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Complete Asymptotic Expansions for Bernstein-Chlodovsky Polynomials

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1. Introduction

Let f be a real function on $[0,\infty)$ which is bounded on every finite subinterval of $[0,\infty)$. For $b>0$, we define the function f_b on $[0,1]$ by $f_b(t)=f(bt)$. Furthermore, we put

$$\|f\|_b = \sup_{0\le t\le b} |f(t)|.$$

Obviously, we have $\|f\|_b = \|f_b\|_1$.

The Bernstein-Chlodovsky polynomials are defined by

$$(C_{n,b}f)(x) = (B_n f_b)\Big(\frac{x}{b}\Big),$$

where B_n stands for the Bernstein polynomials

$$(B_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\Big(\frac{\nu}{n}\Big),$$

with Bernstein basis polynomials

$$p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}, \qquad 0\le \nu\le n.$$

Obviously, we have $C_{n,1}\equiv B_n$.

In the following we suppose that parameter b depends on n, i.e., $b=b_n$. Since the difference between two nodes of $C_{n,b}$ is at least b/n, it is clear that the condition $b_n = o(n)$ as $n\to\infty$ is necessary for having convergence of $(C_{n,b_n}f)(x)$ to $f(x)$. Throughout the paper we assume that the sequence (b_n) satisfies

$$b_n>0, \quad \lim_{n\to\infty} b_n = \infty, \qquad \text{and} \qquad \lim_{n\to\infty}\frac{b_n}{n} = 0. \tag{1}$$

These polynomials were introduced by Chlodovsky [8] in 1937 in order to approximate functions on infinite intervals. He showed that under condition (1), if a function f satisfies

$$\lim_{n\to\infty} \exp\Big(-\frac{\sigma n}{b_n}\Big)\|f\|_{b_n} = 0 \quad \text{for every } \sigma > 0, \tag{2}$$

then

$$\lim_{n\to\infty} (C_{n,b_n} f)(x) = f(x)$$

at each point x of continuity of f. Moreover, he proved convergence in each continuity point for the wide class of functions f satisfying the growth condition $f(t) = \mathcal{O}(\exp(t^p))$ as $t \to \infty$, if the sequence (b_n) satisfies the condition

$$b_n = \mathcal{O}\big(n^{1/(p+1+\eta)}\big), \qquad (n \to \infty), \tag{3}$$

for an arbitrary small $\eta > 0$. For more results on Chlodovsky operators see the survey article [12] by Karsli.

The purpose of this note is a pointwise complete asymptotic expansion for the sequence of Bernstein-Chlodovsky operators of the form

$$(C_{n,b_n} f)(x) \sim f(x) + \sum_{k=1}^{\infty} c_k^{[b_n]}(f,x)\Big(\frac{b_n}{n}\Big)^k, \qquad (n \to \infty),$$

for sufficiently smooth functions f satisfying $f(t) = \mathcal{O}(\exp(\alpha t^p))$ as $t \to \infty$, provided that the sequence (b_n) satisfies $b_n = o\big(n^{1/(p+1)}\big)$ as $n \to \infty$. Note that the latter condition is slightly weaker than (3). The coefficients $c_k^{[b_n]}(f,x)$, which depend on f and b_n, are bounded with respect to n.

The latter formula means that, for each fixed $x > 0$ and for all positive integers q,

$$(C_{n,b_n} f)(x) = f(x) + \sum_{k=1}^{q} c_k^{[b_n]}(f,x)\Big(\frac{b_n}{n}\Big)^k + o\Big(\Big(\frac{b_n}{n}\Big)^q\Big), \qquad (n \to \infty).$$

Explicit expressions for coefficients $c_k^{[b_n]}(f,x)$ in terms of the Stirling numbers were given by Karsli [13]. He derived the asymptotic expansion if the function f satisfies condition (2) for every $\sigma > 0$.

Finally, we announce the corresponding result for the Durrmeyer variant of the Bernstein-Chlodovsky operators given by

$$\big(\tilde{C}_{n,b} f\big)(x) = (M_n f_b)\Big(\frac{x}{b}\Big),$$

where M_n, $n \in \mathbb{N}_0$, are the Bernstein-Durrmeyer operators

$$(M_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x)(n+1)\int_0^1 p_{n,\nu}(t) f(t)\,dt, \qquad x \in [0,1].$$

2. The Main Result

For real constants $\alpha \geq 0$ and $p \geq 0$, let $W_{\alpha,p}$ denote the class of functions $f \in C[0,\infty)$ satisfying the growth condition

$$f(t) = \mathcal{O}(\exp(\alpha t^p)), \qquad (t \to \infty).$$

Note that in the special case $p = 0$ the class $W_{\alpha,0}$ consists of the bounded continuous functions on $[0,\infty)$. Since $W_{0,p}$ and $W_{\alpha,0}$ coincide, we consider only the case $\alpha > 0$.

