CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2016
(K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 51-58
Prof. Marin Drinov Academic Publishing House, Sofia, 2018

Moduli of Smoothness and Polynomial Approximation on the Unit Sphere

Feng Dai*

There are several different well studied moduli of smoothness on the unit sphere, including the classical one defined via the translation operators (i.e., averages over rims of spherical caps), the one introduced by Z. Ditzian via the group of rotations, and the recent one introduced by Y. Xu and myself via finite order differences over Euler angles.

This paper surveys some properties of these three different moduli of smoothness and some related results obtained recently, such as the direct Jackson inequality and its Stechkin type inverse, the strong inverse inequality of type A and the equivalence with different K-functionals. I will also compare different moduli of smoothness and show that they are in fact equivalent in L^p spaces with $1 < p < \infty$.

Keywords and Phrases: Spherical polynomials, moduli of smoothness, K-functionals, direct and inverse theorems.

Mathematics Subject Classification 2010: 33C50, 33C52, 42B15, 42C10.

1. Basic Notations

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ denote the unit sphere of the Euclidean space $\mathbb{R}^d$ equipped with the usual surface Lebesgue measure $d\sigma(x)$. Denote by $L^p(\mathbb{S}^{d-1})$ the Lebesgue L^p space defined with respect to the measure $d\sigma(x)$ on $\mathbb{S}^{d-1}$ with norm $\|\cdot\|_p$. We will use the space $C(\mathbb{S}^{d-1})$ of continuous functions to replace the space $L^\infty(\mathbb{S}^{d-1})$. Let $C(x,\theta) := \{y \in \mathbb{S}^{d-1} : \arccos(x\cdot y) \le \theta\}$ denote the spherical cap with center $x \in \mathbb{S}^{d-1}$ and radius $\theta \in (0,\pi]$. Denote by Π_n^d the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$ (i.e., the restrictions to $\mathbb{S}^{d-1}$ of all algebraic polynomials in d variables of degree at most n). We use the notation $A \sim B$ to mean that there exists a general constant $c > 0$, called the constant of equivalence, such that $c^{-1}A \le B \le cA$.

*The author was partially supported by NSERC Canada under Grant RGPIN 04702.

2. The First Pair of Moduli of Smoothness and K–functionals

The translation $S_\theta f$ of $f \in L^1(\mathbb{S}^{d-1})$ with step θ is defined by

$$S_\theta f(x) := \int_{\partial(C(x,\theta))} f(y)\, d\sigma_{x,\theta}(y), \qquad x \in \mathbb{S}^{d-1},$$

where $\partial(C(x,\theta)) = \{y \in \mathbb{S}^{d-1} : x \cdot y = \cos\theta\}$ and $d\sigma_{x,\theta}$ denotes the $(d-2)$-dimensional Lebesgue measure on $\partial(C(x,\theta))$ normalized so that $S_\theta 1 = 1$. For $1 \le p \le \infty$ and $r \in \mathbb{N}$, we define the r-th order modulus of smoothness on $\mathbb{S}^{d-1}$ by

$$\omega_r^{(1)}(f,t)_p = \sup_{0<\theta\le t} \left\| (I - S_\theta)^{r/2} f \right\|_p, \tag{1}$$

where $(I - S_t)^{r/2} = \sum_{j=0}^{\infty} (-1)^j \binom{r/2}{j} S_t^j$.

Many researchers had made contributions to these first moduli $\omega_r^{(1)}(f,t)_p$: Kušnirenko (1958, $d = 3$, $r = 2$, $p = \infty$), Butzer and Jansche (1971, $r = 2$, $1 \le p \le \infty$), Pawelke (1972, $r = 2$, $1 \le p \le \infty$), Lizorkin and Nikolskii (1988, $r > 0$ and $p = 2$), Kalyabin (1987) and Rustamov (1992, $r > 0$, $1 < p < \infty$), K. Y. Wang (1995, $r > 0$, $p = 1, \infty$). For details, we refer to [12, 17] and the references therein.

