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An Exact Strong Converse Inequality for the Weighted Simultaneous Approximation by the Bernstein Operator

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Recently, it has been shown that the rate of simultaneous approximation by means of the Bernstein operator B_n satisfies the direct estimate

$$\|w(B_nf-f)^{(s)}\| \le c \inf_{g\in C^{s+2}[0,1]} \big\{\|w(f^{(s)}-g^{(s)})\| + n^{-1}\|w(Dg)^{(s)}\|\big\},$$

where $\|\circ\|$ is the supremum norm on the interval $[0,1]$, w is a Jacobi weight whose exponents are non-negative and less than s, and $Dg(x) = x(1-x)g''(x)$. In this paper we establish the converse inequality that exactly matches the direct one above for low-order derivatives and lower upper bounds on the weight exponents.

Keywords and Phrases: Bernstein polynomials, strong converse inequality, modulus of smoothness, K-functional.

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1. The Converse Inequality

The Bernstein operator is defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_nf(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)p_{n,k}(x), \qquad p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}.$$

It is well-known that if $f \in C^s[0,1]$, then

$$\lim_{n\to\infty}(B_nf)^{(s)}(x) = f^{(s)}(x) \quad \text{uniformly on } [0,1]; \tag{1.1}$$

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see e.g. [2, Chapter 10, Theorem 2.1]. Moreover, López-Moreno, Martínez-Moreno, Muñoz-Delgado [19], and Floater [9] found the asymptotics

$$\lim_{n\to\infty} n(B_n f(x) - f(x))^{(s)} = \frac{1}{2}(Df(x))^{(s)} \quad \text{uniformly on } [0,1], \tag{1.2}$$

where $f \in C^{s+2}[0,1]$, $Df = \varphi^2 f''$ and $\varphi(x) = \sqrt{x(1-x)}$.

Quantitative estimates of the convergence rate in (1.1) and (1.2) attracted much attention. C. Badea, I. Badea and H. Gonska [1] made a very helpful review of earlier results on that subject. There they also improved previous estimates. In [5, 6, 7] we recalled several more recent results; see also [11, 12, 13, 21].

Quite recently, we characterized the rate of the simultaneous approximation by the Bernstein operator with Jacobi weights in L_p-norm, $1 < p \le \infty$, (see [7]). To state that result in the essential supremum norm, we define the K-functional

$$K_s(f,t)_w = \inf_{g\in C^{s+2}[0,1]} \left\{ \|w(f - g^{(s)})\| + t\|w(Dg)^{(s)}\| \right\},$$

where $\| \circ \|$ denotes the ess sup norm on the interval $[0,1]$ and

$$w(x) = w(\gamma_0, \gamma_1; x) = x^{\gamma_0}(1-x)^{\gamma_1}, \qquad x \in [0,1], \tag{1.3}$$

as $\gamma_0, \gamma_1 \ge 0$.

We showed in [7] (see Theorem 1.1 there with $p = \infty$ and $r = 1$), that if $0 \le \gamma_0, \gamma_1 < s$, then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\|w(B_n f - f)^{(s)}\| \le c\, K_s(f^{(s)}, n^{-1})_w; \tag{1.4}$$

and conversely,

$$K_s(f^{(s)}, n^{-1})_w \le c\left(\|w(B_n f - f)^{(s)}\| + \|w(B_{Rn} f - f)^{(s)}\|\right) \tag{1.5}$$

with some $R \in \mathbb{N}$ independent of f and n. Here and henceforth c denotes a positive constant, not necessarily the same at each occurrence, which is independent of the functions involved and the degree n of the operators; and $AC^r_{loc}(0,1)$ is the space of functions on $[0,1]$, which along with their derivatives up to order r are absolutely continuous on any interval $[a,b] \subset (0,1)$.

We shall strengthen the converse inequality (1.5), showing that the second term on the right is redundant under certain restrictions.

Theorem 1.1. *Let $s \in \mathbb{N}$ as $s \le 6$, and let $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *with $\gamma_0, \gamma_1 \in [0, s/2]$. Then there exists $n_0 \in \mathbb{N}$ such that for all $f \in C[0,1]$ with $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}$ with $n \ge n_0$ there holds*

$$K_s(f^{(s)}, n^{-1})_w \le c\,\|w(B_n f - f)^{(s)}\|.$$

Remark 1.1. The proof of the theorem is based on a number of very technical results. In establishing just a small fragment of them (namely (3.25) for $j = 0$) we imposed an upper bound on s—all the other ones are verified for all positive integers s. Refinements of the calculations can yield the validity of the theorem for s larger than 6. However, it seems that settling the general case requires much effort or another approach.

Remark 1.2. It seems that the restriction $\gamma_0, \gamma_1 \in [0, s)$ on the exponents of the weight w, under which (1.4) and (1.5) were established, is sharp; whereas the assumption $\gamma_0, \gamma_1 \in [0, s/2]$ in Theorem 1.1 is due to the method of proof we use. It is quite plausible that Theorem 1.1 remains valid for all $\gamma_0, \gamma_1 \in [0, s)$.

Let us explicitly mention that $\mathbb{N}$ denotes the set of the *positive* integers throughout the paper. Theorem 1.1 holds for $s = 0$ (see [14, 22]). Its assertion for $s = 1$ and $w = 1$ has already been established in [10].

Combining (1.4) with Theorem 1.1, we verify that the error of the weighted simultaneous approximation by the Bernstein operator is equivalent to the K-functional $K_s(f^{(s)}, n^{-1})_w$. More precisely, we say that $\Phi(f, t)$ and $\Psi(f, t)$ are equivalent and write $\Phi(f, t) \sim \Psi(f, t)$ if there exists a positive constant c such that $c^{-1}\Phi(f, t) \le \Psi(f, t) \le c\,\Phi(f, t)$ for all f and t under consideration. Thus the following characterization of the rate of the weighted simultaneous approximation by the Bernstein operator holds true.

Corollary 1.1. *Let $s \in \mathbb{N}$ as $s \le 6$, and let $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *with $\gamma_0, \gamma_1 \in [0, s/2]$. Then there exists $n_0 \in \mathbb{N}$ such that for all $f \in C[0, 1]$ with $f \in AC^{s-1}_{loc}(0, 1)$ and $wf^{(s)} \in L_\infty[0, 1]$, and all $n \in \mathbb{N}$ with $n \ge n_0$ there holds*

$$\|w(B_n f - f)^{(s)}\| \sim K_s(f^{(s)}, n^{-1})_w.$$

Theorem 1.1 and Corollary 1.1 with $s = 1$ imply a characterization of the weighted approximation by the Kantorovich operator. To recall, the Kantorovich operator is defined for $f \in L[0, 1]$ and $x \in [0, 1]$ by

$$K_n f(x) = \sum_{k=0}^{n} (n + 1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt\, p_{n,k}(x).$$

It is expressed by the Bernstein operator as follows

$$K_n f(x) = (B_{n+1}F(x))', \qquad F(x) = \int_0^x f(t)\, dt.$$

Corollary 1.1 with $s = 1$, F in place of f, and $n + 1$ in place of n implies the following characterization.

Corollary 1.2. *Let $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *with $\gamma_0, \gamma_1 \in [0, 1/2]$. Then there exists $n_0 \in \mathbb{N}$ such that for all $f \in L[0, 1]$ with $wf \in L_\infty[0, 1]$, and all $n \in \mathbb{N}$ with $n \ge n_0$ there holds*

$$\|w(K_n f - f)\| \sim K_1(f, n^{-1})_w.$$

The K-functional $K_s(f,t)_w$ is equivalent to a sum of simpler K-functionals of the type

$$K_{m,\psi}(f,t)_w = \inf_{g\in AC_{loc}^{m-1}(0,1)} \left\{\|w(f-g)\| + t\,\|w\psi^m g^{(m)}\|\right\}.$$

These K-functionals are equivalent to the well-known Ditzian-Totik moduli [4] (see also [8, Chapter 3, Section 10] and [15, 16, 17] for a recent modification of them). In [7, Theorems 1.2 and 1.3 and Remark 1.4] it was shown that

$$K_s(f,t)_w \sim \begin{cases} K_{2,\varphi}(f,t)_w + K_{1,1}(f,t)_w, & s=1,\ 0\le \gamma_0,\gamma_1<1,\\ K_{2,\varphi}(f,t)_1 + K_{1,1}(f,t)_1 + t\,\|f\|, & s\ge 2,\ \gamma_0=\gamma_1=0,\\ K_{2,\varphi}(f,t)_w + t\,\|wf\|, & s\ge 2,\ 0<\gamma_0,\gamma_1<s. \end{cases}$$

Combining this result with Corollary 1.1 and the equivalence of the K-functionals to the ordinary moduli of smoothness or the Ditzian-Totik moduli, we arrive at the following characterization of the weighted simultaneous approximation by the Bernstein operator.