Recall that the Stirling numbers $s(n,k)$ and $S(n,k)$ of the first and the second kind, respectively, are defined by the relations

$$z^{\underline{n}} = \sum_{k=0}^{n} s(n,k) z^k \qquad \text{and} \qquad z^n = \sum_{k=0}^{n} S(n,k) z^{\underline{k}}, \qquad (z \in \mathbb{C}),$$

where $z^{\underline{0}} = 1$ and $z^{\underline{n}} = z(z-1)\cdots(z-n+1)$, for $n \in \mathbb{N}$, denote the falling factorials.

The following theorem is our main result.

Theorem 1. *Let α, $p \geq 0$. Suppose that the function $f \in W_{\alpha,p}$ is $2q$-times differentiable at the point $x > 0$. Let (b_n) be a sequence of positive reals satisfying the growth condition*

$$b_n = o\big(n^{1/(p+1)}\big), \qquad (n \to \infty). \tag{4}$$

Then, for any positive integer q, the Bernstein-Chlodovsky operators C_{n,b_n} possess the asymptotic expansion

$$(C_{n,b_n} f)(x) = f(x) + \sum_{k=1}^{q} c_k^{[b_n]}(f,x) \Big(\frac{b_n}{n}\Big)^k + o\Big(\Big(\frac{b_n}{n}\Big)^q\Big), \qquad (n \to \infty).$$

where

$$c_k^{[b_n]}(f,x) = \mathcal{O}(1), \qquad (n \to \infty). \tag{5}$$

The coefficients $c_k^{[b_n]}(f,x)$ have the explicit representation

$$c_k^{[b_n]}(f,x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{s} a(k,s,j) b_n^{j-k} x^{s-j}$$

with

$$a(k,s,j) = \sum_{r=\max\{j,k\}}^{s} (-1)^{s-r} \binom{s}{r} s(r-j, r-k) S(r, r-j). \tag{6}$$

Remark 1. Note that the coefficients $c_k^{[b_n]}(f,x)$ depend on n but are bounded with respect to n.

Remark 2. Our assumption (4) on (b_n) is weaker than Chlodovsky's condition (3).

Remark 3. Explicit formulas for the coefficients $c_k^{[b_n]}(f,x)$, for $k = 1, 2, 3$, can be found in [13, p. 1220].

Remark 4. Since $c_1^{[b_n]}(f,x) = \frac{f''(x)}{2!b_n}\, x(b_n - x)$, the special case $q = 1$, i.e.,

$$(C_{n,b_n} f)(x) = f(x) + \frac{f''(x)}{2n}\, x(b_n - x) + o\Big(\frac{b_n}{n}\Big), \qquad (n \to \infty),$$

contains the Voronovskaja-type result

$$\lim_{n\to\infty} \frac{n}{b_n} \big((C_{n,b_n} f)(x) - f(x)\big) = \frac{1}{2}\, x f''(x) \tag{7}$$

by Albrycht and Radecki [6]. They proved the latter formula for the subclass of functions f satisfying the growth condition (2).

Remark 5. When taking $b_n = 1$ for all $n \in \mathbb{N}$, the expansion in Theorem 1 reduces to the (pointwise) complete asymptotic expansion

$$(B_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} c_k^{[1]}(f,x)\, n^{-k} + o(n^{-q}), \qquad (n \to \infty),$$

for the classical Bernstein polynomials, which is valid for all bounded functions $f : [0,1] \to \mathbb{R}$ being sufficiently smooth in $x \in [0,1]$. Note that the Voronovskaja formula

$$\lim_{n\to\infty} n\big((B_n f)(x) - f(x)\big) = \frac{1}{2}\, x(1-x) f''(x)$$

is different from Eq. (7).

3. Auxiliary Results and Proof of the Main Theorem

Our starting-point is an explicit representation of the central moments of the Bernstein polynomials in terms of Stirling numbers of the first and second kind. In the following we write $e_m(x) = x^m$, $m \in \mathbb{N}_0$, for the m-th monomial and $\psi_x(t) = t - x$ for $x \in \mathbb{R}$.