For $f \in C^{2r}(\mathbb{S}^{d-1})$, we have

$$\lim_{t\to 0} \frac{(I - S_t)^r f(x)}{t^{2r}} = c_d(-\Delta_0)^r f(x), \qquad x \in \mathbb{S}^{d-1},$$

where Δ_0 denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. This leads to the following definition of K-functionals: for $r > 0$ and $1 \le p \le \infty$,

$$K_r^{(1)}(f,t)_p := \inf\left\{ \|f - g\|_p + t^r \|(-\Delta_0)^{r/2} g\|_p : (-\Delta_0)^{r/2} g \in L^p \right\}.$$

A classical result on spherical polynomial approximation asserts that the K-functional $K_r^{(1)}(f,t)_p$ is equivalent to the modulus of smoothness $\omega_r^{(1)}(f,t)_p$:

Theorem 1 ([12, 17]). *For $r > 0$ and $1 \le p \le \infty$,*

$$K_r^{(1)}(f,t)_p \sim \omega_r^{(1)}(f,t)_p.$$

We end this section with some comments on the first moduli:

1. It was shown in [4] that the supremum in the definition (1) can be dropped; namely, the following is true:

$$\sup_{0<t\le\theta} \|(I - S_t)^{r/2} f\|_p \sim \|(I - S_\theta)^{r/2} f\|_p, \qquad \theta \in [0, \frac{\pi}{2}],\ 1 \le p \le \infty.$$

2. The following central difference with respect to the step θ of the translation operator S_θ was introduced in [4]:

$$\triangle_t^{2\ell} f(x) := \sum_{j=0}^{2\ell} \binom{2\ell}{j} (-1)^j S_{(\ell-j)t} f(x), \qquad \ell \in \mathbb{N}.$$

It was shown in [4] that for $0 < t < t_{d,\ell} < \frac{\pi}{2}$,

$$\sup_{0<\theta\le t} \|(I - S_\theta)^\ell f\|_p \sim \|\triangle_t^{2\ell} f\|_p, \qquad 1 \le p \le \infty.$$

3. The Second Pair of Modulus of Smoothness and K-functional

Let $SO(d)$ denote the group of rotations on $\mathbb{R}^d$. The following modulus of smoothness was introduced by Ditzian [14]:

Definition 1 ([14]). *For $r \in \mathbb{N}$, $t \in (0, \pi]$ and $0 < p \le \infty$,*

$$\omega_r^{(2)}(f, t)_p := \sup_{Q\in O_t} \|(I - T_Q)^r f\|_p,$$

where $T_Q f(x) := f(Qx)$ for $Q \in SO(d)$, and

$$O_t := \Big\{ Q \in SO(d) \,:\, \|I - Q\| := \max_{x\in\mathbb{S}^{d-1}} |Qx - x| \le t \Big\}, \qquad t \in (0, \pi).$$

For $r = 1$ and $p = 1$, $\omega_1^{(2)}(f, t)_p$ was introduced by Calderòn, Weiss and Zygmund [2, 1966] in their paper on singular integrals (see also [3]), where they also considered them as the most natural moduli on the sphere. Note that $\omega_r^{(2)}(f, t)_p$ can be defined for $0 < p < 1$ as well, while $\omega_r^{(1)}(f, t)_p$ cannot.

It turns out that the moduli of smoothness $\omega_r^{(1)}(f, t)_p$ and $\omega_r^{(2)}(f, t)_p$ are equivalent for $1 < p < \infty$, as was shown in [8].

Theorem 2 ([8]). *For $r \in \mathbb{N}$ and $0 < t < \frac{1}{2}$,*

$$\omega_r^{(1)}(f, t)_p \sim \omega_r^{(2)}(f, t)_p, \qquad 1 < p < \infty.$$

A counter-example was given in [16] to show that the above equivalence fails at the endpoints $p = 1, \infty$.

For the proof of Theorem 2, one need to consider the cases of even dimension d and odd dimension d separately. Let us end this section with a few interesting results obtained in the proof of Theorem 2.

