\varphi(x+n) &= \lim_{N\to\infty} \sum_{n=-N}^{N} f^{(l)}(-n)\varphi(x+n) \\
&= \lim_{N\to\infty} \int_{-\infty}^{\infty} \sum_{n=-N}^{N} \varphi(x+n)e^{-2\pi i n\xi}(2\pi i\xi)^l g(-\xi)\,d\xi \\
&= \int_{-\infty}^{\infty} \lim_{N\to\infty} \sum_{n=-N}^{N} \varphi(x+n)e^{-2\pi i n\xi}(2\pi i\xi)^l g(-\xi)\,d\xi \\
&= \int_{-\infty}^{\infty} e^{2\pi i x\xi} \sum_{s\in\mathbb{Z}} \theta(\xi+s)e^{2\pi i x s}(2\pi i\xi)^l g(-\xi)\,d\xi.
\end{aligned}$$

Replacing x by $2^j x$ and f by $f(2^{-j}\cdot)$, after a change of variable, we obtain

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} 2^{-jl} f^{(l)}(-2^{-j}n)\varphi_{jn}(x) \\
&\qquad = 2^{j/2} \int_{-\infty}^{\infty} e^{2\pi i x\xi} \sum_{s\in\mathbb{Z}} \theta(2^{-j}\xi+s)e^{2\pi i 2^j x s}(2\pi i 2^{-j}\xi)^l g(-\xi)\,d\xi.
\end{aligned}$$

This yields

$$\begin{aligned}
&2^{-j/2} \sum_{n=-\infty}^{\infty} Lf(2^{-j}\cdot)(-n)\varphi_{jn}(x) \\
&\qquad = \int_{-\infty}^{\infty} e^{2\pi i x\xi} \sum_{s\in\mathbb{Z}} \theta(2^{-j}\xi+s)e^{2\pi i 2^j x s} \sum_{l=0}^{m} \overline{\alpha}_l (2\pi i 2^{-j}\xi)^l g(-\xi)\,d\xi. \qquad (12)
\end{aligned}$$

Set $g_1 = g\big|_{[-2^j\delta, 2^j\delta]}$, $g_2 = g - g_1$, $f_1 = \widehat{g}_1$, $f_2 = f - f_1$. If $\xi \in [-2^j\delta, 2^j\delta]$, then

$$\sum_{s\in\mathbb{Z}} \theta(2^{-j}\xi+s)e^{2\pi i 2^j x s} = \widehat{\varphi}(2^{-j}\xi).$$

Taking into account that $\widehat{\varphi}(2^{-j}\xi)\overline{\widehat{\widetilde{\varphi}}(2^{-j}\xi)} = 1$ for almost all $\xi \in [-2^j\delta, 2^j\delta]$, we can replace $\sum_{s\in\mathbb{Z}} \theta(2^{-j}\xi + s)e^{2\pi i 2^j xs} \sum_{l=0}^{m} \overline{\alpha_l}(2\pi i 2^{-j}\xi)^l$ by 1.

Hence, it follows from (12) that

$$2^{-j/2} \sum_{n=-\infty}^{\infty} Lf_1(2^{-j}\cdot)(-n)\varphi_{jn}(x) = \int_{-2^j\delta}^{2^j\delta} e^{2\pi i x\xi} g_1(-\xi)\, d\xi = f_1(x). \tag{13}$$

Since $|e^{2\pi i x\xi} \sum_{s\in\mathbb{Z}} \theta(2^{-j}\xi + s)e^{2\pi i 2^j xs}| \le C_1$, where C_1 depends only on φ, using (12) for f_2 and (13), we have

$$\begin{aligned}
\Big|f(x) - 2^{-j/2} \sum_{n=-\infty}^{\infty} Lf(2^{-j}\cdot)(-n)\varphi_{jn}(x)\Big| \\
&= \Big|f_2(x) - 2^{-j/2} \sum_{n=-\infty}^{\infty} Lf_2(2^{-j}\cdot)(-n)\varphi_{jn}(x)\Big| \\
&\le \int\limits_{|\xi|>2^j\delta} \Big(1 + C_1 \sum_{l=0}^{m} |\alpha_l(2\pi 2^{-j}\xi)^l)|\Big)|g_2(\xi)|\, d\xi \\
&\le C2^{-jm} \int\limits_{|\xi|>2^j\delta} |\xi|^m |g(\xi)| d\xi.
\end{aligned}$$