Theorem 1.2. *Let $s\in\mathbb{N}$, as $s\le 6$, and $w=w(\gamma_0,\gamma_1)$ be given by (1.3). Then there exists $n_0\in\mathbb{N}$ such that for all $f\in C[0,1]$ with $f\in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)}\in L_\infty[0,1]$, and all $n\in\mathbb{N}$ with $n\ge n_0$ there holds*

$$\|w(B_nf-f)'\| \sim \omega_\varphi^2(f',n^{-1/2})_w + \omega_1(f',n^{-1})_w, \quad s=1,\ 0\le\gamma_0,\gamma_1\le 1/2,$$

$$\|(B_nf-f)^{(s)}\| \sim \omega_\varphi^2(f^{(s)},n^{-1/2}) + \omega_1(f^{(s)},n^{-1}) + n^{-1}\|f^{(s)}\|, \quad 2\le s\le 6,\ \gamma_0=\gamma_1=0,$$

$$\|w(B_nf-f)^{(s)}\| \sim \omega_\varphi^2(f^{(s)},n^{-1/2})_w + n^{-1}\|wf^{(s)}\|, \quad 2\le s\le 6,\ 0<\gamma_0,\gamma_1\le s/2.$$

Above $\omega_1(g,t)$ is the ordinary modulus of continuity in $L_\infty[0,1]$, $\omega_1(g,t)_w$ its analogue in the space $L_\infty(w)[0,1]=\{f : wf\in L_\infty[0,1]\}$ (see e.g. [4, Appendix B]), and $\omega_\varphi^2(g,t)_w$ is the Ditzian-Totik modulus of smoothness of order 2 with a step-weight φ in $L_\infty(w)[0,1]$ (see [4, Chapter 6]); the subscript w is omitted if $w=1$.

To compare, the characterization in the case $s=0$ is of the form

$$\|B_nf-f\| \sim \omega_\varphi^2(f,n^{-1/2}).$$

The contents of the paper are organized as follows. In the next section we complement certain estimates established in [7]. That will enable us to strengthen the converse inequality given there to the form stated above. The last section contains a number of technical lemmas used in Section 2.

2. Strengthened Bernstein-type Inequalities

To prove the converse inequality of Theorem 1.1, we apply the method developed by Ditzian and Ivanov [3]. It allows us to establish such converse estimates by means of several other basic estimates concerning the approximation properties of the operator. All but one of them were established in [7, Section 4]. What remains to be shown is that the more iterates of B_n we apply, the smaller constant we can take on the right-hand side of the Bernstein-type inequalities in [7, Proposition 4.13 and Corollary 4.16].

By virtue of [7, Proposition 4.1(a)] with $p = \infty$ we have that if $0 \le \gamma_0, \gamma_1 < s$, then

$$\|w(B_n f)^{(s)}\| \le c\,\|wf^{(s)}\| \tag{2.1}$$

for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$. We shall need a stronger form of that estimate that gives an upper bound of the order by which the constant c can increase when we take iterates of the Bernstein operator.

Proposition 2.1. *Let $m, s \in \mathbb{N}$ as $m \ge 2$, and let $w = w(\gamma_0, \gamma_1)$ be given by (1.3) with $\gamma_0, \gamma_1 \in [0, s)$. Then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}$ such that $n \ge m + s$ there holds*

$$\|w(B_n^m f)^{(s)}\| \le c \log m\,\|wf^{(s)}\|.$$

The constant c is independent of m, n and f.

Proof. There holds (see [20], or [2, Chapter 10, (2.3)], or [18, p. 12])

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}^s_{1/n} f\Big(\frac{k}{n}\Big) p_{n-s,k}(x), \qquad x \in [0,1], \tag{2.2}$$

where $\overrightarrow{\Delta}^s_h f(x)$ is the forward difference of order s with step $h > 0$ of the function f, defined by $\Delta_h f(x) = f(x+h) - f(x)$, $x \in [0, 1-h]$, and $\Delta^s_h f(x) = \Delta_h(\Delta^{s-1}_h f)(x)$. It is known that

$$\overrightarrow{\Delta}^s_h f(x) = \int_0^h \cdots \int_0^h f^{(s)}(x + u_1 + \cdots + u_s)\, du_1 \cdots du_s, \quad x \in [0, 1-sh]. \tag{2.3}$$

Note that, under the assumptions of the proposition, $f^{(s)}(x + u_1 + \cdots + u_s)$ is a summable function of the variables $(u_1, \ldots, u_s)$ on the cube $[0,h]^s$ for each $x \in [0, 1-sh]$.

Identities (2.2) and (2.3) yield the representation

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!} \times \sum_{k=0}^{n-s} \int_0^{1/n} \cdots \int_0^{1/n} f^{(s)}\Big(\frac{k}{n} + u_1 + \cdots + u_s\Big) du_1 \cdots du_s\, p_{n-s,k}(x), \quad x \in [0,1].$$

Iterating it, we arrive at the formula

$$(B_n^m f)^{(s)}(x) = \frac{n!}{(n-s)!} \times \sum_{\bar{k}} \int_0^{1/n} \cdots \int_0^{1/n} f^{(s)}\Big(\frac{k_1}{n} + u_1 + \cdots + u_s\Big)\, du_1 \cdots du_s \, P_{n,s,\bar{k}}\, p_{n-s,k_m}(x), \tag{2.4}$$

where the summation is carried over $k_j = 0, \dots, n-s$, $j = 1, \dots, m$, and we have set $\bar{k} = (k_1, \dots, k_m)$,

$$\begin{aligned} P_{n,s,\bar{k}} &= \prod_{j=1}^{m-1} p_{n,s,k_j}\Big(\frac{k_{j+1}}{n}\Big), \\ p_{n,i,k}(x) &= \frac{n!}{(n-i)!} \int_0^{1/n} \cdots \int_0^{1/n} p_{n-i,k}(x + u_1 + \cdots + u_i)\, du_1 \cdots du_i. \end{aligned} \tag{2.5}$$

Taking into account (2.3), we can write (2.4) in the form

$$(B_n^m f)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{\bar{k}} \overrightarrow{\Delta}_{1/n}^{s} f\Big(\frac{k_1}{n}\Big) P_{n,s,\bar{k}}\, p_{n-s,k_m}(x), \quad x \in [0,1]. \tag{2.6}$$

As it follows from [7, (4.2) and (4.7)] with $p = \infty$ and symmetry, there holds

$$\Big|\overrightarrow{\Delta}_{1/n}^{s} f\Big(\frac{k_1}{n}\Big)\Big| \le \frac{c}{n^s}\, w\Big(\frac{k_1+1}{n}\Big)^{-1} \|wf^{(s)}\|, \qquad k_1 = 0, \dots, n-s. \tag{2.7}$$

We shall establish in (3.1) of Lemma 3.1 that

$$w(x) \sum_{\bar{k}} w\Big(\frac{k_1+1}{n}\Big)^{-1} P_{n,s,\bar{k}}\, p_{n-s,k_m}(x) \le c \log m, \qquad x \in [0,1],$$

for $m \ge 2$ and $n \ge m + s$ with a constant c independent of m and n. Now, (2.6), (2.7) and the last estimate imply the assertion of the proposition. □

Next, we proceed to the Bernstein-type inequalities for the iterated Bernstein operator.

Proposition 2.2. *Let $m, s \in \mathbb{N}$ as $m \ge 2$, and let $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *with $\gamma_0, \gamma_1 \in [0, s/2]$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}$ such that $n \ge m + s$ there hold:*

(a) $\|w\varphi(B_n^m f)^{(s+1)}\| \le c\sqrt{\frac{\log m}{m}}\sqrt{n}\, \|wf^{(s)}\|$, $2 \le s \le 9$;

(b) $\|w\varphi^2(B_n^m f)^{(s+2)}\| \le c\, \frac{\log m}{m}\, n\, \|wf^{(s)}\|$, $2 \le s \le 8$;

(c) $\|w(B_n^m f)^{(s+1)}\| \le c\sqrt{\frac{\log m}{m}}\, n\, \|wf^{(s)}\|$, $2 \le s \le 9$.

