CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2013
(K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 265-273
Prof. Marin Drinov Academic Publishing House, Sofia, 2014

On Approximation by Algebraic Version of the Trigonometric Jackson Integrals $G_{S,N}$ in Weighted Integral Metric

Teodora Zapryanova

We characterize the errors of the algebraic version of trigonometric Jackson integrals $G_{s,n}$ in weighted integral metric. We prove direct and strong converse theorem in terms of a weighted K-functional.

Keywords and Phrases: Linear operator, direct theorem, strong converse theorem, K-functional.

Mathematics Subject Classification (2010): 41A36, 41A25, 41A17.

1. Introduction

We study linear approximation process together with the characterization of the rate of convergence of the algebraic version of the trigonometric Jackson integrals $G_{s,n}$ defined by

$$G_{s,n}(f,x) = \int_{-\pi}^{\pi} f(\cos(\arccos x + v))K_{s,n}(v)\,dv,$$

where

$$K_{s,n}(v) = \mu_{s,n}\Big(\frac{\sin(nv/2)}{\sin(v/2)}\Big)^{2s}, \qquad \int_{-\pi}^{\pi} K_{s,n}(v)\,dv = 1.$$

In [4, 5] we have established the equivalence between the approximation error in uniform norm $\|\cdot\|$ of the operator $G_{s,n}$, a proper K-functional and a proper modulus of smoothness

$$\|f - G_{s,n}f\| \sim K\big(f, \frac{1}{n^2}; C[-1,1], C^2, H\big) \sim \Omega_2\big(f, \frac{1}{n}\big).$$

In this equivalence the K-functional is defined for every $f \in C[-1,1]$ and $t > 0$ by

$$K(f,t;C[-1,1],C^2,H) := \inf\big\{\|f-g\| + t\|Hg\| \,:\, g \in C^2\big\},$$

where $C^2 = C^2[-1,1]$, the differential operator is given by

$$H := (H_1)^2, \qquad (H_1 g)(x) := \sqrt{1-x^2}\,\frac{d}{dx}\,g(x)$$

and the modulus is defined by

$$\Omega_2(f,t) := \sup_{0<h\le t} \|f(\cos(\arccos(\cdot)+h)) + f(\cos(\arccos(\cdot)-h)) - 2f(\cdot)\|.$$

The notation $A(n) \sim B(n)$ means that there exists a positive constant c independent of n such that $\frac{1}{c}B(n) \le A(n) \le cB(n)$. The equivalence $\|f - G_{s,n}f\| \sim K(f, \frac{1}{n^2}; C[-1,1], C^2, H)$ consists of a direct inequality and a strong converse inequality of type A in the sense of [2]. Ditzian and Ivanov have shown that the converse inequality follows from several inequalities of Bernstein and Voronovskaya type. We apply their method.

Let $L_p(u)[-1,1]$, $1 \le p < \infty$, $u(x) = (1-x^2)^{-1/(2p)}$, be the weighted L_p space with the norm

$$\|f\|_{p,u} := \|f\|_{L_p(u)[-1,1]} = \Big(\int_{-1}^{1} |u(x)f(x)|^p dx\Big)^{1/p}.$$

The approximation error $\|f - G_{s,n}f\|_{p,u}$ of $G_{s,n}$ in $L_p(u)[-1,1]$ will be compared with the K-functional, which for every $f \in L_p(u)[-1,1]$ and $t > 0$ is defined by

$$K(f,t; L_p(u)[-1,1], C^2, H) := \inf\big\{\|f-g\|_{p,u} + t\|Hg\|_{p,u} \,:\, g \in C^2\big\}.$$

Our main result states the following:

Theorem 1. *For every $f \in L_p(u)[-1,1]$, $1 \le p < \infty$, and $s, n \in \mathbb{N}$, $s \ge 3$, we have*

$$\|f - G_{s,n}f\|_{L_p(u)[-1,1]} \sim K\big(f, \frac{1}{n^2}; L_p(u)[-1,1], C^2, H\big).$$

In Section 2 we state and prove some auxiliary lemmas. The proof of Theorem 1 is given in Section 3.

