Complete Asymptotic Expansions for Bernstein-Chlodovsky Polynomials

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1. Introduction

Let \( f \) be a real function on \([0, \infty)\) which is bounded on every finite subinterval of \([0, \infty)\). For \( b > 0 \), we define the function \( f_b \) on \([0, 1]\) by \( f_b(t) = f(bt) \). Furthermore, we put

\[
\|f\|_b = \sup_{0 \leq t \leq b} |f(t)|.
\]

Obvioulsy, we have \( \|f\|_b = \|f_b\|_1 \).

The Bernstein-Chlodovsky polynomials are defined by

\[
(C_{n,b}f)(x) = (B_n f_b) \left( \frac{x}{b} \right),
\]

where \( B_n \) stands for the Bernstein polynomials

\[
(B_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f \left( \frac{\nu}{n} \right),
\]

with Bernstein basis polynomials

\[
p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}, \quad 0 \leq \nu \leq n.
\]

Obvisously, we have \( C_{n,1} \equiv B_n \).

In the following we suppose that parameter \( b \) depends on \( n \), i.e., \( b = b_n \). Since the difference between two nodes of \( C_{n,b} \) is at least \( b/n \), it is clear that the condition \( b_n = o(n) \) as \( n \to \infty \) is necessary for having convergence of \( (C_{n,b_n}f)(x) \) to \( f(x) \). Throughout the paper we assume that the sequence \( (b_n) \) satisfies

\[
b_n > 0, \quad \lim_{n \to \infty} b_n = \infty, \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{n} = 0. \quad (1)
\]
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These polynomials were introduced by Chlodovsky [8] in 1937 in order to approximate functions on infinite intervals. He showed that under condition (1), if a function $f$ satisfies

$$\lim_{n \to \infty} \exp \left( - \frac{\sigma n}{b_n} \right) \|f\|_{b_n} = 0 \quad \text{for every } \sigma > 0,$$

(2)

then

$$\lim_{n \to \infty} (C_{n,b_n}f)(x) = f(x)$$

at each point $x$ of continuity of $f$. Moreover, he proved convergence in each continuity point for the wide class of functions $f$ satisfying the growth condition $f(t) = O(\exp(t^p))$ as $t \to \infty$, if the sequence $(b_n)$ satisfies the condition

$$b_n = O(n^{1/(p+1)+\eta}), \quad (n \to \infty),$$

(3)

for an arbitrary small $\eta > 0$. For more results on Chlodovsky operators see the survey article [12] by Karsli.

The purpose of this note is a pointwise complete asymptotic expansion for the sequence of Bernstein-Chlodovsky operators of the form

$$(C_{n,b_n}f)(x) \sim f(x) + \sum_{k=1}^{\infty} c_k^{b_n}(f,x) \left( \frac{b_n}{n} \right)^k, \quad (n \to \infty),$$

for sufficiently smooth functions $f$ satisfying $f(t) = O(\exp(t^p))$ as $t \to \infty$, provided that the sequence $(b_n)$ satisfies $b_n = o(n^{1/(p+1)})$ as $n \to \infty$. Note that the latter condition is slightly weaker than (3). The coefficients $c_k^{b_n}(f,x)$, which depend on $f$ and $b_n$, are bounded with respect to $n$.

The latter formula means that, for each fixed $x > 0$ and for all positive integers $q$,

$$(C_{n,b_n}f)(x) = f(x) + \sum_{k=1}^{q} c_k^{b_n}(f,x) \left( \frac{b_n}{n} \right)^k + o\left( \left( \frac{b_n}{n} \right)^q \right), \quad (n \to \infty).$$

Explicit expressions for coefficients $c_k^{b_n}(f,x)$ in terms of the Stirling numbers were given by Karsli [13]. He derived the asymptotic expansion if the function $f$ satisfies condition (2) for every $\sigma > 0$.