The constant c is independent of m, n and f.

Proof. To prove assertion (a), we follow the argument in [14, pp. 318–320]. We differentiate (2.6) in x and apply the formula (see e.g. [2, Chapter 10, (2.1)])

$$p'_{n,k}(x) = n[p_{n-1,k-1}(x) - p_{n-1,k}(x)], \tag{2.8}$$

where we have set for convenience $p_{n,k} = 0$ if $k < 0$ or $k > n$. Then we use the Abel transform to derive $m-1$ different representations of $(B_n^m f)^{(s+1)}$. This is the key step in the considerations of Knoop and Zhou in [14, pp. 318–320].

Thus we arrive at the formula

$$(B_n^m f)^{(s+1)}(x) = \frac{1}{m-1}\,\frac{n!}{(n-s)!}\sum_{\bar{k}} \overrightarrow{\Delta}^s_{1/n} f\Big(\frac{k_1}{n}\Big) P_{n,s,\bar{k}}\, Q_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x), \tag{2.9}$$

where the summation is carried over $k_j = 0, \dots, n-s$ and $j = 1, \dots, m$, $P_{n,s,\bar{k}}$ is given in (2.5), and we have set

$$Q_{n,s,\bar{k}} = \sum_{j=1}^{m-1} Q_{n,s,j,\bar{k}}, \qquad Q_{n,s,m-1,\bar{k}} = \ell^*_{n,s,k_{m-1}}\Big(\frac{k_m}{n}\Big),$$

$$Q_{n,s,j,\bar{k}} = \ell^*_{n,s,k_j}\Big(\frac{k_{j+1}}{n}\Big)\ell_{n,s,k_{j+1}}\Big(\frac{k_{j+2}}{n}\Big)\cdots \ell_{n,s,k_{m-1}}\Big(\frac{k_m}{n}\Big), \quad j = 1, \dots, m-2,$$

$$\ell^*_{n,s,k}(x) = \frac{(n-s)\int_0^{1/n} p'_{n,s,k}(x+u)\,du}{p_{n,s,k}(x)}, \qquad \ell_{n,s,k}(x) = \frac{p_{n,s+1,k}(x)}{p_{n,s,k}(x)}.$$

Further, we apply Cauchy's inequality and (2.7) to derive from (2.9) the estimate

$$|w(x)\varphi(x)(B_n^m f)^{(s+1)}(x)| \le \frac{c}{m}\Big(\varphi^2(x)\sum_{\bar{k}} P_{n,s,\bar{k}}\, Q^2_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x)\Big)^{1/2}$$
$$\times \|wf^{(s)}\|\Big(w^2(x)\sum_{\bar{k}} w^{-2}\Big(\frac{k_1+1}{n}\Big) P_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x)\Big)^{1/2}. \tag{2.10}$$

We shall show in (3.12) of Lemma 3.2 below that

$$\varphi^2(x)\sum_{\bar{k}} P_{n,s,\bar{k}}\, Q^2_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\,mn, \qquad x \in [0,1],$$

for $2 \le s \le 9$, $m \ge 2$, and $n \ge m+s$. Also, (3.2) of Lemma 3.1 with w^2 in place of w yields

$$w^2(x)\sum_{\bar{k}} w^{-2}\Big(\frac{k_1+1}{n}\Big) P_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\log m, \qquad x \in [0,1],$$

for $m \ge 2$, and $n \ge m+s$. In view of these two inequalities, (2.10) implies assertion (a).

To prove (b) for even $m \ge 4$ we just apply (a) twice with $m/2$ in place of m, as the first time we take $w\varphi$ in place of w, and $s+1$ in place of s.

We reduce the case of odd $m \ge 5$ to the case of even m's greater than or equal to 4 by applying (2.1) with $B_n^{m-1}f$ in place of f, $s+2$ in place of s, and $w\varphi^2$ in place of w. Assertion (b) for $m = 2, 3$ follows directly from (2.1) and [7, Proposition 4.13(a)] with $p = \infty$ and $\ell = 1$.

Assertion (c) is verified similarly to (a) as instead of (3.12) we use (3.13). □

Corollary 2.1. *Let $m, s \in \mathbb{N}$ as $s \le 6$ and $m \ge 2$, and let $w = w(\gamma_0, \gamma_1)$ be given by (1.3) with $\gamma_0, \gamma_1 \in [0, s/2]$. Then for all $f \in C^{s+2}[0,1]$ and $n \in \mathbb{N}$ such that $n \ge m + s + 2$ there holds*

$$\|w(D^2 B_n^m f)^{(s)}\| \le c' \sqrt{\frac{\log m}{m}}\, n \|w(Df)^{(s)}\|.$$

The constant c' is independent of m, n and f.

Proof. By [7, (4.84)] with $r = 2$, $p = \infty$ and $g = B_n^m f$, we have

$$\begin{aligned} &\|w(D^2 B_n^m f)^{(s)}\| \\ &\qquad \le c\big(\|w(B_n^m f)^{(s')}\| + \|w(B_n^m f)^{(s+2)}\| + \|w\varphi^4 (B_n^m f)^{(s+4)}\|\big), \end{aligned} \tag{2.11}$$

where $s' = \max\{2, s\}$.

We shall show that each of the terms on the right above is estimated from above by $c\sqrt{\frac{\log m}{m}}\, n\|w(Df)^{(s)}\|$, where the constant c is independent of m, n and f.

By [7, Proposition 2.4] with $r = 1$ and $p = \infty$ we have

$$\|wf^{(s')}\| \le c\,\|w(Df)^{(s)}\|, \tag{2.12}$$

$$|wf^{(s+1)}\| \le c\,\|w(Df)^{(s)}\| \tag{2.13}$$

and

$$\|w\varphi^2 f^{(s+2)}\| \le c\,\|w(Df)^{(s)}\|. \tag{2.14}$$

Proposition 2.1 with s' in place of s and inequality (2.12) imply for $m \le n$ the estimates

$$\begin{aligned} \|w(B_n^m f)^{(s')}\| &\le c\,\log m \|wf^{(s')}\| \le c\,\log m \|w(Df)^{(s)}\| \\ &\le c\sqrt{\frac{\log m}{m}}\, n\|w(Df)^{(s)}\|. \end{aligned} \tag{2.15}$$

Next, Proposition 2.2(c) with $s+1$ in place of s and (2.13) imply

$$\|w(B_n^m f)^{(s+2)}\| \le c\sqrt{\frac{\log m}{m}}\, n\,\|wf^{(s+1)}\| \le c\sqrt{\frac{\log m}{m}}\, n\,\|w(Df)^{(s)}\|. \tag{2.16}$$

Finally, by means of Proposition 2.2(b) with $s+2$ in place of s and $w\varphi^2$ in place of w, and (2.14) we arrive at

$$\|w\varphi^4(B_n^m f)^{(s+4)}\| \le c\,\frac{\log m}{m}\,n\,\|w\varphi^2 f^{(s+2)}\| \le c\sqrt{\frac{\log m}{m}}\,n\,\|w(Df)^{(s)}\|. \quad (2.17)$$

Estimates (2.11) and (2.15)–(2.17) imply the assertion of the corollary. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We apply [3, Theorem 4.1] with the operator $Q_\alpha = B_n$ on the space

$$X = \left\{f \in C[0,1] : f \in AC_{loc}^{s-1}(0,1), wf^{(s)} \in L_\infty[0,1]\right\}$$

with the semi-norm $\|f\|_X = \|wf^{(s)}\|$. Let us note that [3, Theorem 4.1] continues to hold for semi-norms. Let also $Y = C^{s+2}[0,1]$ and $Z = C^{s+4}[0,1]$.

Inequality (2.1) shows that B_n is a bounded operator on X, so that [3, (3.3)] holds.

Next, [7, Corollary 4.12] with $r=1$ and $p=\infty$ yields

$$\left\|w\Big(B_n f - f - \frac{1}{2n}\,Df\Big)^{(s)}\right\| \le \frac{c''}{n^2}\,\|w(D^2 f)^{(s)}\|, \qquad f \in Z,$$

where c'' is a positive constant, which is independent of f and n. Thus [3, (3.4)] with $\Phi(f) = \|w(D^2 f)^{(s)}\|$, $\lambda(n) = 1/(2n)$ and $\lambda_1(n) = c''/n^2$ is valid.