2. Auxiliary Lemmas

The convolution between a summable on $\mathbb{R}$ function F and a 2π-periodic function G is given by

$$F * G(x) := \int_{-\infty}^{\infty} F(x-v)G(v)\,dv.$$

The following three lemmas follow immediately by Fubini's theorem and Minkowski's inequality.

Lemma 1. *Let f be summable on $\mathbb{R}$ and $g \in L_p[-\pi,\pi]$ be a 2π-periodic function, $1 \le p < \infty$. Then the following holds true:*

$$\|f * g\|_{L_p[-\pi,\pi]} \le \|f\|_{L_1(-\infty,\infty)}\|g\|_{L_p[-\pi,\pi]}.$$

Lemma 2. *For a 2π-periodic integrable on $[-\pi,\pi]$ function g and every $v \neq 0$ we have*

$$\Big\|\frac{1}{v}\int_z^{z+v} |g(\xi)|\,d\xi\Big\|_{L_1[-\pi,\pi]} = \|g\|_{L_1[-\pi,\pi]}.$$

Lemma 3. *For a 2π-periodic function $g \in L_p[-\pi,\pi]$, $1 \le p < \infty$, and every $v \neq 0$ we have*

$$\Big\|\frac{1}{v}\int_z^{z+v} |g(\xi)|\,d\xi\Big\|_{L_p[-\pi,\pi]} \le \|g\|_{L_p[-\pi,\pi]}.$$

Let us set

$$Y := \big\{g \in C[-1,1] \,:\, H_1 g \in C[-1,1],\ Hg \in C[-1,1],\ H_1 g(\pm 1) = 0\big\}, \quad (1)$$
$$Z := \big\{g \in Y \,:\, H_1^3 g \in C[-1,1],\ H^2 g \in C[-1,1],\ H_1^3 g(\pm 1) = 0\big\}. \quad (2)$$

The following lemma is proved in [5, p. 402].

Lemma 4. *Let Y be the space defined in* (1), *$g \in Y$ and $\widetilde{g}(\sigma) := g(\cos\sigma)$. Then $\widetilde{g} \in C^2(\mathbb{R})$ and $\widetilde{g}''(\sigma) = Hg(\cos\sigma)$ for $\sigma \in \mathbb{R}$.*

The last statement in this section is

Lemma 5. *Let Y be the space defined in* (1). *Then for every function $f \in L_p(u)[-1,1]$ and $t > 0$, we have*

$$K(f,t;L_p(u)[-1,1],Y,H) = K(f,t;L_p(u)[-1,1],C^2,H).$$

Proof. From $C^2 \subset Y$ we see that

$$K(f,t;L_p(u)[-1,1],Y,H) \le K(f,t;L_p(u)[-1,1],C^2,H).$$

In order to prove the opposite inequality

$$K(f,t;L_p(u)[-1,1],C^2,H) \le K(f,t;L_p(u)[-1,1],Y,H)$$

it is sufficient to show (see [3, Lemma 2, p. 116]) that for every $g \in Y$ and $\varepsilon > 0$ there exists $G \in C^2$ such that

$$\|G - g\|_{L_p(u)[-1,1]} \le \varepsilon, \qquad \|HG\|_{L_p(u)[-1,1]} \le \|Hg\|_{L_p(u)[-1,1]} + \varepsilon.$$

Let $g(x) \in Y$. We put $x = \cos\sigma$ and consider $\widetilde{g}(\sigma) := g(\cos\sigma)$. Since $g(x) \in Y$, $\widetilde{g}(s) \in C^2$ (see Lemma 4). We use the Jackson integrals of the following type

$$J_n(\widetilde{g}, \sigma) := \int_{-\pi}^{\pi} \widetilde{g}(\sigma + v) K_{1,s,n}(v)\, dv = \int_{-\pi}^{\pi} \widetilde{g}(v) K_{1,s,n}(\sigma - v)\, dv, \qquad (3)$$

where $s > 0$, $n > 0$ and

$$K_{1,s,n}(v) := \lambda_{s,n} \Big(\frac{\sin(mv/2)}{\sin(v/2)}\Big)^{2s}, \qquad \int_{-\pi}^{\pi} K_{1,s,n}(v)\, dv = 1 \qquad (4)$$

for $m = [n/s] + 1$.