First, we define the operator A_θ by

$$A_\theta f(x) = \int_{SO(d)} f(Q^{-1} M_\theta Q x)\, dQ, \qquad x \in \mathbb{S}^{d-1},$$

where dQ is the Haar measure on the group $SO(d)$ normalized by the condition $\int_{SO(d)} 1\,dQ = 1$, and M_θ is a $d\times d$ matrix defined as follows:

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta & & & \\ -\sin\theta & \cos\theta & & & \\ & & \ddots & & \\ & & & \cos\theta & \sin\theta \\ & & & -\sin\theta & \cos\theta \end{pmatrix}, \qquad \text{if } d \text{ is even,}$$

and

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta & & & & \\ -\sin\theta & \cos\theta & & & & \\ & & \ddots & & & \\ & & & \cos\theta & \sin\theta & \\ & & & -\sin\theta & \cos\theta & \\ & & & & & 1 \end{pmatrix}, \qquad \text{if } d \text{ is odd.}$$

We then have

Theorem 3 ([8]). (i) *If d is even, then*

$$A_\theta f(x) = S_\theta f(x), \qquad x\in\mathbb{S}^{d-1},\ \theta\in\mathbb{R}.$$

(ii) *If $d\ge 3$ is odd, then for $f\in L^1(\mathbb{S}^{d-1})$ and $\theta\in(0,\pi)$,*

$$\operatorname{proj}_n(A_\theta f) = \frac{P_n^{(\lambda,\lambda-1)}(\cos\theta)}{P_n^{(\lambda,\lambda-1)}(1)} \operatorname{proj}_n f, \qquad n = 0,1,\dots,$$

where proj_n *denotes the orthogonal projection onto the space of spherical harmonics of degree n, and $\lambda = \frac{d-2}{2}$.*

Next, we define $\mathcal{M}$ to be the set of all $d\times d$ skew-symmetric matrices M of the form

$$M = \begin{pmatrix} 0 & \alpha_1 & & & & & & \\ -\alpha_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \alpha_k & & & \\ & & & -\alpha_k & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

with $k\in\mathbb{N}$ and $0<\alpha_k\le\alpha_{k-1}\le\cdots\le\alpha_1\le 1$. It is known that given a rotation $\rho\in SO(d)$ satisfies the condition $\min_{x\in\mathbb{S}^{d-1}} \rho x\cdot x \ge \cos\theta$ if and only if it can be represented as $\rho = e^{\theta Q M Q^{-1}}$ for some $Q\in SO(d)$ and $M\in\mathcal{M}$.

Now we define the K-functional $K_r^{(2)}(f,t)_p$ by

$$K_r^{(2)}(f,t)_p = \inf_{g\in C^r(\mathbb{S}^{d-1})} \Big\{ \|f-g\|_p + t^r \sup_{\substack{M\in\mathcal{M},\\ Q\in SO(d)}} \Big\| \Big(\frac{\partial}{\partial u}\Big)^r g\big(e^{uQMQ^{-1}}\cdot\big)\Big\|_p \Big\}.$$

Then we have

Theorem 4 ([8]). *Suppose that $f\in L^p(\mathbb{S}^{d-1})$, $1\le p<\infty$, or $f\in C(\mathbb{S}^{d-1})$ for $p=\infty$. Then*

$$\omega_r^{(2)}(f,t)_{L^p(\mathbb{S}^{d-1})} \sim K_r^{(2)}(f,t)_{L^p(\mathbb{S}^{d-1})}.$$

4. The Third Pair of Moduli of Smoothness and K-functionals

For $1\le i\ne j\le d$, denote by $Q_{i,j,t}$ the rotation by an oriented angle t in the (x_i,x_j)-plane. For example,

$$Q_{1,2,t}x = \begin{pmatrix} \cos t & -\sin t & 0 & \cdots & 0\\ \sin t & \cos t & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} x = (r\cos(\theta+t), r\sin(\theta+t), x_3,\dots,x_d),$$

where $(x_1,x_2)=(r\cos\theta, r\sin\theta)$.