Finally, we announce the corresponding result for the Durrmeyer variant of the Bernstein-Chlodovsky operators given by

$$(\tilde{C}_{n,b}f)(x) = (M_n f_b)\left( \frac{x}{b} \right),$$

where $M_n, n \in \mathbb{N}_0$, are the Bernstein-Durrmeyer operators

$$(M_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x)(n+1) \int_0^1 p_{n,\nu}(t)f(t) \, dt, \quad x \in [0,1].$$
2. The Main Result

For real constants $\alpha \geq 0$ and $p \geq 0$, let $W_{\alpha,p}$ denote the class of functions $f \in C[0, \infty)$ satisfying the growth condition

$$f(t) = \mathcal{O}(\exp(\alpha t^p)), \quad (t \to \infty).$$

Note that in the special case $p = 0$ the class $W_{\alpha,0}$ consists of the bounded continuous functions on $[0, \infty)$. Since $W_{0,p}$ and $W_{\alpha,0}$ coincide, we consider only the case $\alpha > 0$.

Recall that the Stirling numbers $s(n,k)$ and $S(n,k)$ of the first and the second kind, respectively, are defined by the relations

$$z^n = \sum_{k=0}^{n} s(n,k) z^k$$

and

$$z^n = \sum_{k=0}^{n} S(n,k) z^k, \quad (z \in \mathbb{C}),$$

where $z^0 = 1$ and $z^n = z(z-1)\cdots(z-n+1)$, for $n \in \mathbb{N}$, denote the falling factorials.

The following theorem is our main result.

**Theorem 1.** Let $\alpha, p \geq 0$. Suppose that the function $f \in W_{\alpha,p}$ is $2q$-times differentiable at the point $x > 0$. Let $(b_n)$ be a sequence of positive reals satisfying the growth condition

$$b_n = o(n^{1/(p+1)}), \quad (n \to \infty).$$

Then, for any positive integer $q$, the Bernstein-Chlodovsky operators $C_{n,b_n}$ possess the asymptotic expansion

$$(C_{n,b_n}f)(x) = f(x) + \sum_{k=1}^{q} c_k^{[b_n]}(f,x) \left( \frac{b_n}{n} \right)^k + o\left( \left( \frac{b_n}{n} \right)^q \right), \quad (n \to \infty).$$

where

$$c_k^{[b_n]}(f,x) = \mathcal{O}(1), \quad (n \to \infty).$$

The coefficients $c_k^{[b_n]}(f,x)$ have the explicit representation

$$c_k^{[b_n]}(f,x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{s} a(k,s,j)b_n^{j-k} x^{s-j}$$

with

$$a(k,s,j) = \sum_{r=\max\{j,k\}}^{s} (-1)^{s-r} \binom{s}{r} s(r,j,r-k) S(r,r-j).$$

**Remark 1.** Note that the coefficients $c_k^{[b_n]}(f,x)$ depend on $n$ but are bounded with respect to $n$. 
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Remark 2. Our assumption (4) on \( (b_n) \) is weaker than Chlodovsky’s condition (3).

Remark 3. Explicit formulas for the coefficients \( c_k^{[b_n]}(f,x) \), for \( k = 1, 2, 3 \), can be found in [13, p. 1220].

Remark 4. Since \( c_1^{[b_n]}(f,x) = \frac{f''(x)}{2n} x(b_n - x) \), the special case \( q = 1 \), i.e.,

\[
(C_n, b_n f)(x) = f(x) + \frac{f''(x)}{2n} x(b_n - x) + o\left(\frac{b_n}{n}\right), \quad (n \to \infty),
\]

contains the Voronovskaja-type result

\[
\lim_{n \to \infty} \frac{n}{b_n} ((C_n, b_n f)(x) - f(x)) = \frac{1}{2} x f''(x)
\]

by Albrycht and Radecki [6]. They proved the latter formula for the subclass of functions \( f \) satisfying the growth condition (2).