Lemma 1. *The central moments of the Bernstein polynomials possess the representation*

$$\big(B_n \psi_x^s\big)(x) = \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} n^{-k} \sum_{j=0}^{s} a(k,s,j) x^{s-j}, \qquad (s = 0, 1, 2, \ldots),$$

where coefficients $a(k,s,j)$ are given by Eq. (6).

For a proof see, e.g., [1].

Lemma 2. *The central moments of the Bernstein-Chlodovsky operators possess the representation*

$$\left(C_{n,b}\psi_x^s\right)(x) = \sum_{k=\lfloor (s+1)/2\rfloor}^{s} n^{-k} \sum_{j=0}^{s} a(k,s,j) b^j x^{s-j},$$

where coefficients $a(k,s,j)$ are given by Eq. (6)*. Furthermore, for $k \le s \le 2k$, there holds*

$$\sum_{j=0}^{s} a(k,s,j) b_n^j x^{s-j} = \mathcal{O}\left(b_n^k\right), \qquad (n \to \infty). \tag{8}$$

Proof. We have

$$\begin{aligned}\left(C_{n,b}\psi_x^s\right)(x) &= \sum_{\nu=0}^{n} p_{n,\nu}\left(\frac{x}{b}\right)\left(b\,\frac{\nu}{n} - x\right)^s = b^s \sum_{\nu=0}^{n} p_{n,\nu}\left(\frac{x}{b}\right)\left(\frac{\nu}{n} - \frac{x}{b}\right)^s \\ &= b^s\left(B_n\psi_{x/b}^s\right)\left(\frac{x}{b}\right)\end{aligned}$$

and the first part of the lemma follows by Lemma 1. For the second part, we assume that $j > k$. Then, by Eq. (6),

$$a(k,s,j) = \sum_{r=j}^{s} (-1)^{s-r}\binom{s}{r} s(r-j, r-k) S(r, r-j) = 0,$$

because $s(r-j, r-k) = 0$, if $r - j < r - k$. This proves Eq. (8). □

As a consequence we obtain the following result:

Lemma 3 (Butzer and Karsli [7]). *For $s = 0, 1, 2, \ldots$, there holds*

$$\left(C_{n,b_n}\psi_x^s\right)(x) = \mathcal{O}\left(\left(\frac{b_n}{n}\right)^{\lfloor (s+1)/2\rfloor}\right), \qquad (n \to \infty).$$

A crucial tool is the following estimate due to Bernstein (see [14, Theorem 1.5.3, p. 18ff]).

Lemma 4 (Bernstein). *For $0 \le t \le 1$, the inequality*

$$0 \le z \le \frac{3}{2}\sqrt{nt(1-t)}$$

implies

$$\sum_{|\nu - nt| \ge 2z\sqrt{nt(1-t)}} p_{n,\nu}(t) \le 2\exp(-z^2).$$

The next lemma presents a form of Lemma 4, which is more useful for application to Chlodovsky operators.

Lemma 5 (Albrycht and Radecki [6]). *If* $b > 0$ *and* $0 < \delta < x \le 2b/3$, *then there holds*

$$\sum_{|b\frac{\nu}{n}-x|\ge\delta} p_{n,\nu}\Big(\frac{x}{b}\Big) \le 2\exp\Big(-\frac{n\delta^2}{4xb}\Big).$$

Since the paper [6] is hardly available, for sake of completeness we give a proof.

Proof of Lemma 5. By putting $t = x/b$ in Lemma 4, we have

$$\sum_{|\nu-n\frac{x}{b}|\ge 2z\sqrt{n\frac{x}{b}\left(1-\frac{x}{b}\right)}} p_{n,\nu}\Big(\frac{x}{b}\Big) = \sum_{|b\frac{\nu}{n}-x|\ge 2z\sqrt{n^{-1}x(b-x)}} p_{n,\nu}\Big(\frac{x}{b}\Big) \le 2\exp(-z^2)$$

if $0 \le z \le \frac{3}{2}\sqrt{n\frac{x}{b}\left(1-\frac{x}{b}\right)}$. Put $\delta = 2z\sqrt{n^{-1}x(b-x)}$, then