□

6. Wavelet Frame-type Decompositions

Due to Theorems 1, 5, 7, appropriate continuous functions f can be approximated (in different senses) by the expansions $\sum_{k\in\mathbb{Z}} c_{jk}\varphi_{jk}$ with the coefficients interpolating f at equidistant points. Now we are interested in construction of decompositions with a similar property. Let us find the corresponding wavelets. Assume that the function φ is refinable, m_0 is its mask. The distribution $\widetilde{\varphi} = \delta$ is also refinable, its mask is $\widetilde{m}_0 \equiv 1$. Set

$$m_1(\xi) = \widetilde{m}_1(\xi) = \frac{1}{\sqrt{2}}e^{2\pi i\xi}, \qquad m_2(\xi) = \frac{1}{\sqrt{2}}(1 - 2m_0(\xi)), \qquad \widetilde{m}_2(\xi) = \frac{1}{\sqrt{2}}.$$

The columns of the matrices

$$\begin{pmatrix} m_0(\xi) & m_0(\xi + 1/2) \\ m_1(\xi) & m_1(\xi + 1/2) \\ m_2(\xi) & m_2(\xi + 1/2) \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{m}_0(\xi) & \widetilde{m}_0(\xi + 1/2) \\ \widetilde{m}_1(\xi) & \widetilde{m}_1(\xi + 1/2) \\ \widetilde{m}_2(\xi) & \widetilde{m}_2(\xi + 1/2) \end{pmatrix}$$

form a biorthonormal system. Define wavelet functions by

$$\widehat{\psi^{(\nu)}}(\xi) = m_\nu(\xi/2)\widehat{\varphi}(\xi/2), \qquad \widehat{\widetilde{\psi}^{(\nu)}}(\xi) = \widetilde{m}_\nu(\xi/2)\widehat{\widetilde{\varphi}}(\xi/2), \qquad \nu = 1, 2.$$

To construct these wavelet functions we apply a well-known matrix extension principle which often leads to wavelet frames. However our wavelet functions $\psi^{(\nu)}$, $\nu = 1, 2$, do not generate a frame because $\widehat{\psi^{(\nu)}}(0) \neq 0$, which is necessary for frames (see [31, Theorem 1]). Nevertheless, we will obtain desirable frame-type expansion. Indeed, for any j, $j' \in \mathbb{Z}$, $j' < j$, the following identity holds (see, e.g., [27, Lemma 11] for the proof):

$$\sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} - \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{j'k}\rangle \varphi_{j'k} = \sum_{i=j'}^{j-1} \sum_{\nu=1}^{2} \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\psi}_{ik}^{(\nu)}\rangle \psi_{ik}^{(\nu)}. \tag{14}$$

If now $\sum\limits_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}$ converges in some sense to f as $j \to +\infty$, then

$$f = \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} + \sum_{i=0}^{\infty} \sum_{\nu=1}^{2} \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\psi}_{ik}^{(\nu)}\rangle \psi_{ik}^{(\nu)}, \tag{15}$$

where the series converges in the same sense. As above, $\langle f, \widetilde{\varphi}_{jk}\rangle$ can be replaced by $\langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle$ or by $\langle g^{-}, \widehat{\widetilde{\varphi}_{jk}}\rangle$ for the corresponding classes of functions f.

Taking into account that $\widetilde{\psi}^{(1)}(x) = \sqrt{2}\delta(2x+1)$, $\widetilde{\psi}^{(2)}(x) = \sqrt{2}\delta(2x)$, we can rewrite (15) as

$$\begin{aligned} f(x) = \sum_{k\in\mathbb{Z}} f(-k)\varphi(x+k) + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \Big(& f(-2^{-j-1}(2k+1))\psi^{(1)}(2^j x + k) \\ & + f(-2^{-j-1}(2k))\psi^{(2)}(2^j x + k)\Big), \end{aligned} \tag{16}$$

where $\psi^{(1)}(x) = \sqrt{2}\varphi(2x+1)$, $\psi^{(2)}(x) = \sqrt{2}(\varphi(2x) - \varphi(x))$.

To construct these wavelets, we used a standard approach based on the refinability of functions φ, $\widehat{\varphi}$. Unfortunately, a class of refinable band-limited functions is not rich. As far as we know, there exists only one idea suggested by Meyer in [25]: if $\operatorname{supp}\widehat{\varphi} \subset [-\frac{2}{3}, \frac{2}{3}]$, and $\widehat{\varphi} \equiv 1$ on $[-\frac{1}{3}, \frac{1}{3}]$, then φ satisfies the refinement equation $\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi)$, where $m_0(\xi) = \sum_{l\in\mathbb{Z}} \widehat{\varphi}(2(\xi + l))$. However, (16) can be extended to a wider class of functions. Observe that if $\widetilde{\varphi} = \delta$ and the functions $\psi^{(1)}, \psi^{(2)}, \widetilde{\psi}^{(1)}, \widetilde{\psi}^{(2)}$ are defined as above, then (14) holds true for any φ (without assumption of its refinability!). Thus, for every φ satisfying all conditions of Theorem 5 or Theorem 7 with $m = 0$ and $\alpha_0 = 1$, identity (16) holds.