Remark 5. When taking \( b_n = 1 \) for all \( n \in \mathbb{N} \), the expansion in Theorem 1 reduces to the (pointwise) complete asymptotic expansion

\[
(B_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} c_k^{[1]}(f,x) n^{-k} + o(n^{-q}), \quad (n \to \infty),
\]

for the classical Bernstein polynomials, which is valid for all bounded functions \( f : [0, 1] \to \mathbb{R} \) being sufficiently smooth in \( x \in [0, 1] \). Note that the Voronovskaja formula

\[
\lim_{n \to \infty} n ((B_n f)(x) - f(x)) = \frac{1}{2} x(1-x)f''(x)
\]

is different from Eq. (7).

3. Auxiliary Results and Proof of the Main Theorem

Our starting-point is an explicit representation of the central moments of the Bernstein polynomials in terms of Stirling numbers of the first and second kind. In the following we write \( e_m(x) = x^m \), \( m \in \mathbb{N}_0 \), for the \( m \)-th monomial and \( \psi_x(t) = t - x \) for \( x \in \mathbb{R} \).

Lemma 1. The central moments of the Bernstein polynomials possess the representation

\[
(B_n \psi_x^s)(x) = \sum_{k=\lceil (s+1)/2 \rceil}^s n^{-k} \sum_{j=0}^s a(k,s,j)x^{s-j}, \quad (s = 0, 1, 2, \ldots),
\]

where coefficients \( a(k,s,j) \) are given by Eq. (6).
Lemma 2. The central moments of the Bernstein-Chlodovsky operators possess the representation
\[ (C_{n,b}^s \psi \chi(x)) = \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} a(k, s, j) b_j x^{s-j}, \]
where coefficients \( a(k, s, j) \) are given by Eq. (6). Furthermore, for \( k \leq s \leq 2k \), there holds
\[ \sum_{j=0}^{s} a(k, s, j) b_j^s x^{s-j} = O(b_n^k), \quad (n \to \infty). \] 

Proof. We have
\[ (C_{n,b}^s \psi \chi(x)) = \sum_{\nu=0}^{n} p_{n,\nu} \left( \frac{b}{n} \right)^{\nu} \left( b_{\nu} - x \right)^s = b^s \sum_{\nu=0}^{n} p_{n,\nu} \left( \frac{b}{n} \right)^{\nu} \left( \frac{n}{b} - \frac{x}{b} \right)^s = b^s \left( B_{n,b}^s \psi \chi(b) \right) \left( \frac{x}{b} \right), \]
and the first part of the lemma follows by Lemma 1. For the second part, we assume that \( j > k \). Then, by Eq. (6),
\[ a(k, s, j) = \sum_{r=j}^{s} (-1)^{s-r} \begin{pmatrix} s \\ r \end{pmatrix} s(r-j, r-k) S(r, r-j) = 0, \]
because \( s(r-j, r-k) = 0 \), if \( r-j < r-k \). This proves Eq. (8). \( \square \)

As a consequence we obtain the following result:

Lemma 3 (Butzer and Karsli [7]). For \( s = 0, 1, 2, \ldots \), there holds
\[ (C_{n,b}^s \psi \chi(x)) = O\left( \left( \frac{b_n}{n} \right)^{\lfloor (s+1)/2 \rfloor} \right), \quad (n \to \infty). \]

A crucial tool is the following estimate due to Bernstein (see [14, Theorem 1.5.3, p. 18ff]).

Lemma 4 (Bernstein). For \( 0 \leq t \leq 1 \), the inequality
\[ 0 \leq z \leq \frac{3}{2} \sqrt{nt(1-t)} \]
implies
\[ \sum_{|\nu-nt| \geq 2z \sqrt{nt(1-t)}} p_{n,\nu}(t) \leq 2 \exp(-z^2). \]
The next lemma presents a form of Lemma 4, which is more useful for application to Chlodovsky operators.