Further, we apply Corollary 2.1 with $B_n f$ in place of f to obtain

$$\|w(D^2 B_n^{m+1} f)^{(s)}\| \le c'\sqrt{\frac{\log m}{m}}\,n\|w(DB_n f)^{(s)}\|, \qquad f \in X.$$

Hence [3, (3.5)] is established with $m+1$ in place of m, $\ell = 1$, and

$$A = 2c'c''\sqrt{\frac{\log m}{m}}.$$

We fix $m \ge 2$ so large that $A < 1$.

Finally, [7, Corollary 4.15] with $r=1$ and $p=\infty$ implies

$$\|w(DB_n f)^{(s)}\| \le c\,n\|wf^{(s)}\|, \qquad f \in X,$$

which is [3, (3.6)] with $\ell = 1$.

Now, [3, Theorem 4.1] implies the converse estimate for $n \ge m+s+2$. □

3. Auxiliary Lemmas

Here we shall provide proofs of the technical lemmas we used to verify Propositions 2.1 and 2.2.

Lemma 3.1. *Let $m, n, s \in \mathbb{N}$, $n \geq m+s$, $m \geq 2$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *with* $0 \leq \gamma_0, \gamma_1 \leq s$. *Then*

$$w(x) \sum_{\bar{k}} w\Big(\frac{k_1+1}{n}\Big)^{-1} P_{n,s,\bar{k}}\, p_{n-s,k_m}(x) \leq c \log m, \qquad x \in [0,1], \tag{3.1}$$

and

$$w(x) \sum_{\bar{k}} w\Big(\frac{k_1+1}{n}\Big)^{-1} P_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \leq c \log m, \qquad x \in [0,1], \tag{3.2}$$

where the summation is carried over $k_j = 0, \ldots, n-s$ and $j = 1, \ldots, m$. The constant c is independent of m, n and x.

Proof. We follow the considerations of Knoop and Zhou [14] (see the proof of Lemma 3.1 there). Throughout c denotes a constant whose value is independent of m, n and x in the specified ranges.

By means of the inequalities:

$$\frac{1}{2}\left(x^{-\gamma_0} + (1-x)^{-\gamma_1}\right) \leq w(x)^{-1} \leq 2^{s-1}\left(x^{-\gamma_0} + (1-x)^{-\gamma_1}\right), \qquad x \in (0,1),$$

Hölder's inequality and the relations

$$\sum_{\bar{k}} P_{n,s,\bar{k}} p_{n-s-r,k_m}(x) \equiv \Big(\frac{n!}{(n-s)!\, n^s}\Big)^{m-1} \leq 1, \qquad r = 0, 1,$$

we reduce the assertion of the lemma to the estimates

$$\sum_{\bar{k}} (k_1+1)^{-s} P_{n,s,\bar{k}}\, p_{n-s-r,k_m}(x) \leq c \log m\, (nx)^{-s}, \qquad x \in (0,1),$$

and

$$\sum_{\bar{k}} (n-k_1)^{-s} P_{n,s,\bar{k}}\, p_{n-s-r,k_m}(x) \leq c \log m\, (n(1-x))^{-s}, \qquad x \in (0,1),$$

where $r = 0, 1$.

We set for $\tau \in [0,1]$

$$F_{n,0}(\tau) = 1-\tau, \qquad F_{n,j}(\tau) = 1 - e^{-\frac{n-s}{n} F_{n,j-1}(\tau)}, \qquad j = 1, 2, \ldots.$$

Just as in [14, pp. 322–324] we show that

$$\sum_{\bar{k}} (k_1+1)^{-s} P_{n,s,\bar{k}}\, p_{n-s-r,k_m}(x) \leq \Big(\frac{n}{n-s}\Big)^{s(m-1)}$$
$$\times \int_0^1 \cdots \int_0^1 \frac{F^s_{n,m-1}(\tau_1 \cdots \tau_s)}{F^s_{n,0}(\tau_1 \cdots \tau_s)} e^{-(n-s-r)F_{n,m-1}(\tau_1 \cdots \tau_s)x} d\tau_1 \cdots d\tau_s$$

and

$$\sum_{\bar k}(n-k_1)^{-s}P_{n,s,\bar k}\,p_{n-s-r,k_m}(x) \le \Big(\frac{n}{n-s}\Big)^{s(m-1)}$$
$$\times\int_0^1\cdots\int_0^1 \frac{F^s_{n,m-1}(\tau_1\cdots\tau_s)}{F^s_{n,0}(\tau_1\cdots\tau_s)}e^{-(n-s-r)F_{n,m-1}(\tau_1\cdots\tau_s)(1-x)}d\tau_1\cdots d\tau_s.$$

Since

$$\Big(\frac{n}{n-s}\Big)^m \le e^s, \qquad n\ge m+s, \tag{3.3}$$

to complete the proof of the lemma it is sufficient to show

$$\int_0^1\cdots\int_0^1 \frac{F^s_{n,m-1}(\tau_1\cdots\tau_s)}{F^s_{n,0}(\tau_1\cdots\tau_s)}\,e^{-(n-s-1)F_{n,m-1}(\tau_1\cdots\tau_s)x}d\tau_1\cdots\tau_s$$
$$\le c\,\log m\,(nx)^{-s} \tag{3.4}$$

for all $n\ge m+s$, $m\ge 2$ and $x\in(0,1]$.

Using that $y^se^{-y}\le c$, $y\ge 0$, we get

$$F^s_{n,m-1}(\tau)e^{-(n-s-1)F_{n,m-1}(\tau)x}\le c(nx)^{-s}, \qquad x\in(0,1],\ \tau\in[0,1]. \tag{3.5}$$

Also, we clearly have $F_{n,0}(\tau)\ge 1/2$ for $\tau\in[0,1/2]$. Therefore, if $\mathcal{D}\subset[0,1]^s$ is a parallelepiped with at least one side of the form $[0,1/2]$, then

$$\int_{\mathcal{D}} \frac{F^s_{n,m-1}(\tau_1\cdots\tau_s)}{F^s_{n,0}(\tau_1\cdots\tau_s)}\,e^{-(n-s-1)F_{n,m-1}(\tau_1\cdots\tau_s)x}\,d\tau_1\cdots d\tau_s\le c(nx)^{-s} \tag{3.6}$$

for all $n\ge m+s$, $m\ge 2$ and $x\in(0,1]$.

In order to estimate the integral on the cube $[1/2,1]^s$, we set

$$F_{n,m-1}(\tau,x)=\frac{F^s_{n,m-1}(\tau)}{F^s_{n,0}(\tau)}\,e^{-(n-s-1)F_{n,m-1}(\tau)x},$$

make the change of the variables, defined by the formulae $\sigma_j=\tau_1\cdots\tau_j$, $j=1,\dots,s$, and arrange the order of integration from σ_1 to σ_s to get

$$\int_{1/2}^1\cdots\int_{1/2}^1 F_{n,m-1}(\tau_1\cdots\tau_s,x)\,d\tau_1\cdots d\tau_s$$
$$\le\int_{2^{-s}}^1\Big[F_{n,m-1}(\sigma_s,x)\int_{\sigma_s}^1\Big(\cdots\Big(\frac{1}{\sigma_3}\int_{\sigma_3}^1\Big(\frac{1}{\sigma_2}\int_{\sigma_2}^1\frac{1}{\sigma_1}\,d\sigma_1\Big)\,d\sigma_2\Big)\cdots\Big)\,d\sigma_{s-1}\Big]d\sigma_s$$
$$\le c\int_{2^{-s}}^1\Big[F_n(\sigma_s,x)\int_{\sigma_s}^1\Big(\cdots\Big(\int_{\sigma_3}^1\Big(\int_{\sigma_2}^1 d\sigma_1\Big)\,d\sigma_2\Big)\cdots\Big)\,d\sigma_{s-1}\Big]\,d\sigma_s$$
$$\le c\int_{2^{-s}}^1 F_{n,m-1}(\sigma,x)(1-\sigma)^{s-1}\,d\sigma.$$

We make the change of the variable $\sigma = 1 - t$ and set $G_{n,j}(t) = F_{n,j}(1-t)$. Thus we arrive at

$$\int_{1/2}^{1}\cdots\int_{1/2}^{1}\frac{F^s_{n,m-1}(\tau_1\cdots\tau_s)}{F^s_{n,0}(\tau_1\cdots\tau_s)}\,e^{-(n-s-1)F_{n,m-1}(\tau_1\cdots\tau_s)x}\,d\tau_1\cdots d\tau_s \le c\int_0^1 t^{-1}G^s_{n,m-1}(t)\,e^{-(n-s-1)G_{n,m-1}(t)x}\,dt. \quad (3.7)$$

By means of induction on m we show that (cf. [14, (4.7)])

$$\Big(\frac{n-s}{n}\Big)^m\Big(t-\frac{m}{2}\,t^2\Big) \le G_{n,m-1}(t) \le t, \qquad t\in[0,1]. \quad (3.8)$$

We split the integral on the right-hand side of (3.7) by means of the intermediate point $1/m$. For the one between 0 and $1/m$ we apply (3.3) and (3.8) to get

$$\int_0^{1/m} t^{-1}G^s_{n,m-1}(t)\,e^{-(n-s-1)G_{n,m-1}(t)x}dt \le \int_0^1 t^{s-1}e^{-cnxt}dt \le c(nx)^{-s}, \quad (3.9)$$

as the last estimate is verified by integration by parts.