Since $\frac{1}{2m}\Big(\frac{\sin(mv/2)}{\sin(v/2)}\Big)^2 = \frac{1}{2} + \sum\limits_{k=0}^{m-1}(1 - k/m)\cos kv$, it follows that $K_{1,s,n}$ is an even non-negative trigonometric polynomial of degree at most n. Moreover, $J_n(\widetilde{g}, \sigma)$ is a trigonometric polynomial of degree at most n, which is even as $\widetilde{g}$ is even. From Jackson's theorem (see [1, Chap. 7, Theorem 2.2]) we get

$$\|\widetilde{g} - J_n(\widetilde{g})\|_{L_p[0,\pi]} \le c\,\omega_2(\widetilde{g}, \frac{1}{n})_{L_p[0,\pi]} = O\big(\frac{1}{n^2}\big). \qquad (5)$$

By the substitution $\sigma = \arccos x$ in $J_n(\widetilde{g}, \sigma)$ we obtain an algebraic polynomial, which is the desired function from C^2. We set

$$G(x) = J_n(\widetilde{g}, \arccos x).$$

From $\|g - G\|_{L_p(u)[-1,1]} = \|\widetilde{g} - J_n(\widetilde{g})\|_{L_p[0,\pi]}$ and (5) we get

$$\|g - G\|_{L_p(u)[-1,1]} \le c\omega_2(\widetilde{g}, \frac{1}{n})_{L_p[0,\pi]} = O(\frac{1}{n^2}).$$

From (3) and (4) it follows that

$$\frac{d^2}{d\sigma^2} J_n(\widetilde{g}, \sigma) = \int_{-\pi}^{\pi} \widetilde{g}''(\sigma - v) K_{1,s,n}(v)\, dv = \int_{-\pi}^{\pi} \widetilde{g}''(v) K_{1,s,n}(\sigma - v)\, dv = J_n(\widetilde{g}'', \sigma).$$

Using the Jackson theorem, we get

$$\|\widetilde{g}'' - J_n(\widetilde{g}'')\|_{L_p[0,\pi]} \le c\,\omega_2(\widetilde{g}'', \frac{1}{n})_{L_p[0,\pi]}. \qquad (6)$$

Since $(Hg)(x) = \frac{d^2}{d\sigma^2}\widetilde{g}(\sigma)$ and $(HG)(x) = \frac{d^2}{d\sigma^2} J_n(\widetilde{g}, \sigma)$, inequality (6) implies

$$\|Hg - HG\|_{L_p(u)[-1,1]} \le c\,\omega_2(\widetilde{g}'', \frac{1}{n})_{L_p[0,\pi]}.$$

For a given $\varepsilon > 0$ we choose n such that $(1 + c)\omega_2(\widetilde{g}'', \frac{1}{n})_{L_p[0,\pi]} < \varepsilon$ and $(1 + c)\omega_2(\widetilde{g}, \frac{1}{n})_{L_p[0,\pi]} < \varepsilon$ to obtain

$$\|HG\|_{L_p(u)[-1,1]} < \|Hg\|_{L_p(u)[-1,1]} + \varepsilon \quad \text{and} \quad \|G - g\|_{L_p(u)[-1,1]} < \varepsilon.$$

This completes the proof of the lemma. □

3. Proof of Theorem 1

In view of Lemma 5 the theorem will be proved if we show that

$$\|f - G_{s,n}f\|_{p,u} \sim K(f, \frac{1}{n^2}; L_p(u)[-1,1], Y, H).$$

First we prove the converse result

$$K(f, \frac{1}{n^2}; L_p(u)[-1,1], Y, H) \leq c\|f - G_{s,n}f\|_{L_p(u)[-1,1]},$$

which is a strong converse inequality of type A in terms of [2]. We utilize [2, Theorems 3.1 and 4.1] with

$$\begin{aligned} Q_\alpha = G_{s,n}, \qquad Df = Hf, \qquad \Phi(f) = \|H^2 f\|_{L_p(u)[-1,1]}, \\ \lambda(n) = \frac{1}{2}\int_{-\pi}^{\pi} v^2 K_{s,n}(v)\,dv \sim n^{-2} \qquad \text{for } s \geq 2, \\ \lambda_1(n) = \frac{1}{3!}\int_{-\pi}^{\pi} v^4 K_{s,n}(v)\,dv \sim n^{-4} \qquad \text{for } s \geq 3. \end{aligned} \tag{7}$$