Similarly, if $Lf := \sum_{l=0}^{m} \overline{\alpha_l} f^{(l)}$, under the conditions of Theorem 6 or Theorem 7, an appropriate function f can be approximated by scaling expansions $\sum_{k\in\mathbb{Z}} c_{jk}\varphi_{jk}$ with coefficients interpolating $2^{-j/2}Lf(2^{-j}\cdot)$ at the integer points. Now we construct the corresponding wavelet decompositions. For

$l = 0, \dots, m$, we set

$$\widetilde{\psi}^{(2l+1)}(x) = \sqrt{2}\sum_{l=0}^{m} \alpha_l (-1)^l \delta^{(l)}(2x+1), \qquad \psi^{(2l+1)}(x) = \sqrt{2}\varphi(2x+1),$$

$$\widetilde{\psi}^{(2l+2)}(x) = \sqrt{2}\sum_{l=0}^{m} \alpha_l (-1)^l \delta^{(l)}(2x), \qquad \psi^{(2l+2)}(x) = \sqrt{2}(\varphi(2x) - 2^l\varphi(x)).$$

It is not difficult to check that again (14) is satisfied (with replacing $\sum_{\nu=1}^{2}$ by $\sum_{\nu=1}^{2m+2}$), whenever $\alpha_0 = 1$. This leads to the following decomposition:

$$\begin{aligned} f(x) = & \sum_{k\in\mathbb{Z}}\sum_{l=0}^{m} \alpha_l f^{(l)}(-k)\varphi(x+k) \\ & + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{l=0}^{m} 2^{-(j+1)l}\alpha_l \Big(f^{(l)}(-2^{-j-1}(2k+1))\psi^{(2l+1)}(2^j x + k) \\ & \qquad + f^{(l)}(-2^{-j-1}(2k))\psi^{(2l+2)}(2^j x + k)\Big). \end{aligned}$$

Bibliography

[1] C. de Boor C., R. DeVore and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, *J. Funct. Anal.* **119** (1994), no. 1, 37–78.

[2] C. de Boor C., R. DeVore and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, *Trans. Amer. Math. Soc.* **341** (1994), no. 2, 787–806.

[3] C. de Boor C., R. DeVore and A. Ron, Approximation orders of FSI spaces in $L_2(\mathbb{R}^d)$, *Constr. Approx.* **14** (1998), no. 4, 631–652.

[4] C. Bardaro, P. L. Butzer, R. L. Stens and G. Vinti, Approximation error of the Whittaker cardinal series in terms of an averaged modulus of smoothness covering discontinuous signals, *Math. Anal. Appl.* **316** (2006), no. 1, 269–306.

[5] C. Bardaro, P. L. Butzer, R. L. Stens and G. Vinti, Prediction by samples from the past with error estimates covering discontinuous signals, *IEEE Trans. Inform. Theory* **56** (2010), no. 1, 614–633.

[6] E. Borel, Sur l'interpolation, *C. R. Acad. Sci. Paris* **124** (1897), 673–676.

[7] J. L. Brown, Jr., On the error in reconstructing a non-bandlimited function by means of the bandpass sampling theorem, *J. Math. Anal. Appl.* **18** (1967), 75-84.

[8] P. L. Butzer, J. R. Higgins and R. L. Stens, Classical and approximate sampling theorems: studies in the $L_p(\mathbb{R})$ and the uniform norm, *J. Approx. Theory* **137** (2005), no. 2, 250–263.

[9] P. L. Butzer, P. J. S. G. Ferreira, G. Schmeisser and R. L. Stens, The summation formulae of Euler-Maclaurin, Abel-Plana, Poisson, and their interconnections with the approximate sampling formula of signal analysis, *Results Math.* **59** (2011), no. 3–4, 359–400.

[10] P. L. Butzer, M. M. Dodson, P. J. S. G. Ferreira, J. R. Higgins, O. Lange, P. Seidler and R. L. Stens, Multiplex signal transmission and the development of sampling techniques: the work of Herbert Raabe in contrast to that of Claude Shannon, *Appl. Anal.* **90** (2011), no. 3–4, 643–688.

[11] P. L. Butzer, P. J. S. G. Ferreira, J. R. Higgins, G. Schmeisser and R. L. Stens, The sampling theorem, Poisson's summation formula, general Parseval formula, reproducing kernel formula and the Paley-Wiener theorem for bandlimited signalstheir interconnections, *Appl. Anal.* **90** (2011), no. 3–4, 431–461.

[12] P. L. Butzer and R. L. Stens, Reconstruction of signals in $L_p(\mathbb{R})$-space by generalized sampling series based on linear combinations of B-splines, *Integral Transforms Spec. Funct.* **19** (2008), no. 1, 35–58.

[13] A. L. Cauchy, Me'moire sur diverses formules d'analyse, *C. R. Acad. Sci.* **12** (1841), 283–298.