For the other integral we again use (3.5) to derive

$$\int_{1/m}^{1} t^{-1}G^s_{n,m-1}(t)\,e^{-(n-s-1)G_{n,m-1}(t)x}\,dt \le c(nx)^{-s}\int_{1/m}^{1}\frac{dt}{t} = c\,\log m\,(nx)^{-s}. \quad (3.10)$$

Estimates (3.7), (3.9) and (3.10) yield

$$\int_{1/2}^{1}\cdots\int_{1/2}^{1}\frac{F^s_{n,m-1}(\tau)}{F^s_{n,0}(\tau)}\,e^{-(n-s-1)F_{n,m-1}(\tau)x}\,d\tau_1\cdots d\tau_s \le c\,\log m\,(nx)^{-s} \quad (3.11)$$

for all $n \ge m+s$, $m \ge 2$ and $x \in (0,1]$.

Now, (3.6) and (3.11) imply (3.4). □

Lemma 3.2. *Let $m, n, s \in \mathbb{N}$ as $2 \le s \le 9$, $m \ge 2$, and $n \ge m+s$. Then*

$$\varphi^2(x)\sum_{\bar{k}} P_{n,s,\bar{k}}\,Q^2_{n,s,\bar{k}}\,p_{n-s-1,k_m}(x) \le c\,mn, \qquad x\in[0,1], \quad (3.12)$$

and

$$\sum_{\bar{k}} P_{n,s,\bar{k}}\,Q^2_{n,s,\bar{k}}\,p_{n-s-1,k_m}(x) \le c\,mn^2, \qquad x\in[0,1], \quad (3.13)$$

where the summation is carried over $k_j = 0,\dots,n-s$ and $j = 1,\dots,m$. The constant c is independent of m, n and x.

Remark 3.1. The proof of the lemma is reduced to several simpler inequalities. All but one of them is verified for all $s \ge 2$ (see Remark 3.2).

Proof of Lemma 3.2. Both estimates are verified just like [14, Lemma 3.2], where the case $s = 2$ was considered. We shall indicate the modifications we need to make. Often that amounts only to replacing $n-2$ with $n-s$.

To establish (3.12) it is enough to verify (see [14, p. 328]) that

$$\sum_{\bar{k}} P_{n,s,\bar{k}}\, Q^2_{n,s,j,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\, n\, \varphi^{-2}(x), \quad x \in (0,1), \quad j = 1, \dots, m-1.$$

It follows from the estimates:

$$\sum_{k=0}^{n-s} p_{n,s,k}\Big(\frac{j}{n}\Big) \ell^*_{n,s,k}\Big(\frac{j}{n}\Big)^2 \le c\, n\, \varphi^{-2}\Big(\frac{j+1}{n-s+1}\Big) = \frac{c\, n(n-s+1)^2}{(j+1)(n-s-j)} \tag{3.14}$$

and

$$\sum_{k=0}^{n-s-1} \frac{p^2_{n,s+1,k}\big(\frac{j}{n}\big)}{(k+1)(n-s-k)p_{n,s,k}\big(\frac{j}{n}\big)} \le \frac{1+\frac{c}{n}}{(j+1)(n-s-j)}, \tag{3.15}$$

for $j = 0, \dots, n-s-1$, and also

$$\sum_{k=0}^{n-s-1} \frac{p_{n-s-1,k}(x)}{(k+1)(n-s-k)} \le \frac{c}{n^2}\, \varphi^{-2}(x), \qquad x \in (0,1). \tag{3.16}$$

Inequalities (3.14) and (3.15) are established in Lemmas 3.3 and 3.4 below, and (3.16) directly follows from [14, (4.21)] with $n-2$ replaced with $n-s$.

Similarly, (3.13) follows from (3.14) and (3.15) and the trivial inequality

$$\sum_{k=0}^{n-s-1} \frac{p_{n-s-1,k}(x)}{(k+1)(n-s-k)} \le \frac{c}{n}, \qquad x \in [0,1]. \qquad \square$$

Lemma 3.3. *Let $n, s \in \mathbb{N}$, as $n \ge s+2$. Then*

$$\sum_{k=0}^{n-s} p_{n,s,k}\Big(\frac{j}{n}\Big) \ell^*_{n,s,k}\Big(\frac{j}{n}\Big)^2 \le c\, n\, \varphi^{-2}\Big(\frac{j+1}{n-s+1}\Big) \tag{3.17}$$

for $j = 0, \dots, n-s-1$. The constant c is independent of n.

Proof. We estimate each of the summands on the left-hand side as we consider two cases: $j = 0, n-s-1$ and $1 \le j \le n-s-2$.

For $j = 0$ we apply (2.8) to derive

$$\begin{aligned} n^{s+1} \int_0^{1/n} \cdots \int_0^{1/n} & |p'_{n-s,k}(u_1 + \cdots + u_{s+1})|\, du_1 \cdots du_{s+1} \\ & \le (n-s)\left[\binom{n-s-1}{k-1}\Big(\frac{s+1}{n}\Big)^{k-1} + \binom{n-s-1}{k}\Big(\frac{s+1}{n}\Big)^{k}\right] \end{aligned} \tag{3.18}$$

for $k=0,\dots,n-s$, as we set for convenience $\binom{\alpha}{-1}=0$.

Using that

$$(u_1+\cdots+u_s)^k \ge u_1^k$$

and

$$(1-u_1-\cdots-u_s)^{n-s-k} \ge \Big(1-\frac{s}{n}\Big)^n \ge c,$$

we estimate the denominators of the terms on the left-hand side of (3.17) by

$$n^s\int_0^{1/n}\cdots\int_0^{1/n} p_{n-s,k}(u_1+\cdots+u_s)\,du_1\cdots du_s \ge \frac{c}{k+1}\binom{n-s}{k}\frac{1}{n^k}. \quad (3.19)$$

Estimates (3.18) and (3.19) yield (cf. [14, p. 326])

$$\sum_{k=0}^{n-s} p_{n,s,k}\Big(\frac{j}{n}\Big)\ell^*_{n,s,k}\Big(\frac{j}{n}\Big)^2 \le c\,n^2\Bigg\{\sum_{k=1}^{n-s}\frac{(k+1)\binom{n-s-1}{k-1}^2}{\binom{n-s}{k}}\left[\frac{(s+1)^2}{n}\right]^{k-2} + \sum_{k=0}^{n-s-1}\frac{(k+1)\binom{n-s-1}{k}^2}{\binom{n-s}{k}}\left[\frac{(s+1)^2}{n}\right]^{k}\Bigg\}.$$

To complete the proof of the lemma for $j=0$, it remains to show that the two sums on the right above are bounded on n. For the first one we have

$$\begin{aligned}\sum_{k=1}^{n-s}\frac{(k+1)\binom{n-s-1}{k-1}^2}{\binom{n-s}{k}}\left[\frac{(s+1)^2}{n}\right]^{k-2} &\le c\left(1+\sum_{k=3}^{n-s}\binom{n-s-3}{k-3}\left[\frac{(s+1)^2}{n}\right]^{k-3}\right)\\ &= c\left(1+\sum_{k=0}^{n-s-3}\binom{n-s-3}{k}\left[\frac{(s+1)^2}{n}\right]^{k}\right)\\ &\le c\left(1+\frac{(s+1)^2}{n}\right)^{n-s-3} \le c\,e^{(s+1)^2}.\end{aligned}$$

The other sum is treated in a similar way.