The result needed for inequality (3.3) from Theorem 3.1 in [2] with $M = 1$ is given by

$$\|G_{s,n}f\|_{L_p(u)[-1,1]} \leq \|f\|_{L_p(u)[-1,1]}. \tag{8}$$

In order to prove (8), we set $\widetilde{f}(z+v) := f(\cos(z+v)) = f(\cos(\arccos x + v))$, $z = \arccos x$, and recall the representation

$$\begin{aligned} (G_{s,n}f)(x) &= \int_{-\pi}^{\pi} f(\cos(\arccos x + v))K_{s,n}(v)\,dv = \int_{-\pi}^{\pi} \widetilde{f}(z+v)K_{s,n}(v)\,dv \\ &= \int_{-\pi}^{\pi} \widetilde{f}(z-v)K_{s,n}(v)\,dv = \widetilde{f} * K(z) = K * \widetilde{f}(z) = (\widetilde{G}_{s,n}\widetilde{f})(z), \end{aligned}$$

where

$$K(v) = K_1(v) = \begin{cases} \mu_{s,n}\Big(\dfrac{\sin(nv/2)}{\sin(v/2)}\Big)^{2s}, & \text{if } |v| \leq \pi; \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\|G_{s,n}f\|_{L_p(u)[-1,1]} = \big\|\widetilde{G}_{s,n}\widetilde{f}\big\|_{L_p[0,\pi]}.$$

Using Lemma 1 as $\widetilde{f}$ and $\widetilde{G}_{s,n}\widetilde{f}$ are even 2π-periodic functions, we obtain

$$\begin{aligned} \big\|\widetilde{G}_{s,n}\widetilde{f}\big\|_{L_p[0,\pi]} &= \big\|K * \widetilde{f}\big\|_{L_p[0,\pi]} \\ &\leq \|K\|_{L_1(-\infty,\infty)}\big\|\widetilde{f}\big\|_{L_{p[0,\pi]}} = \big\|\widetilde{f}\big\|_{L_{p[0,\pi]}} = \|f\|_{L_p(u)[-1,1]} \end{aligned}$$

to complete the proof of (8).

Let Z be the space defined in (2). We will show now that for $f \in Z$ the following Voronovskaya-type estimate holds true:

$$\|f - G_{s,n}f + \lambda(n)Hf\|_{L_p(u)[-1,1]} \le \lambda_1(n)\Phi(f), \tag{9}$$

with $\Phi(f) = \|H^2 f\|_{L_p(u)[-1,1]} = \big\|\widetilde{f}^{(4)}\big\|_{L_p[0,\pi]}$ and $\lambda(n)$, $\lambda_1(n)$ defined in (7). Inequality (9) will serve for inequality (3.4) from Theorem 3.1 in [2]. We note that $G_{s,n}f \in Z$ as $G_{s,n}f$ is an algebraic polynomial. Let $f \in Z$. From $Z \subset Y$ and Lemma 4 it follows that $\widetilde{f} \in C^2(\mathbb{R})$. We apply Lemma 4 for Hf to obtain that $\widetilde{f} \in C^4(\mathbb{R})$. We have

$$\begin{aligned} f(x) - (G_{s,n}f)(x) + \lambda(n)Hf(x) &= \widetilde{f}(z) - (\widetilde{G}_{s,n}\widetilde{f})(z) + \lambda(n)\widetilde{f}''(z) \\ &= \int_{-\pi}^{\pi} \Big[\widetilde{f}(z) - \widetilde{f}(z+v) + \frac{1}{2}v^2\widetilde{f}''(z)\Big] K_{s,n}(v)dv. \end{aligned}$$