$$\sum_{|b\frac{\nu}{n}-x|\ge\delta} p_{n,\nu}\Big(\frac{x}{b}\Big) \le 2\exp\Big(-\frac{n\delta^2}{4x(b-x)}\Big)$$

if $0 \le \delta/\big(2\sqrt{n^{-1}x(b-x)}\big) \le \frac{3}{2}\sqrt{n\frac{x}{b}\left(1-\frac{x}{b}\right)}$. The latter inequality is equivalent to

$$\delta \le 3x\Big(1-\frac{x}{b}\Big).$$

By assumption, we have $3x(1-x/b) \ge x$. The assertion follows by observing that

$$\exp\Big(-\frac{n\delta^2}{4x(b-x)}\Big) \le \exp\Big(-\frac{n\delta^2}{4xb}\Big).$$

□

Lemma 6. *Let* $b > 0$ *and* $0 < \delta < x \le 2b/3$. *If a bounded function* $f : [0,b] \to \mathbb{R}$ *satisfies* $f(t) = 0$ *for all* $t \in (x-\delta, x+\delta) \cap [0,b]$, *then the following estimate holds true:*

$$|(C_{n,b}f)(x)| \le 2\exp\Big(-\frac{n\delta^2}{4xb}\Big)\|f\|_b\,.$$

Proof. Since $f\big(b\frac{\nu}{n}\big) = 0$ for all $\nu \in \{1,\dots,n\}$ with $\big|b\frac{\nu}{n}-x\big| < \delta$, we have

$$|(C_{n,b}f)(x)| = \Bigg|\sum_{|b\frac{\nu}{n}-x|\ge\delta} p_{n,\nu}\Big(\frac{x}{b}\Big) f\Big(b\,\frac{\nu}{n}\Big)\Bigg| \le \|f\|_b \sum_{\substack{\nu \\ |b\frac{\nu}{n}-x|\ge\delta}} p_{n,\nu}\Big(\frac{x}{b}\Big),$$

and the assertion follows by application of Lemma 5. □

A direct consequence is the following localization result for Bernstein–Chlodovsky polynomials, which is interesting in itself.

Proposition 1 (Localization theorem). *Let α, $p \geq 0$ be fixed constants and suppose that $f \in W_{\alpha,p}$ satisfies*

$$|f(t)| \leq K \exp(\alpha t^p), \qquad (t \geq 0).$$

If for a fixed $x > 0$ and $\delta > 0$, $f(t) = 0$ for all $t \in (x - \delta, x + \delta) \cap [0, \infty)$, then

$$(C_{n,b_n} f)(x) = 2K \exp\Big(\alpha b_n^p - \frac{n\delta^2}{4xb_n}\Big), \qquad (n \to \infty).$$

Proof of Theorem 1. Suppose that f is continuous on $[0, \infty)$ being $2q$-times differentiable at the point $x > 0$. Define the function h_x by

$$f = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \psi_x^s + h_x \psi_x^{2q} \tag{9}$$

and $h_x(x) = 0$. It is a consequence of Taylor's theorem that h_x is continuous at x. Hence, $h_x \in C[0, \infty)$. Applying the operator $C_{n,b}$ to both sides of Eq. (9) we obtain

$$(C_{n,b} f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \big(C_{n,b} \psi_x^s\big)(x) + \big(C_{n,b}\big(h_x \psi_x^{2q}\big)\big)(x).$$

The sum in the right-hand side is equal to

$$\begin{aligned}
\sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \big(C_{n,b} \psi_x^s\big)(x) &= \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} n^{-k} \sum_{j=0}^{s} a(k, s, j) b^j x^{s-j} \\
&= \sum_{k=0}^{2q} n^{-k} \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{s} a(k, s, j) b^j x^{s-j} \\
&= \sum_{k=0}^{2q} c_k^{[b]}(f, x) \Big(\frac{b}{n}\Big)^k.
\end{aligned}$$

Note that $a_0^{[b]}(f, x) = 1$. Eq. (5) is a consequence of Eq. (8). We conclude that

$$\sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \big(C_{n,b_n} \psi_x^s\big)(x) = \sum_{k=0}^{q} c_k^{[b_n]}(f, x) \Big(\frac{b_n}{n}\Big)^k + o\Big(\Big(\frac{b_n}{n}\Big)^q\Big), \qquad (n \to \infty).$$

In order to complete the proof, we have to show that the remainder satisfies

$$\big(C_{n,b_n}\big(h_x \psi_x^{2q}\big)\big)(x) = o\Big(\Big(\frac{b_n}{n}\Big)^q\Big), \qquad (n \to \infty).$$