Definition 2 ([10]). *For $r\in\mathbb{N}$, $t>0$, and $1\le p\le\infty$,*

$$\omega_r^{(3)}(f,t)_p := \sup_{|\theta|\le t}\max_{1\le i<j\le d}\big\|\Delta_{i,j,\theta}^r f\big\|_p,$$

where

$$\triangle_{i,j,\theta}^r f(x) := (I-T_{Q_{i,j,\theta}})^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f(Q_{i,j,k\theta}x), \quad 1\le i<j\le d.$$

Note that $Q_{i,j,t}x\cdot x\ge \cos t$ for all $x\in\mathbb{S}^{d-1}$. Hence, it is clear that

$$\omega_r^{(3)}(f,t)_p \le \omega_r^{(2)}(f,t)_p.$$

On the other hand, it was shown in [8] that for $0<t<\frac12$ and $r=1,2$,

$$\omega_r^{(3)}(f,t)_p \sim \omega_r^{(2)}(f,t)_p, \qquad 1<p<\infty.$$

The main advantage of this third modulus of smoothness is that it reduces to forward differences in Euler angles, which live on two dimensional circles on the sphere and are easier to compute.

For $r \in \mathbb{N}$, $f \in C^r(\mathbb{S}^{d-1})$ and $x \in \mathbb{S}^{d-1}$, we have

$$\lim_{t\to 0} \frac{\triangle_{i,j,t}^r f(x)}{t^r} = D_{i,j}^r f(x),$$

where $D_{i,j} = x_i\partial_j - x_j\partial_i$ denotes the angular derivative in the (x_i, x_j)-plane. For $(i, j) = (1, 2)$, $(x_1, x_2) = (s\cos\phi, s\sin\phi)$ and $r \in \mathbb{N}$, we have

$$D_{1,2}^r f(x) = \Big(-\frac{\partial}{\partial\phi}\Big)^r f(s\cos\phi, s\sin\phi, x_3, \ldots, x_d). \tag{2}$$

Theorem 5 ([10]). *For $r \in \mathbb{N}$, $0 < t < \pi$, and $1 \le p \le \infty$,*

$$\omega_r^{(3)}(f, t)_p \sim K_r^{(3)}(f, t)_p,$$

where

$$K_r^{(3)}(f, t)_p := \inf_{g\in C^r(\mathbb{S}^{d-1})} \Big\{\|f - g\|_p + t^r \max_{1\le i<j\le d} \|D_{i,j}^r g\|_p\Big\}.$$

To conclude this section, we collect several useful properties of the angular derivatives in the following proposition.

Proposition 1. (i) $\Delta_0 = \sum_{1\le i<j\le d} D_{i,j}^2$.

(ii) *For $f \in C^1(\mathbb{S}^{d-1})$ and $x \in \mathbb{S}^{d-1}$,*

$$|\nabla_0 f(x)|^2 = \sum_{1\le i<j\le d} |D_{i,j} f(x)|^2,$$

where $\nabla_0 f = \nabla f(\frac{x}{|x|})\Big|_{\mathbb{S}^{d-1}}$.

(iii) *If f is a spherical harmonic of degree n, so is $D_{i,j} f$.*

(iv) *(Integration by parts) For $f, g \in C^1(\mathbb{S}^{d-1})$,*

$$\langle D_{i,j} f, g\rangle_{L^2(\mathbb{S}^{d-1})} = -\langle f, D_{i,j} g\rangle_{L^2(\mathbb{S}^{d-1})}.$$

5. Properties of Moduli of Smoothness

Below, $\omega_r(f,t)_p$ stands for either of the moduli of smoothness $\omega_r^{(i)}(f,t)_p$, $i=1,2,3$.

Proposition 2. *Let $1\le p\le\infty$. Then:*

(i) *For $0<s<r$, $\omega_r(f,t)_p\le C_r\omega_s(f,t)_p$.*

(ii) *For $\lambda>0$, $\omega_r(f,\lambda t)_p\le C(\lambda+1)^r\omega_r(f,t)_p$.*

(iii) *For $0<t<\frac{1}{2}$ and every $m>r$,*

$$\omega_r(f,t)_p\le c_m t^r\int_t^1\frac{\omega_m(f,u)_p}{u^{r+1}}\,du.$$

The proof of Proposition 2 can be found in [17, 12] for the first moduli of smoothness, in [14] for the second moduli, and in [10] for the third moduli.