Expanding $\widetilde{f}(z+v)$ by Taylor's formula and using that $\int_{-\pi}^{\pi} vK_{s,n}(v)\,dv = 0$, $\int_{-\pi}^{\pi} v^3K_{s,n}(v)\,dv = 0$, we obtain

$$f(x)-(G_{s,n}f)(x)+\lambda(n)Hf(x) = -\int_{-\pi}^{\pi}\int_z^{z+v} \frac{1}{3!}\widetilde{f}^{(4)}(\xi)(z+v-\xi)^3 d\xi K_{s,n}(v)\,dv.$$

We recall that $\widetilde{f}^{(4)}(\xi) = (H^2f)(\cos\xi)$. As for $\xi \in [z, z+v]$, $z+v-\xi \in [0,v]$ and the sign of $\int_z^{z+v} |\widetilde{f}^{(4)}(\xi)|\,d\xi$ is constant and coincides with the sign of v (if the integral is not zero), via Minkowski's inequality we get

$$\begin{aligned} \|f - G_{s,n}f + \lambda(n)Hf\|_{L_p(u)[-1,1]} &= \big\|\widetilde{f} - \widetilde{G}_{s,n}\widetilde{f} + \lambda(n)\widetilde{f}''\big\|_{L_p[0,\pi]} \\ &= \Big\{\int_0^{\pi}\Big|\int_{-\pi}^{\pi}\int_z^{z+v} \frac{1}{3!}\widetilde{f}^{(4)}(\xi)(z+v-\xi)^3 d\xi K_{s,n}(v)\,dv\Big|^p dz\Big\}^{1/p} \\ &\le \frac{1}{3!}\Big\{\int_0^{\pi}\Big(\int_{-\pi}^{\pi} v^4 K_{s,n}(v)\Big|\frac{1}{v}\int_z^{z+v} \big|\widetilde{f}^{(4)}(\xi)\big|\,d\xi\Big|dv\Big)^p dz\Big\}^{1/p} \\ &\le \frac{1}{3!}\int_{-\pi}^{\pi} v^4 K_{s,n}(v)\Big\|\frac{1}{v}\int_z^{z+v}\big|\widetilde{f}^{(4)}(\xi)\big|d\xi\Big\|_{L_p[0,\pi]} dv \\ &\le \frac{1}{3!}\int_{-\pi}^{\pi} v^4 K_{s,n}(v)\big\|\widetilde{f}^{(4)}\big\|_{L_p[0,\pi]} dv \\ &= \lambda_1(n)\Phi(f). \end{aligned}$$

In the last inequality above we have applied Lemma 3 to the even 2π-periodic function $\widetilde{f}^{(4)}$ to obtain $\big\|\frac{1}{v}\int_z^{z+v}\big|\widetilde{f}^{(4)}(\xi)\big|d\xi\big\|_{L_p[0,\pi]} \le \big\|\widetilde{f}^{(4)}\big\|_{L_p[0,\pi]}$. This establishes (9).

To obtain results corresponding to (3.5) and (3.6) from Theorem 3.1 in [2], we need a weighted Bernstein-type inequality for the power of the operator

$G_{s,n}$ like inequality (6.10) in [2]. We use representations

$$(G_{s,n}f)(x) = \int_{-\pi}^{\pi} \widetilde{f}(z-v)K_{s,n}(v)\,dv = \widetilde{f} * K(z) = K * \widetilde{f}(z) = (\widetilde{G}_{s,n}\widetilde{f})(z),$$

$$G_{s,n}(G_{s,n}f)(x) = K * K * \widetilde{f}(z) = K_2 * \widetilde{f}(z),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(G^m_{s,n}f)(x) = K * K_{m-1} * \widetilde{f}(z) = K_m * \widetilde{f}(z),$$

($K_m = K * K_{m-1}$ for $m = 2, 3, \ldots$), to obtain

$$H_1 G^m_{s,n} f = -\{K'\} * K_{m-1} * \widetilde{f},$$

where

$$\{K'\}(v) = \begin{cases} \mu_{s,n}\dfrac{d}{dv}\Big(\dfrac{\sin(nv/2)}{\sin(v/2)}\Big)^{2s}, & \text{if } |v| \le \pi; \\ 0, & \text{otherwise.} \end{cases}$$