**Lemma 5 (Albrycht and Radecki [6]).** If $b > 0$ and $0 < \delta < x \leq 2b/3$, then there holds

$$\sum_{|b\nu - x| \geq \delta} p_{n,\nu}(\frac{x}{b}) \leq 2 \exp\left(-\frac{n\delta^2}{4xb}\right).$$

Since the paper [6] is hardly available, for sake of completeness we give a proof.

**Proof of Lemma 5.** By putting $t = x/b$ in Lemma 4, we have

$$\sum_{|\nu - n\frac{x}{b}| \geq 2z\sqrt{n\frac{b}{b-x}(1-x)}} p_{n,\nu}(\frac{x}{b}) \leq 2 \exp(-z^2)$$

if $0 \leq z \leq \frac{3}{2}\sqrt{n\frac{b}{b-x}(1-x)}$. Put $\delta = 2z\sqrt{n^{-1}x(b-x)}$, then

$$\sum_{|b\nu - x| \geq \delta} p_{n,\nu}(\frac{x}{b}) \leq 2 \exp\left(-\frac{n\delta^2}{4xb}\right)$$

if $0 \leq \delta/(2\sqrt{n^{-1}x(b-x)}) \leq \frac{3}{2}\sqrt{n\frac{b}{b-x}(1-x)}$. The latter inequality is equivalent to

$$\delta \leq 3x\left(1 - \frac{x}{b}\right).$$

By assumption, we have $3x(1-x/b) \geq x$. The assertion follows by observing that

$$\exp\left(-\frac{n\delta^2}{4xb}\right) \leq \exp\left(-\frac{n\delta^2}{4xb}\right).$$

**Lemma 6.** Let $b > 0$ and $0 < \delta < x \leq 2b/3$. If a bounded function $f : [0, b] \to \mathbb{R}$ satisfies $f(t) = 0$ for all $t \in (x - \delta, x + \delta) \cap [0, b]$, then the following estimate holds true:

$$|(C_{n,b}f)(x)| \leq 2 \exp\left(-\frac{n\delta^2}{4xb}\right)\|f\|_b.$$

**Proof.** Since $f(b\frac{\nu}{n}) = 0$ for all $\nu \in \{1, \ldots, n\}$ with $|b\frac{\nu}{n} - x| < \delta$, we have

$$|(C_{n,b}f)(x)| = \left|\sum_{|b\frac{\nu}{n} - x| \geq \delta} p_{n,\nu}(\frac{x}{b}) f(b\frac{\nu}{n})\right| \leq \|f\|_b \sum_{\nu} p_{n,\nu}(\frac{x}{b}),$$

and the assertion follows by application of Lemma 5. □

A direct consequence is the following localization result for Bernstein–Chlodovsky polynomials, which is interesting in itself.
Proposition 1 (Localization theorem). Let $\alpha, p \geq 0$ be fixed constants and suppose that $f \in W_{\alpha,p}$ satisfies
\[ |f(t)| \leq K \exp(\alpha t^p), \quad (t \geq 0). \]
If for a fixed $x > 0$ and $\delta > 0$, $f(t) = 0$ for all $t \in (x - \delta, x + \delta) \cap [0, \infty)$, then
\[ (C_{n,b} f)(x) = 2K \exp\left(\alpha b^p n - \frac{n\delta^2}{4x b_n}\right), \quad (n \to \infty). \]