To this end let (δ_n) be a sequence of positive numbers such that

$$\delta_n^2 = 4x\Big(\alpha \frac{b_n^{p+1}}{n} - q \frac{b_n}{n} \log \frac{b_n}{n} + \Big(\frac{b_n}{n}\Big)^{1/2}\Big), \qquad (n \in \mathbb{N}). \tag{10}$$

Note that conditions (1) and (4) imply that $\delta_n = o(1)$ as $n \to \infty$. Define

$$\varepsilon_n = \sup\big\{|h_x(t)| \,:\, t \in (x-\delta_n, x+\delta_n) \cap [0,+\infty)\big\}.$$

Since h_x is continuous with $h_x(x) = 0$, we have $\varepsilon_n = o(1)$ as $n \to \infty$. We split the remainder into two parts

$$\begin{aligned}\big(C_{n,b_n}\big(h_x\psi_x^{2q}\big)\big)(x) &= \sum_{\substack{\nu \\ \left|b_n\frac{\nu}{n}-x\right|<\delta_n}} p_{n,\nu}\Big(\frac{x}{b_n}\Big)\big(h_x\psi_x^{2q}\big)\Big(b_n\frac{\nu}{n}\Big) \\ &\quad + \sum_{\substack{\nu \\ \left|b_n\frac{\nu}{n}-x\right|\ge\delta_n}} p_{n,\nu}\Big(\frac{x}{b_n}\Big)\big(h_x\psi_x^{2q}\big)\Big(b_n\frac{\nu}{n}\Big) \\ &=: \sum\nolimits_1 + \sum\nolimits_2 .\end{aligned}$$

Let us start with the estimate of the first sum:

$$\begin{aligned}\Big|\sum\nolimits_1\Big| &\le \varepsilon_n \sum_{\substack{\nu \\ \left|b_n\frac{\nu}{n}-x\right|<\delta_n}} p_{n,\nu}\Big(\frac{x}{b_n}\Big)\psi_x^{2q}\Big(b_n\frac{\nu}{n}\Big) \\ &\le \varepsilon_n\big(C_{n,b_n}\psi_x^{2q}\big)(x) \\ &= \varepsilon_n\, \mathcal{O}\Big(\Big(\frac{b_n}{n}\Big)^q\Big) = o\Big(\Big(\frac{b_n}{n}\Big)^q\Big)\end{aligned}$$

as $n \to \infty$, where we have used Lemma 3. By Taylor's formula (9), the second sum can be rewritten as

$$\sum\nolimits_2 = \sum_{\substack{\nu \\ \left|b_n\frac{\nu}{n}-x\right|\ge\delta_n}} p_{n,\nu}\Big(\frac{x}{b_n}\Big)\left(f\Big(b_n\frac{\nu}{n}\Big) - \sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}\,\psi_x^s\Big(b_n\frac{\nu}{n}\Big)\right)$$

and we obtain

$$\Big|\sum\nolimits_2\Big| \le 2\exp\Big(-\frac{n\delta_n^2}{4xb_n}\Big)\left(\|f\|_{b_n} + \sum_{s=0}^{2q}\frac{|f^{(s)}(x)|}{s!}\,b_n^s\right),$$

where in the last step Lemma 5 was applied. Note that

$$\sum_{s=0}^{2q}\frac{|f^{(s)}(x)|}{s!}\,b_n^s = \mathcal{O}\big(b_n^{2q}\big), \qquad (n \to \infty).$$

Hence,

$$\sum\nolimits_2 = \mathcal{O}\Big(\exp\Big(\alpha b_n^p - \frac{n\delta_n^2}{4xb_n}\Big)\Big) + \mathcal{O}\Big(\exp\Big(2q\log b_n - \frac{n\delta_n^2}{4xb_n}\Big)\Big), \qquad (n \to \infty).$$

In the case $p = 0$, i.e., f is bounded on $[0, \infty)$, we have

$$\sum\nolimits_2 = \mathcal{O}\Big(\exp\Big(2q \log b_n - \frac{n\delta_n^2}{4xb_n}\Big)\Big), \qquad (n \to \infty).$$

We may assume that $\alpha > 0$. Therefore, in the case $p > 0$, we have

$$\sum\nolimits_2 = \mathcal{O}\Big(\exp\Big(\alpha b_n^p - \frac{n\delta_n^2}{4xb_n}\Big)\Big), \qquad (n \to \infty).$$