We now estimate the action of H_1 on the m-th degree $G^m_{s,n}$ of the operator. Using Lemma 1, we obtain

$$\begin{aligned} \big\|H_1 G^m_{s,n} f\big\|_{L_p(u)[-1,1]} &= \big\|\{K'\} * K_{m-1} * \widetilde{f}\big\|_{L_p[0,\pi]} \\ &\le \|\{K'\} * K_{m-1}\|_{L_1(-\infty,\infty)} \big\|\widetilde{f}\big\|_{L_p[0,\pi]} \\ &= \|\{K'\} * K_{m-1}\|_{L_1(-\infty,\infty)} \|f\|_{L_p(u)[-1,1]}. \end{aligned}$$

We have proved in [4, Assertion 1.2] that

$$\|\{K'\} * K_{m-1}\|_{L_1(-\infty,\infty)} \le c\frac{n}{\sqrt{m}}$$

and therefore

$$\big\|H_1 G^m_{s,n} f\big\|_{L_p(u)[-1,1]} \le c\frac{n}{\sqrt{m}} \|f\|_{L_p(u)[-1,1]}. \tag{10}$$

As $G_{s,n}$ commutes with the operator H_1, using estimation (10) we observe that

$$\begin{aligned} \big\|H^2 G^{4m}_{s,n} f\big\|_{L_p(u)[-1,1]} &\le c\,n^2 m^{-1} \big\|H G^{2m}_{s,n} f\big\|_{L_p(u)[-1,1]} \\ &= A\frac{\lambda(n)}{\lambda_1(n)} \big\|H G^{2m}_{s,n} f\big\|_{L_p(u)[-1,1]}, \end{aligned} \tag{11}$$

$$\big\|H G^{2m}_{s,n} f\big\|_{L_p(u)[-1,1]} \le c\,n^2 m^{-1} \|f\|_{L_p(u)[-1,1]} = B\frac{1}{\lambda(n)} \|f\|_{L_p(u)[-1,1]}. \tag{12}$$

Estimations (11) and (12) correspond to (3.5) and (3.6) from Theorem 3.1 in [2]. To match the conditions of Theorem 4.1 in [2], we need the constant A in (11) to satisfy $A < 1$. This is true for large m because

$$A = c\,n^2 m^{-1}\frac{\lambda_1(n)}{\lambda(n)} \sim c\,m^{-1} < 1.$$

From (8), (9), (11) and (12) we obtain

$$K\big(f, \frac{1}{n^2}; L_p(u)[-1,1], Y, H\big) \le c\|f - G_{s,n}f\|_{L_p(u)[-1,1]}.$$

To prove the direct result, we need (8) and the inequality

$$\|g - G_{s,n}g\|_{L_p(u)[-1,1]} \le c\frac{1}{n^2}\|Hg\|_{L_p(u)[-1,1]} \qquad \text{for } g \in Y \tag{13}$$

to be satisfied (see Theorem 3.4 in [2]).

Let $g \in Y$. We set $\widetilde{g}(z+v) := g(\cos(z+v)) = g(\cos(\arccos x + v))$, $z = \arccos x$. We have

$$(G_{s,n}g - g)(x) = \int_{-\pi}^{\pi} \big[\widetilde{g}(z+v) - \widetilde{g}(z)\big] K_{s,n}(v)\, dv.$$

Expanding $\widetilde{g}(z+v)$ by Taylor's formula (note that $\widetilde{g} \in C^2(\mathbb{R})$ by Lemma 4)

$$\widetilde{g}(z+v) - \widetilde{g}(z) = v\widetilde{g}'(z) + \int_{z}^{z+v} \widetilde{g}''(\xi)(z+v-\xi)\, d\xi,$$

and using that $\int_{-\pi}^{\pi} vK_{s,n}(v)\, dv = 0$, we obtain

$$(G_{s,n}g - g)(x) = \int_{-\pi}^{\pi}\int_{z}^{z+v} \widetilde{g}''(\xi)(z+v-\xi)\, d\xi K_{s,n}(v)\, dv.$$