Proof of Theorem 1. Suppose that $f$ is continuous on $[0, \infty)$ being $2q$-times differentiable at the point $x > 0$. Define the function $h_x$ by
\[ f = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \psi_x^s + h_x \psi_x^{2q} \quad (9) \]
and $h_x(0) = 0$. It is a consequence of Taylor’s theorem that $h_x$ is continuous at $x$. Hence, $h_x \in C[0, \infty)$. Applying the operator $C_{n,b}$ to both sides of Eq. (9) we obtain
\[ (C_{n,b} f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (C_{n,b} \psi_x^s)(x) + (C_{n,b} (h_x \psi_x^{2q}))(x). \]
The sum in the right-hand side is equal to
\[ \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (C_{n,b} \psi_x^s)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} n^{-k} \sum_{j=0}^{s} a(k, s, j) b^j x^{s-j} \]
\[ = \sum_{k=0}^{2q} n^{-k} \sum_{s=k}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{s} a(k, s, j) b^j x^{s-j} \]
\[ = \sum_{k=0}^{2q} \binom{b}{k} (f, x) \left(\frac{b}{n}\right)^k. \]
Note that $a_0^0(f, x) = 1$. Eq. (5) is a consequence of Eq. (8). We conclude that
\[ \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (C_{n,b} \psi_x^s)(x) = \sum_{k=0}^{q} \binom{b_n}{k} (f, x) \left(\frac{b_n}{n}\right)^k + o\left(\left(\frac{b_n}{n}\right)^q\right), \quad (n \to \infty). \]
In order to complete the proof, we have to show that the remainder satisfies
\[ (C_{n,b} (h_x \psi_x^{2q}))(x) = o\left(\left(\frac{b_n}{n}\right)^q\right), \quad (n \to \infty). \]
To this end let $(\delta_n)$ be a sequence of positive numbers such that
\[ \delta_n^2 = 4x \left(\alpha \frac{b_n^{p+1}}{n} - \frac{b_n}{n} \log \frac{b_n}{n} + \left(\frac{b_n}{n}\right)^{1/2}\right), \quad (n \in \mathbb{N}). \quad (10) \]
Note that conditions (1) and (4) imply that $\delta_n = o(1)$ as $n \to \infty$. Define
\[ \epsilon_n = \sup \{ |h_x(t)| : t \in (x - \delta_n, x + \delta_n) \cap [0, +\infty) \} \].
Since $h_x$ is continuous with $h_x(x) = 0$, we have $\epsilon_n = o(1)$ as $n \to \infty$. We split the remainder into two parts
\[ (C_{n,b_n} (h_x \psi_x^{2q}))(x) = \sum_{\nu} p_{n,\nu} \left( \frac{x}{b_n} \right) \left( h_x \psi_x^{2q} \right) \left( b_n \frac{\nu}{n} \right) \]

Let us start with the estimate of the first sum:
\[ \left| \sum_1 \right| \leq \epsilon_n \sum_{\nu} p_{n,\nu} \left( \frac{x}{b_n} \right) \psi_x^{2q} \left( b_n \frac{\nu}{n} \right) \]

and we obtain
\[ \left| \sum_2 \right| \leq 2 \exp \left( -\frac{n b_n^2}{4 x b_n} \right) \left( \| f \|_{b_n} + \sum_{s=0}^{2q} \frac{|f^{(s)}(x)|}{s!} b_n^s \right) \]

where in the last step Lemma 5 was applied. Note that
\[ \sum_{s=0}^{2q} \frac{|f^{(s)}(x)|}{s!} b_n^s = O(b_n^{2q}), \quad (n \to \infty). \]

Hence,
\[ \sum_2 = O \left( \exp \left( ab_n^p - \frac{n b_n^2}{4 x b_n} \right) \right) + O \left( \exp \left( 2q \log b_n - \frac{n b_n^2}{4 x b_n} \right) \right), \quad (n \to \infty). \]
In the case \( p = 0 \), i.e., \( f \) is bounded on \([0, \infty)\), we have
\[
\sum_2 = O\left( \exp \left( 2q \log b_n - \frac{n \delta_n^2}{2xb_n} \right) \right), \quad (n \to \infty).
\]
We may assume that \( \alpha > 0 \). Therefore, in the case \( p > 0 \), we have
\[
\sum_2 = O\left( \exp \left( \alpha b_n - \frac{n \delta_n^2}{2xb_n} \right) \right), \quad (n \to \infty).
\]
Obviously, it is sufficient to estimate the latter expression. By Eq. (10) we infer that
\[
\sum_2 = O\left( \exp \left( q \log \frac{b_n}{n} - \left( \frac{n}{b_n} \right)^{1/2} \right) \right) = O\left( \left( \frac{b_n}{n} \right)^q e^{-\sqrt{n/b_n}} \right), \quad (n \to \infty).
\]
Finally, we conclude that the remainder can be estimated by
\[
(C_{n,b_n}(h_x \psi_x^q))(x) = o\left( \left( \frac{b_n}{n} \right)^q \right), \quad (n \to \infty),
\]
which completes the proof of the theorem. \( \square \)