Next, we reduce the case $j=n-s-1$ to $j=0$. More precisely, we make the change of the variables $v_i=1/n-u_i$, $i=1,\dots,s+1$, and apply (3.20) to arrive at

$$\begin{aligned}&\int_0^{1/n}\cdots\int_0^{1/n} p'_{n-s,k}\Big(\frac{n-s-1}{n}+u_1+\cdots+u_{s+1}\Big)\,du_1\cdots du_{s+1}\\ &\quad=\int_0^{1/n}\cdots\int_0^{1/n} p'_{n-s,k}(1-v_1-\cdots-v_{s+1})\,dv_1\cdots dv_{s+1}\\ &\quad=-\int_0^{1/n}\cdots\int_0^{1/n} p'_{n-s,n-s-k}(v_1+\cdots+v_{s+1})\,dv_1\cdots dv_{s+1};\end{aligned}$$

similarly, using the same change of the variables and the inequality

$$\Big(\frac{1-v_1-\cdots-v_s-\frac{1}{n}}{1-v_1-\cdots-v_s}\Big)^k \ge \Big(1-\frac{1}{n-s}\Big)^{n-s} \ge c,$$

we deduce

$$\begin{aligned}
&p_{n,s,k}\Big(\frac{n-s-1}{n}\Big)\\
&\qquad = \frac{n!}{(n-s)!}\int_0^{1/n}\cdots\int_0^{1/n} p_{n-s,n-s-k}\Big(v_1+\cdots+v_s+\frac{1}{n}\Big)\,dv_1\cdots dv_s\\
&\qquad \ge c\,p_{n,s,n-s-k}(0).
\end{aligned}$$

Consequently,

$$p_{n,s,k}\Big(\frac{n-s-1}{n}\Big)\ell^*_{n,s,k}\Big(\frac{n-s-1}{n}\Big)^2 \le c\,p_{n,s,n-s-k}(0)\ell^*_{n,s,n-s-k}(0)^2.$$

It only remains to observe that

$$\varphi^2\Big(\frac{n-s}{n-s+1}\Big) = \varphi^2\Big(\frac{1}{n-s+1}\Big)$$

to derive the assertion of the lemma for $j = n-s-1$ from the one for $j = 0$.

Let $1 \le j \le n-s-2$. Set $U = j/n + u_1 + \cdots + u_{s+1}$.

It is known that (see e.g. [2, Chapter 10, (2.1)])

$$p'_{n,k}(x) = \varphi^{-2}(x)(k-nx)p_{n,k}(x). \tag{3.20}$$

By means of that identity and Cauchy's inequality, we get

$$\begin{aligned}
p_{n,s,k}\Big(\frac{j}{n}\Big)\ell^*_{s,n,k}\Big(\frac{j}{n}\Big)^2 &\le c\,n^{s+3}\int_0^{1/n}\cdots\int_0^{1/n}\varphi^{-4}(U)\\
&\quad\times\Big(\frac{k}{n-s}-U\Big)^2\frac{p^2_{n-s,k}\left(\frac{j}{n}+u_1+\cdots+u_{s+1}\right)}{p_{n-s,k}\left(\frac{j}{n}+u_1+\cdots+u_s\right)}\,du_1\cdots du_{s+1}.
\end{aligned}$$

Further, we set

$$A = \frac{\left(\frac{j}{n}+u_1+\cdots+u_{s+1}\right)^2}{\frac{j}{n}+u_1+\cdots+u_s}, \qquad B = \frac{\left(1-\frac{j}{n}-u_1-\cdots-u_{s+1}\right)^2}{1-\frac{j}{n}-u_1-\cdots-u_s}. \tag{3.21}$$

There hold

$$\varphi^2(U) \ge c\,\varphi^2\Big(\frac{j+1}{n-s+1}\Big) \ge \frac{c}{n}, \tag{3.22}$$

$$A+B = 1+\frac{u_{s+1}^2}{\varphi^2\left(\frac{j}{n}+u_1+\cdots+u_s\right)} \le 1+\frac{c}{n} \tag{3.23}$$

and

$$\Big(\frac{k}{n-s}-U\Big)^2 \le 2\Big(\frac{k}{n-s}-A\Big)^2+\frac{c}{n^2}$$

for $0 \le u_i \le 1/n$, $i = 1, \dots, s+1$.

Consequently, if we denote the sum at the left-hand side of (3.17) by S, we get

$$S \le \frac{c\, n^{s+3}}{\varphi^4\big(\frac{j+1}{n-s+1}\big)} \times \Bigg[\int_0^{1/n} \cdots \int_0^{1/n} \Bigg(\sum_{k=0}^{n-s} \Big(\frac{k}{n-s} - A\Big)^2 \binom{n-s}{k} A^k B^{n-s-k} + \frac{1}{n^2}(A+B)^{n-s} \Bigg)\, du_1 \cdots du_{s+1} \Bigg]. \tag{3.24}$$

Using (3.22) and (3.23), we readily get

$$\frac{c\, n^{s+1}}{\varphi^4\big(\frac{j+1}{n-s+1}\big)} \int_0^{1/n} \cdots \int_0^{1/n} (A+B)^{n-s}\, du_1 \cdots du_{s+1} \le c\, n\, \varphi^{-2}\Big(\frac{j+1}{n-s+1}\Big).$$

So, to complete the proof of (3.17) for $1 \le j \le n-s-2$, it remains to estimate the first multiple integral on the right of (3.24). To this end, we apply the identity (cf. [14, (4.18)])

$$\sum_{k=0}^{n-s} \Big(\frac{k}{n-s} - A\Big)^2 \binom{n-s}{k} A^k B^{n-s-k} = (A+B)^{n-s-2}\Big(A^2(A+B-1)^2 + \frac{AB}{n-s}\Big),$$

inequality (3.23) and the estimate (cf. [14, (4.19)])

$$A^2(A+B-1)^2 + \frac{AB}{n-s} \le \frac{c}{n}\, \varphi^2\Big(\frac{j+1}{n-s+1}\Big).$$

The latter follows from the inequalities

$$A \le c\, \frac{j+1}{n}, \qquad B \le c\, \frac{n-s-j}{n},$$

(3.22) and (3.23). □

Lemma 3.4. *Let $n, s \in \mathbb{N}$ as $2 \le s \le 9$ and $n \ge s+2$. Then*

$$\sum_{k=0}^{n-s-1} \frac{p^2_{n,s+1,k}\left(\frac{j}{n}\right)}{(k+1)(n-s-k)p_{n,s,k}\left(\frac{j}{n}\right)} \le \frac{1+\frac{c}{n}}{(j+1)(n-s-j)} \tag{3.25}$$

for $j = 0, \dots, n-s-1$. The constant c is independent of n.

Remark 3.2. The assertion of the lemma for $j = 1, \dots, n-s-1$ is verified for any positive integer $s \ge 2$ in the proof below.

Proof of Lemma 3.4. The assertion of the lemma was verified for $s = 2$ in [14, (4.20)]. So, we can assume that $s \geq 3$.

First, let $j = 0$. In order to estimate the denominators of the terms on the left-hand side of (3.25), we expand $(u_1 + \cdots + u_s)^k$ by the binomial formula to get

$$(u_1 + \cdots + u_s)^k = \sum_{i=0}^{k} \binom{k}{i} (u_1 + \cdots + u_{s-1})^{k-i} u_s^i,$$

apply the trivial estimate

$$(1 - u_1 - \cdots - u_s)^{n-s-k} \geq \Big(1 - u_1 - \cdots - u_{s-1} - \frac{1}{n}\Big)^{n-s-k}$$

for $u_s \in [0, 1/n]$ and integrate on $u_s \in [0, 1/n]$. Thus we get

$$\begin{aligned} p_{n,s,k}(0) \geq \frac{n!}{(n-s)!} \, \frac{1}{n(k+1)} \\ \times \int_0^{1/n} \cdots \int_0^{1/n} \binom{n-s}{k} \sum_{i=0}^{k} \binom{k}{i} (u_1 + \cdots + u_{s-1})^{k-i} \Big(\frac{1}{n}\Big)^i \\ \times \Big(1 - u_1 - \cdots - u_{s-1} - \frac{1}{n}\Big)^{n-s-k} du_1 \ldots du_{s-1}. \end{aligned}$$