Obviously, it is sufficient to estimate the latter expression. By Eq. (10) we infer that

$$\sum\nolimits_2 = \mathcal{O}\Big(\exp\Big(q \log \frac{b_n}{n} - \Big(\frac{n}{b_n}\Big)^{1/2}\Big)\Big) = \mathcal{O}\Big(\Big(\frac{b_n}{n}\Big)^q e^{-\sqrt{n/b_n}}\Big), \qquad (n \to \infty).$$

Finally, we conclude that the remainder can be estimated by

$$\big(C_{n,b_n}\big(h_x \psi_x^{2q}\big)\big)(x) = o\Big(\Big(\frac{b_n}{n}\Big)^q\Big), \qquad (n \to \infty),$$

which completes the proof of the theorem. □

4. Bernstein-Durrmeyer-Chlodovsky Polynomials

The Bernstein-Durrmeyer operators M_n, $n \in \mathbb{N}_0$, were introduced by Durrmeyer [10] and independently by Lupaş [15] in order to approximate integrable functions on finite intervals. For a function $f \in L^1[0,1]$ they are defined by

$$(M_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x)(n+1) \int_0^1 p_{n,\nu}(t) f(t)\, dt, \qquad x \in [0,1],$$

where $p_{n,\nu}$ denote the Bernstein basis polynomials.

In [2, Theorem 1] one of the authors derived a complete asymptotic expansion for the Bernstein-Durrmeyer operators as $n \to \infty$. The representation is given in terms of reciprocals of $(n+2)^{\overline{k}}$. The rising factorials are defined by $z^{\overline{0}} = 1$ and $z^{\overline{n}} = z(z+1)\cdots(z+n-1)$, for $n \in \mathbb{N}$.

Theorem 2. *Let $q \in \mathbb{N}$. Then for every function $f \in L^\infty[0,1]$ which is $2q$-times differentiable at $x \in [0,1]$, the Bernstein-Durrmeyer operators M_n satisfy the asymptotic relation*

$$(M_n f)(x) = f(x) + \sum_{k=1}^{q} \frac{1}{k!(n+2)^{\overline{k}}} \big[x^k (1-x)^k f^{(k)}(x)\big]^{(k)} + o\big(n^{-q}\big)$$

as $n \to \infty$.

Generalizations for the weighted one-dimensional and multivariate Bernstein-Durrmeyer operators were obtained in [3, 4, 5].

The Bernstein-Durrmeyer-Chlodovsky polynomials are defined by

$$\big(\tilde{C}_{n,b}f\big)(x) = (M_n f_b)\Big(\frac{x}{b}\Big).$$

Without giving a proof, we announce the following result.

Theorem 3. *Let α, $p \geq 0$. Suppose that the function $f \in W_{\alpha,p}$ is $2q$-times differentiable at the point $x > 0$. Let (b_n) be a sequence of positive reals, which in the case $p > 0$ satisfies the growth condition*

$$b_n = o\big(n^{1/(p+1)}\big), \qquad (n \to \infty), \tag{11}$$

while in the case $p = 0$, (b_n) satisfies the slightly stronger condition

$$b_n = o\Big(\frac{n}{\log n}\Big), \qquad (n \to \infty). \tag{12}$$

Then, for any positive integer q, the Bernstein-Durrmeyer-Chlodovsky operators $\tilde{C}_{n,b_n}$ possess the asymptotic expansion

$$\big(\tilde{C}_{n,b_n}f\big)(x) = f(x) + \sum_{k=1}^{q} \tilde{c}_k^{[b_n]}(f,x)\frac{b_n^k}{(n+2)^{\overline{k}}} + o\Big(\Big(\frac{b_n}{n}\Big)^q\Big), \qquad (n \to \infty),$$

where

$$\tilde{c}_k^{[b_n]}(f,x) = \frac{1}{k!}\Big(x^k\Big(1 - \frac{x}{b_n}\Big)^k f^{(k)}(x)\Big)^{(k)}. \tag{13}$$

Remark 6. Note that Eq. (13) implies that $\tilde{c}_k^{[b_n]}(f,x) = \mathcal{O}(1)$ as $n \to \infty$.

Remark 7. Clearly, if condition (11) is fulfilled with some $p > 0$, then condition (12) is satisfied, too.

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