Next, define

$$E_n(f)_p:=\inf_{g\in\Pi_n^d}\|f-g\|_p.$$

Theorem 6 (Direct and inverse theorem). *For $1\le p\le\infty$ and $r\in\mathbb{N}$,*

$$E_n(f)_p\le c\,\omega_r(f,n^{-1})_p, \tag{3}$$

$$\omega_r(f,n^{-1})_p\le c\,n^{-r}\sum_{k=1}^n k^{r-1}E_{k-1}(f)_p. \tag{4}$$

Several remarks are in order. For the first moduli of smoothness, the proof of Theorem 6 can be found in [17, 12]. In fact, many researchers made important contributions to the proof of the Jackson inequality (3) for the first moduli, including Butzer, Jansche, Lizorkin, Nikolskii, Kalyabin, Rustamov, K. Y. Wang. For details, see [13].

For the second moduli of smoothness, the proof of the inverse inequality (4) can be found in [14], while proofs of the Jackson inequality (3) can be found in [15, 6]. The Jackson inequality (3) for $r=1$ and $0<p<1$ was established in [7].

Finally, for the 3rd moduli of smoothness, Theorem 6 was proved in [10].

References

[1] E. Belinsky, F. Dai and Z. Ditzian, Multivariate approximating averages, *J. Approx. Theory* **125** (2003), no. 1, 85–105.

[2] A. P. Calderón, M. Weiss and A. Zygmund, On the existence of singular integrals, *in* "Singular integrals" (Proc. Sympos. Pure Math., Chicago, Ill., 1966), pp. 56–73, Amer. Math. Soc., Providence, R.I., 1967.

[3] A. P. Calderón and A. Zygmund, A note on singular integrals, *Studia Math.* **65** (1979), no. 1, 77–87.

[4] F. Dai and Z. Ditzian, Combinations of multivariate averages, *J. Approx. Theory* **131** (2004), no. 2, 268–283.

[5] F. Dai and Z. Ditzian, Littlewood-Paley theory and a sharp Marchaud inequality, *Acta Sci. Math. (Szeged)* **71** (2005), no. 1-2, 65–90.

[6] F. Dai and Z. Ditzian, Jackson inequality for Banach spaces on the sphere, *Acta Math. Hungar.* **118** (2008), no. 1-2, 171–195.

[7] F. Dai and Z. Ditzian, Jackson theorem in L_p, $0 < p < 1$ for functions on the sphere, *J. Approx. Theory* **162** (2010), no. 2, 382–391.

[8] F. Dai, Z. Ditzian and H. Huang, Equivalence of measures of smoothness in $L^p(\mathbb{S}^{d-1})$, $1 < p < \infty$, *Studia Math.* **196** (2010), no. 2, 179–205.

[9] F. Dai, Z. Ditzian and S. Tikhonov, Sharp Jackson inequalities, *J. Approx. Theory* **151** (2008), no. 1, 86–112.

[10] F. Dai and Y. Xu, Moduli of smoothness and approximation on the unit sphere and the unit ball, *Adv. Math.* **224** (2010), no. 4, 1233–1310.

[11] F. Dai and Y. Xu, Polynomial approximation in Sobolev spaces on the unit sphere and the unit ball, *J. Approx. Theory* **163** (2011), no. 10, 1400–1418.

[12] F. Dai and Y. Xu, "Approximation Theory and Harmonic Analysis on Spheres and Balls", Springer Monographs in Mathematics, Springer, New York, 2013.

[13] F. Dai and K. Wang, A survey of polynomial approximation on the sphere, *Int. J. Wavelets Multiresolut. Inf. Process.* **7** (2009), no. 6, 749–771.

[14] Z. Ditzian, A modulus of smoothness on the unit sphere, *J. Anal. Math.* **79** (1999), 189–200.

[15] Z. Ditzian, Jackson-type inequality on the sphere, *Acta Math. Hungar.* **102** (2004), no. 1–2, 1–35.

[16] Z. Ditzian, Optimality of the range for which equivalence between certain measures of smoothness holds, *Studia Math.* **198** (2010), no. 3, 271–277.

[17] Wang Kunyang and Li Luoqing, "Harmonic Analysis and Approximation on the Unit Sphere", Science Press, Beijing, 2000.

Feng Dai

Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

E-mail: `fdai@ualberta.ca`