We apply the binomial formula once again and arrive at

$$\begin{aligned} &p_{n,s,k}(0) \\ &\geq \frac{(n-1)!}{(k+1)(n-s)!} \int_0^{1/n} \cdots \int_0^{1/n} p_{n-s,k}\Big(u_1 + \cdots + u_{s-1} + \frac{1}{n}\Big) du_1 \cdots du_{s-1}. \end{aligned}$$

Further, we use Cauchy's inequality to get the estimate

$$\begin{aligned} &\frac{p^2_{n,s+1,k}(0)}{(k+1)(n-s-k)\, p_{n,s,k}(0)} \\ &\quad\leq n^{s+1} \int_0^{1/n} \cdots \int_0^{1/n} \frac{p^2_{n-s-1,k}(u_1 + \cdots + u_{s+1})}{(n-s-k)\, p_{n-s,k}\big(u_1 + \cdots + u_{s-1} + \frac{1}{n}\big)} \, du_1 \cdots du_{s+1}. \end{aligned} \tag{3.26}$$

We set

$$\widetilde{A} = \frac{(u_1 + \cdots + u_{s+1})^2}{u_1 + \cdots + u_{s-1} + \frac{1}{n}}, \qquad \widetilde{B} = \frac{(1 - u_1 - \cdots - u_{s+1})^2}{1 - u_1 - \cdots - u_{s-1} - \frac{1}{n}}.$$

Then (3.26) yields

$$\begin{aligned} &\frac{p^2_{n,s+1,k}(0)}{(k+1)(n-s-k)\, p_{n,s,k}(0)} \\ &\qquad \leq \frac{n^{s+2}}{(n-s)^2} \int_0^{1/n} \cdots \int_0^{1/n} \binom{n-s-1}{k} \widetilde{A}^k \widetilde{B}^{n-s-1-k} \, du_1 \cdots du_{s+1} \end{aligned}$$

and, consequently, for $n \geq s+3$ we have

$$\sum_{k=2}^{n-s-1} \frac{p_{n,s+1,k}^2(0)}{(k+1)(n-s-k)\,p_{n,s,k}(0)} \leq \frac{n^{s+2}}{(n-s)^2}$$
$$\times \int_0^{1/n} \cdots \int_0^{1/n} \big[(\widetilde{A}+\widetilde{B})^{n-s-1} - \widetilde{B}^{n-s-1} - (n-s-1)\widetilde{A}\widetilde{B}^{n-s-2}\big]\,du_1 \cdots du_{s+1}. \tag{3.27}$$

By means of the inequality $1+x \leq e^x$, we get

$$\begin{aligned}
\widetilde{A}+\widetilde{B} &= 1 + \frac{\left(u_s + u_{s+1} - \frac{1}{n}\right)^2}{\left(u_1 + \cdots + u_{s-1} + \frac{1}{n}\right)\left(1 - u_1 - \cdots - u_{s-1} - \frac{1}{n}\right)} \\
&\leq 1 + \frac{\left(u_s + u_{s+1} - \frac{1}{n}\right)^2}{\left(u_1 + \cdots + u_{s-1} + \frac{1}{n}\right)\left(1 - \frac{s}{n}\right)} \\
&\leq e^{\frac{(nu_s + nu_{s+1} - 1)^2}{(n-s)(nu_1 + \cdots + nu_{s-1} + 1)}}.
\end{aligned}$$

Therefore

$$(\widetilde{A}+\widetilde{B})^{n-s-1} \leq e^{\frac{(nu_s + nu_{s+1} - 1)^2}{nu_1 + \cdots + nu_{s-1} + 1}}. \tag{3.28}$$

Similarly, by means of the inequality $1+x \geq (1-x^2)e^x$, $x \in [-1,1]$, we establish

$$\begin{aligned}
\widetilde{B} &\geq 1 + \frac{1}{n} - u_1 - \cdots - u_{s-1} - 2u_s - 2u_{s+1} \\
&\geq \left(1 - \left(\frac{s+2}{n}\right)^2\right) e^{\frac{1}{n} - u_1 - \cdots - u_{s-1} - 2u_s - 2u_{s+1}};
\end{aligned}$$

hence, using Bernoulli's inequality $(1+x)^n \geq 1+nx$ for $x \geq -1$, and $e^x \geq 1+x$, we derive

$$\widetilde{B}^{n-s-j} \geq \left(1 - \frac{c}{n}\right) e^{1 - nu_1 - \cdots - nu_{s-1} - 2nu_s - 2nu_{s+1}}, \qquad j = 1, 2. \tag{3.29}$$

We apply estimates (3.27)–(3.29), make the change of the variables $t_i = nu_i$, $i = 1, \ldots, s+1$, and use the representation

$$\frac{(t_1 + \cdots + t_{s+1})^2}{t_1 + \cdots + t_{s-1} + 1} = -1 + t_1 + \cdots + t_{s-1} + 2t_s + 2t_{s+1} + \frac{(t_s + t_{s+1} - 1)^2}{t_1 + \cdots + t_{s-1} + 1}$$

to obtain

$$\sum_{k=2}^{n-s-1} \frac{p_{n,s+1,k}^2(0)}{(k+1)(n-s-k)\,p_{n,s,k}(0)} \leq \frac{1}{n-s}\left(1 + \frac{c}{n}\right)(I_s' - I_s''), \tag{3.30}$$

where we have set

$$\begin{aligned}
I_s' &= \int_0^1 \cdots \int_0^1 e^{\frac{(t_s + t_{s+1} - 1)^2}{t_1 + \cdots + t_{s-1} + 1}}\,dt_1 \cdots dt_{s+1}, \\
I_s'' &= \int_0^1 \cdots \int_0^1 (t_1 + \cdots + t_{s-1} + 2t_s + 2t_{s+1}) e^{1 - t_1 - \cdots - t_{s-1} - 2t_s - 2t_{s+1}}\,dt_1 \cdots dt_{s+1}.
\end{aligned}$$

We estimate the first integral by means of the inequality

$$e^x \le 1 + x + \frac{e\,x^2}{2}, \qquad x \in [0,1],$$

and direct computations. Thus we get

$$\begin{aligned} &I'_3 \le 1.11327, \quad I'_4 \le 1.08629, \quad I'_5 \le 1.06929, \quad I'_6 \le 1.05773,\\ &I'_7 \le 1.0494, \quad\ \ I'_8 \le 1.04314, \quad I'_9 \le 1.03827. \end{aligned} \tag{3.31}$$

We evaluate I''_s and get

$$I''_s = \frac{e}{4}(1-e^{-1})^s(1+e^{-1})\big[(s-1)(1-2e^{-1})(1+e^{-1}) + 2(1-3e^{-2})\big]; \tag{3.32}$$

hence

$$\begin{aligned} &I''_3 \ge 0.44866, \quad I''_4 \ge 0.33725, \quad I''_5 \ge 0.24709, \quad I''_6 \ge 0.17762,\\ &I''_7 \ge 0.12583, \quad I''_8 \ge 0.0881, \quad\ \ I''_9 \ge 0.0611. \end{aligned} \tag{3.33}$$

We shall now estimate the first two terms in the sum in (3.25). We use the inequalities $(1-x^2)e^x \le 1 + x \le e^x$, $x \in [-1,1]$, to derive

$$\begin{aligned} &\frac{p^2_{n,s+1,0}(0)}{(n-s)p_{n,s,0}(0)}\\ &\quad = \frac{n!}{(n-s-1)!}\frac{\Big(\int_0^{1/n}\cdots\int_0^{1/n}(1-u_1-\cdots-u_{s+1})^{n-s-1}\,du_1\cdots du_{s+1}\Big)^2}{\int_0^{1/n}\cdots\int_0^{1/n}(1-u_1-\cdots-u_s)^{n-s}\,du_1\cdots du_s}\\ &\quad \le \frac{n!}{(n-s-1)!}\Big(1+\frac{c}{n}\Big)\frac{\Big(\int_0^{1/n}\cdots\int_0^{1/n}e^{-(n-s-1)(u_1+\cdots+u_{s+1})}\,du_1\cdots du_{s+1}\Big)^2}{\int_0^{1/n}\cdots\int_0^{1/n}e^{-(n-s)(u_1+\cdots+u_s)}\,du_1\cdots du_s}\\ &\quad \le \frac{n!}{(n-s-1)!}\Big(1+\frac{c}{n}\Big)\frac{\Big(\int_0^{1/n}\cdots\int_0^{1/n}e^{-n(u_1+\cdots+u_{s+1})}\,du_1\cdots du_{s+1}\Big)^2}{\int_0^{1/n}\cdots\int_0^{1/n}e^{-n(u_1+\cdots+u_s)}\,du_1\cdots du_s}\\ &\quad \le \frac{1}{n-s}\Big(1+\frac{c}{n}\Big)\frac{\Big(\int_0^{1}\cdots\int_0^{1}e^{-(t_1+\cdots+t_{s+1})}\,dt_1\cdots dt_{s+1}\Big)^2}{\int_0^{1}\cdots\int_0^{1}e^{-(t_1+\cdots+t_s)}\,dt_1\cdots dt_s}. \end{aligned}$$