4. Bernstein-Durrmeyer-Chlodovsky Polynomials

The Bernstein-Durrmeyer operators \( M_n, n \in \mathbb{N}_0 \), were introduced by Durrmeyer [10] and independently by Lupaş [15] in order to approximate integrable functions on finite intervals. For a function \( f \in L^1[0,1] \) they are defined by
\[
(M_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x)(n+1) \int_0^1 p_{n,\nu}(t)f(t) \, dt, \quad x \in [0,1],
\]
where \( p_{n,\nu} \) denote the Bernstein basis polynomials.

In [2, Theorem 1] one of the authors derived a complete asymptotic expansion for the Bernstein-Durrmeyer operators as \( n \to \infty \). The representation is given in terms of reciprocals of \((n+2)^{\gamma}\). The rising factorials are defined by
\[
z^n = 1 \quad \text{and} \quad z^{\gamma} = z(z+1) \cdots (z+n-1), \quad \text{for} \ n \in \mathbb{N}.
\]

**Theorem 2.** Let \( q \in \mathbb{N} \). Then for every function \( f \in L^\infty[0,1] \) which is \( 2q \)-times differentiable at \( x \in [0,1] \), the Bernstein-Durrmeyer operators \( M_n \) satisfy the asymptotic relation
\[
(M_n f)(x) = f(x) + \sum_{k=1}^{q} \frac{1}{k!(n+2)^{\gamma}} \left[ x^k (1-x)^k f^{(k)}(x) \right]^{(k)} + o(n^{-q})
\]
as \( n \to \infty \).
Generalizations for the weighted one-dimensional and multivariate Bernstein-Durrmeyer operators were obtained in [3, 4, 5].

The Bernstein-Durrmeyer-Chlodovsky polynomials are defined by

\[
(\tilde{C}_{n,b} f)(x) = (M_{n,b}) \left( \frac{x}{b_n} \right).
\]

Without giving a proof, we announce the following result.

**Theorem 3.** Let \(\alpha, p \geq 0\). Suppose that the function \(f \in W_{\alpha,p}\) is \(2q\)-times differentiable at the point \(x > 0\). Let \((b_n)\) be a sequence of positive reals, which in the case \(p > 0\) satisfies the growth condition

\[
b_n = o\left( n^{1/(p+1)} \right), \quad (n \to \infty),
\]

while in the case \(p = 0\), \((b_n)\) satisfies the slightly stronger condition

\[
b_n = o\left( \frac{n}{\log n} \right), \quad (n \to \infty).
\]

Then, for any positive integer \(q\), the Bernstein-Durrmeyer-Chlodovsky operators \(\tilde{C}_{n,b_n}\) possess the asymptotic expansion

\[
(\tilde{C}_{n,b_n} f)(x) = f(x) + \sum_{k=1}^{q} \tilde{c}_{[b_n]}^k (f, x) \frac{b_n^k}{(n+2)^k} + o\left( \left( \frac{b_n}{n} \right)^q \right), \quad (n \to \infty),
\]

where

\[
\tilde{c}_{[b_n]}^k (f, x) = \frac{1}{k!} \left( x^k \left( 1 - \frac{x}{b_n} \right)^k \right)^{(k)}.
\]

**Remark 6.** Note that Eq. (13) implies that \(\tilde{c}_{[b_n]}^k (f, x) = \mathcal{O}(1)\) as \(n \to \infty\).

**Remark 7.** Clearly, if condition (11) is fulfilled with some \(p > 0\), then condition (12) is satisfied, too.

**References**


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