We recall that $\widetilde{g}''(\xi) = (Hg)(\cos\xi)$. Since $z + v - \xi \in [0, v]$ for $\xi \in [z, z+v]$ and the sign of $\int_{z}^{z+v} |\widetilde{g}''(\xi)|\, d\xi$ is constant and coincides with the sign of v (if the integral is not zero), we get

$$\begin{aligned}
\|g - G_{s,n}g\|_{L_p(u)[-1,1]} &= \big\|\widetilde{g} - \widetilde{G}_{s,n}\widetilde{g}\big\|_{L_p[0,\pi]} \\
&= \Big\{\int_0^{\pi}\Big|\int_{-\pi}^{\pi}\int_{z}^{z+v} \widetilde{g}''(\xi)(z+v-\xi)\, d\xi K_{s,n}(v)\, dv\Big|^p dz\Big\}^{1/p} \\
&\le \Big\{\int_0^{\pi}\Big(\int_{-\pi}^{\pi} v^2 K_{s,n}(v)\Big|\frac{1}{v}\int_{z}^{z+v} |\widetilde{g}''(\xi)|\, d\xi\Big| dv\Big)^p dz\Big\}^{1/p}.
\end{aligned}$$

Using Minkowski's inequality, we estimate the last quantity as follows:

$$\begin{aligned}
&\Big\{\int_0^{\pi}\Big(\int_{-\pi}^{\pi} v^2 K_{s,n}(v)\Big|\frac{1}{v}\int_{z}^{z+v} |\widetilde{g}''(\xi)|\, d\xi\Big| dv\Big)^p dz\Big\}^{1/p} \\
&\qquad \le \int_{-\pi}^{\pi} v^2 K_{s,n}(v)\Big\|\frac{1}{v}\int_{z}^{z+v} |\widetilde{g}''(\xi)|\, d\xi\Big\|_{L_p[0,\pi]} dv \\
&\qquad \le \int_{-\pi}^{\pi} v^2 K_{s,n}(v)\, \|\widetilde{g}''\|_{L_p[0,\pi]}\, dv \\
&\qquad \le c\frac{1}{n^2}\big\|\widetilde{g}''\big\|_{L_p[0,\pi]} \\
&\qquad = c\frac{1}{n^2}\|Hg\|_{L_p(u)[-1,1]}.
\end{aligned}$$

In the second inequality above we have applied Lemma 3 to the even 2π-periodic function $\widetilde{g}''$ to obtain $\left\|\frac{1}{v}\int_z^{z+v}\left|\widetilde{g}''(\xi)\right|d\xi\right\|_{L_p[0,\pi]} \leq \left\|\widetilde{g}''\right\|_{L_p[0,\pi]}$. This proves (13) and completes the proof of the direct result

$$\|f - G_{s,n}f\|_{L_p(u)[-1,1]} \leq cK\big(f, \frac{1}{n^2}; L_p(u)[-1,1], C^2, H\big).$$

This concludes the proof of Theorem 1. □

Bibliography

[1] R. A. DeVore and G. G. Lorentz, "Constructive Approximation", Berlin, Springer-Verlag, 1993.

[2] Z. Ditzian and K. G. Ivanov, Strong converse inequalities, *J. Anal. Math.* **61** (1993), 61–111.

[3] K. G. Ivanov, A characterization theorem for the K-functional for Kantorovich and Durrmeyer operators, *in* "Approximation Theory" (G. Nikolov and R. Uluchev, Eds.), pp. 112–125, Prof. Marin Drinov Academic Publishing House, Sofia, 2004.

[4] T. Zapryanova, Approximation by operators of Cao-Gonska type $G_{s,n}$ and $G^*_{s,n}$. Direct and converse theorem, *in* "Mathematics and Education in Mathematics, 2008", pp. 189–194, Proc. of the 37th Spring Conf. of the UBM, Borovetz, April 2–6, 2008.

[5] T. Zapryanova, A characterization of the K-functional for the algebraic version of the trigonometric Jackson integrals $G_{s,n}$ and K-functionals for Cao-Gonska operators $G^*_{s,n}$ and $G^+_{s,n}$, *Results Math.* **54** (2009), 397–413.

T. Zapryanova
University of Economics – Varna
77 Knyaz Boris I Blvd.
9002 Varna
BULGARIA
E-mail: teodorazap@abv.bg