Consequently,

$$\frac{p^2_{n,s+1,0}(0)}{(n-s)p_{n,s,0}(0)} \le \frac{1}{n-s}\Big(1+\frac{c}{n}\Big)\left(1-e^{-1}\right)^{s+2}. \tag{3.34}$$

Similarly, we derive

$$\frac{p_{n,s+1,1}^2(0)}{2(n-s-1)p_{n,s,1}(0)} \le \frac{1}{n-s}\Big(1+\frac{c}{n}\Big)\frac{1}{2}\frac{\Big(\int_0^1\cdots\int_0^1(t_1+\cdots+t_{s+1})\,e^{-(t_1+\cdots+t_{s+1})}\,dt_1\cdots dt_{s+1}\Big)^2}{\int_0^1\cdots\int_0^1(t_1+\cdots+t_s)\,e^{-(t_1+\cdots+t_s)}\,dt_1\cdots dt_s}. \tag{3.35}$$

We have

$$\int_0^1\cdots\int_0^1(t_1+\cdots+t_s)\,e^{-(t_1+\cdots+t_s)}\,dt_1\cdots dt_s = s(1-e^{-1})^{s-1}(1-2e^{-1}).$$

Consequently,

$$\frac{p_{n,s+1,1}^2(0)}{2(n-s-1)p_{n,s,1}(0)} \le \frac{1}{n-s}\Big(1+\frac{c}{n}\Big)\frac{(s+1)^2}{2s}(1-e^{-1})^{s+1}(1-2e^{-1}). \tag{3.36}$$

For

$$J_s = \big(1-e^{-1}\big)^{s+2} + \frac{(s+1)^2}{2s}(1-e^{-1})^{s+1}(1-2e^{-1})$$

we have

$$\begin{aligned} &J_3 \le 0.21343, \quad J_4 \le 0.14714, \quad J_5 \le 0.10102, \quad J_6 \le 0.06901, \\ &J_7 \le 0.04691, \quad J_8 \le 0.03175, \quad J_9 \le 0.0214. \end{aligned} \tag{3.37}$$

By (3.30), (3.34) and (3.35) we have

$$\sum_{k=0}^{n-s-1}\frac{p_{n,s+1,k}^2\big(\frac{j}{n}\big)}{(k+1)(n-s-k)p_{n,s,k}\big(\frac{j}{n}\big)} \le \frac{1}{n-s}\Big(1+\frac{c}{n}\Big)(J_s+I_s'-I_s'').$$

Inequalities (3.31), (3.33) and (3.37) imply

$$J_s + I_s' - I_s'' \le 1, \qquad s = 3,4,\ldots,9.$$

Thus the lemma is established for $j=0$.

Let $s \ge 2$. For $1 \le j \le n-s-1$ we get by means of Cauchy's inequality

$$\frac{p_{n,s+1,k}^2\left(\frac{j}{n}\right)}{p_{n,s,k}\left(\frac{j}{n}\right)} \le n^{s+1}\int_0^{1/n}\cdots\int_0^{1/n}\frac{p_{n-s-1,k}^2\left(\frac{j}{n}+u_1+\cdots+u_{s+1}\right)}{p_{n-s,k}\left(\frac{j}{n}+u_1+\cdots+u_s\right)}\,du_1\cdots du_{s+1}.$$

Therefore,

$$\frac{p^2_{n,s+1,k}\left(\frac{j}{n}\right)}{(k+1)(n-s-k)p_{n,s,k}\left(\frac{j}{n}\right)} \le \frac{n^{s+1}}{(n-s)^2}\int_0^{1/n}\cdots\int_0^{1/n}\frac{\binom{n-s}{k+1}A^{k+1}B^{n-s-k-1}}{A\left(1-\frac{j}{n}-u_1-\cdots-u_s\right)}\,du_1\cdots du_{s+1}$$

with A and B defined in (3.21). We sum up these inequalities for $k=0,\dots,n-s-1$ and apply the binomial formula. Thus we get the estimate

$$S \le \frac{n^{s+1}}{(n-s)^2}\int_0^{1/n}\cdots\int_0^{1/n}\frac{(A+B)^{n-s}}{A\left(1-\frac{j}{n}-u_1-\cdots-u_s\right)}\,du_1\cdots du_{s+1}, \quad (3.38)$$

where S denotes the sum on the left of estimate (3.25)

Further, we again use (3.23) and the inequality $1+x\le e^x$ to deduce

$$(A+B)^{n-s} \le e^{\frac{nu^2_{s+1}}{\varphi^2(j/n+u_1+\cdots+u_s)}};$$

and hence

$$(A+B)^{n-s} \le \left(1+\frac{c}{n}\right)\begin{cases} e^{\frac{n^2u^2_{s+1}}{\xi}}, & 1\le j\le (n-s)/2,\\ e^{\frac{n^2u^2_{s+1}}{n-\xi}}, & (n-s)/2\le j\le n-s-1,\end{cases}$$

where $\xi = j+nu_1+\cdots+nu_s$. We apply that estimate in (3.38) and make the change of the variables $t_i = nu_i$, $i=1,\dots,s+1$. Thus, for $1\le j\le (n-s)/2$, we arrive at

$$S \le \left(1+\frac{c}{n}\right)\frac{1}{n-s-j} \times \int_0^1\cdots\int_0^1 \frac{j+t_1+\cdots+t_s}{(j+t_1+\cdots+t_{s+1})^2}\,e^{\frac{t^2_{s+1}}{j+t_1+\cdots+t_s}}\,dt_1\cdots dt_{s+1}. \quad (3.39)$$

Using that the function $T(T+t)^{-2}e^{t^2/T}$ is decreasing on T in $[1,\infty)$ for any fixed $t\in[0,1]$, we deduce that

$$\frac{j+t_1+\cdots+t_s}{(j+t_1+\cdots+t_{s+1})^2}\,e^{\frac{t^2_{s+1}}{j+t_1+\cdots+t_s}} \le \frac{j+t_1+t_2}{(j+t_1+t_2+t_{s+1})^2}\,e^{\frac{t^2_{s+1}}{j+t_1+t_2}} \quad (3.40)$$

for all $t_i\in[0,1]$, $i=1,\dots,s+1$.

Combining (3.39), (3.40) and [14, (4.10)], we verify (3.25) for $1\le j\le (n-s)/2$ and $s\ge 2$.

Similarly, for $(n-s)/2\le j\le n-s-1$ we have

$$S \le \left(1+\frac{c}{n}\right)\frac{1}{j+1}\int_0^1\cdots\int_0^1 \frac{e^{\frac{t^2_{s+1}}{n-j-t_1-\cdots-t_s}}}{n-j-t_1-\cdots-t_s}\,dt_1\cdots dt_{s+1}. \quad (3.41)$$

Above we used that the function $T/(T+t)^2$ is decreasing on T in $[1,\infty)$ for any fixed $t \in [0,1]$ to derive

$$\frac{j+t_1+\cdots+t_s}{(j+t_1+\cdots+t_{s+1})^2} \le \frac{j}{(j+t_{s+1})^2} \le \frac{1}{j+1}\Big(1+\frac{c}{n}\Big).$$

Next, we make the change of the variables $v_i = 1 - t_i$, $i = 1,\dots,s$, in the integral in (3.41). Thus we arrive at

$$S \le \Big(1+\frac{c}{n}\Big)\frac{1}{j+1}\int_0^1\cdots\int_0^1 \frac{e^{\frac{t_{s+1}^2}{n-s-j+v_1+\cdots+v_s}}}{n-s-j+v_1+\cdots+v_s}\,dv_1\cdots du_s\,dt_{s+1}.$$

Now, (3.25) for $(n-s)/2 \le j \le n-s-1$ and $s \ge 2$ follows from the fact that the function $T^{-1}e^{t^2/T}$ is decreasing on T in $[1,\infty)$ for any fixed $t \in [0,1]$ and [14, (4.11)]. $\square$

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