[14] A. Cohen, I. Daubechies and J.Ċ. Feauveau, Biorthogonal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **45** (1992), no. 5, 485-560.

[15] I. M. Gel'fand and G. E. Shilov, "Generalized Functions, Vol. 1: Properties and Operations", Academic Press, New York, 1964.

[16] K. Jetter and D. X. Zhou, Order of linear approximation from shift invariant spaces, *Constr. Approx.* **11** (1995), no. 4, 423–438.

[17] K. Jetter and D. X. Zhou, Order of linear approximation on finitely generated shift invariant spaces, 1998, preprint.

[18] R. Q. Jia, Refinable shift-invariant spaces: From splines to wavelets, *in* "Approximation Theory VIII. Vol. 2. Wavelets and Multilevel Approximation" (C. K. Chui and L.L̇. Schumaker, Eds.), pp. 179–208, World Scientific, Singapore, 1995.

[19] R. Q. Jia, Convergence rates of cascade algorithms, *Proc. Amer. Math. Soc.* **131** (2003) 1739–1749.

[20] R. Q. Jia, Approximation by quasi-projection operators in Besov spaces, *J. Approx. Theory* **162** (2010), no. 1, 186–200.

[21] A. Kivinukk and G. Tamberg, Interpolating generalized Shannon sampling operators, their norms and approximation properties, *Sampl. Theory Signal Image Process.* **8** (2009), 77-95.

[22] A. Kivinukk and G. Tamberg, On Blackman-Harris windows for Shannon sampling series, *Sampl. Theory Signal Image Process.* **6** (2007), 87-108.

[23] V. A. Kotelnikov, On the transmission capacity of the 'ether'and of cables in electrical communications, *in* "Proc. of the First All-Union Conference on the Technological Reconstruction of the Communications Sector and Low-current Engineering", Izd. Red. Upr. Svyazzi RKKA, 1933 [in Russian].

[24] J. J. Lei, R. Q. Jia and E. W. Cheney, Approximation from shift-invariant spaces by integral operators, *SIAM J. Math. Anal.* **28** (1997), 481–498.

[25] Y. Meyer, "Ondelettes et ope'rateurs", Hermann, Paris, 1990.

[26] K. Ogura, On a certain transcendetal integral function in the theory of interpolation, *Tôhoku Math. J.*, **17** (1920), 64–72.

[27] A. Krivoshein and M. Skopina, Approximation by frame-like wavelet systems, *Appl. Comput. Harmon. Anal.* **31** (2011), no. 3, 410–428.

[28] A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$ II: dual systems, *J. Fourier Anal. Appl.* **3** (1997), 617–637.

[29] A. Ron and Z. Shen, Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$, *Math. Comp.* **67** (1998), 191–207.

[30] Bl. Sendov and V. A. Popov, "The Averaged Moduli of Smoothness", Pure and Applied Mathematics, Wiley, Chichester, 1988.

[31] M. Skopina, On construction of multivariate wavelet frames, *Appl. Comput. Harmon. Anal.* **27** (2009), 55–72.

[32] M. Skopina, Band-limited scaling and wavelet expansions, *Appl. Comput. Harmon. Anal.* **36** (2014), 143–157.

[33] C. E. Shannon, Communication in the presence of noise, *Proc. IRE* **37** (1949), 10–21.

[34] R. L. Stens, Sampling with generalized kernels, *in* "Sampling Theory in Fourier and Signal Analysis: Advanced Topics" (J. R. Higgins and R. L. Stens, Eds.), Clarendon Press, Oxford, 1999.

[35] A. Yu. Trynin, A generalization of the Whittaker-Kotel'nikov-Shannon sampling theorem for continuous functions on a closed interval, *Mat. Sb.* **200** (2009), no. 11, 61–108 [in Russian]; English transl.: *Sb. Math.* **200** (2009), no. 11–12, 1633-1679.

[36] A. Yu. Trynin, On operators of interpolation with respect to solutions of the Cauchy problem and Lagrange-Jacobi polynomials, *Izv. Ross. Akad. Nauk Ser. Mat.* **75** (2011), no. 6, 129–162 [in Russian]; English transl.: *Izv. Math.* **75** (2011), no. 6, 1215-1248.

[37] A. Yu. Trynin, On the divergence of sinc-approximations everywhere on $(0, \pi)$, *Algebra i Analiz* **22** (2010), no. 4, 232–256 [in Russian].

[38] V. S. Vladimirov, "Generalized Functions in Mathematical Physics", 2nd ed., Nauka, Moscow, 1979 [in Russian] English transl.: Mir, Moscow, 1979.

[39] E. T. Whittaker, On the functions which are represented by the expansion of the interpolation-theory, *Proc. Roy. Soc. Edinburgh Sect. A* **35** (1915), 181–194.

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