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A New Characterization of Weighted Peetre *K*-Functionals (III)

Borislav R. Draganov and Kamen G. Ivanov*

We characterize Peetre K-functionals with weights of power-type asymptotics at the ends of the interval by means of the classical moduli of smoothness taken on a proper linear transforms of the function. Negative exponents at finite ends of the interval are included. We also point out applications with regard to weighted approximation by Bernstein-type operators. The paper presents a continuation of the authors' research in [5, 8].

Keywords and Phrases: K-functional, modulus of smoothness, linear operator, degree of approximation, fractional integral, inequalities involving derivatives, Bernstein polynomials.

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1. Introduction

Let I be an open interval on the real line and let w and φ be weights on I as follows:

I	$w(x)$	$\varphi(x)$
(a,b)	$(x-a)^{\gamma_a}(b-x)^{\gamma_b}$	$(x-a)^{\lambda_a}(b-x)^{\lambda_b}$
(a,∞)	$(x-a)^{\gamma_a}(x-a+1)^{\gamma_\infty-\gamma_a}$	$(x-a)^{\lambda_a}(x-a+1)^{\lambda_\infty-\lambda_a}$
$\mathbb{R}=(-\infty,\infty)$	$\begin{cases}\lvert x\rvert^{\gamma_{-\infty}}, & x<-1,\\ 1, & -1\le x\le 1,\\ x^{\gamma_{+\infty}}, & x>1.\end{cases}$	$\begin{cases}\lvert x\rvert^{\lambda_{-\infty}}, & x<-1,\\ 1, & -1\le x\le 1,\\ x^{\lambda_{+\infty}}, & x>1.\end{cases}$

Table 1

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The γ's and λ's are arbitrary real numbers.

Further, we put $L_p(w)(I) = \{f : wf \in L_p(I)\}$, $1 \le p \le \infty$. Instead of $L_\infty(I)$ we can consider the corresponding subspace of continuous functions $C(w)(I) = \big\{f \in C(I) : wf \in L_\infty(I)\big\}$ as the interval I can be closed. We denote the standard L_p-norm on the interval I by $\| \circ \|_{p(I)}$. The set of the absolutely continuous functions on the interval $[a_1, b_1]$ is denoted by $AC[a_1, b_1]$. Then we set $AC^k_{loc}(I) = \big\{g : g, g', \dots, g^{(k)} \in AC[a_1, b_1] \ \forall a_1, b_1 \in I,\ a_1 < b_1\big\}$. We set $D = \frac{d}{dx}$ and $D^r g$ means the r-th derivative of the function g.

The weighted K-functional we shall consider is defined for $f \in L_p(w)(I)$ and $t > 0$ by

$$K(f, t^r; L_p(w)(I), AC^{r-1}_{loc}, \varphi^r D^r) = \inf\big\{\|w(f-g)\|_{p(I)} + t^r \|w\,\varphi^r D^r g\|_{p(I)} : g \in AC^{r-1}_{loc}(I),\ wg, w\varphi^r D^r g \in L_p(I)\big\}. \tag{1.1}$$

In [5, 8, 9, 11] we showed that in a number of cases this K-functional could be characterized by the classical unweighted moduli of smoothness with an unvarying step $\omega_r(F, t)_{p(\tilde I)}$ with appropriate function $F \in L_p(\tilde I)$, related to f, and an interval $\tilde I \subseteq \mathbb{R}$. To recall, $\omega_r(F, t)_{p(\tilde I)}$ is defined for $F \in L_p(\tilde I)$ and $t > 0$ by

$$\omega_r(F, t)_p = \sup_{0<h\le t} \|\Delta^r_h F\|_{p(\tilde I)},$$

where the finite difference with a fixed step h is given by

$$\Delta^r_h F(x) = \begin{cases} \displaystyle\sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} F(x + kh), & \text{if } x, x + rh \in \tilde I, \\ 0, & \text{otherwise.} \end{cases}$$

The simplest form of that characterization is

$$K\big(f, t^r; L_p(w)(I), AC^{r-1}_{loc}, \varphi^r D^r\big) \sim \omega_r(\mathcal{A} f, t)_{p(\tilde I)}, \tag{1.2}$$

where $\mathcal{A} : L_p(I) \to L_p(\tilde I)$ is a bounded linear operator. The relation $\psi_1(F, t) \sim \psi_2(F, t)$ above means that there exists a positive constant c such that for all F and t under consideration there holds

$$c^{-1}\psi_2(F, t) \le \psi_1(F, t) \le c\,\psi_2(F, t).$$

To simplify the operator $\mathcal{A}$ and cover a broader range of exponents in the weights w and φ, we can separate the singularities to get for $0 < t \le t_0$ a characterization of the form

$$K(f, t^r; L_p(w)(I), AC^{r-1}_{loc}, \varphi^r D^r) \sim \omega_r(\mathcal{A}_1 f, t)_{p(\tilde I_1)} + \omega_r(\mathcal{A}_2 f, t)_{p(\tilde I_2)}. \tag{1.3}$$

Let either $\bar a = a$ be finite, or $\bar a = -\infty$; similarly, let $\bar b = b$ be finite, or $\bar b = \infty$. So far we have established (1.3) in the following cases (see [5, 8] and particularly [8, Theorem 7.1]):

- $1 \le p < \infty$, $\gamma_{\bar a}, \gamma_{\bar b} \ne 1 - r - 1/p, 2 - r - 1/p, \dots, -1/p$, $\lambda_{\bar a}, \lambda_{\bar b} \ne 1$;
- $p = \infty$, $\gamma_{\bar a} = \gamma_{\bar b} = 0$, $\lambda_{\bar a}, \lambda_{\bar b} \ne 1$.

In [11, Theorems 1.4, 1.5, 5.13 and 5.14] we established a characterization similar to (1.2) and (1.3) but involving also lower order moduli for the case $\lambda_{\bar a} = \lambda_{\bar b} = 1$, any $\gamma_{\bar a}, \gamma_{\bar b}$ and $1 \le p \le \infty$. In [6, 10] we stated a characterization exactly of type (1.2) for $\gamma_a = \gamma_\infty \in \mathbb{R}$ and $\lambda_a = \lambda_\infty = 1$. We plan to give a proof of this result in [12].

Here we pursue mainly two goals. First, we shall improve results given in [8] in respect to simplification of the operator $\mathcal{A}$ in (1.2). Secondly, we shall construct operators $\mathcal{A}_1, \mathcal{A}_2$ of (1.3) for the cases:

- $p = \infty$, $\gamma_{\bar a}, \gamma_{\bar b} = 1 - r, 2 - r, \dots, -1$, $\lambda_{\bar a}, \lambda_{\bar b} \ne 1$;
- $1 \le p < \infty$, $\gamma_{\bar a}, \gamma_{\bar b} = 1 - r - 1/p, 2 - r - 1/p, \dots, -1/p$ and $\lambda_a, \lambda_b < 1$ or $\lambda_{-\infty}, \lambda_\infty > 1$;
- $p = \infty$, $\gamma_{\bar a}, \gamma_{\bar b} \ne 1 - r, 2 - r, \dots, 0$ and $\lambda_a, \lambda_b < 1$ or $\lambda_{-\infty}, \lambda_\infty > 1$.

Whereas one can use the operators already defined in [5, 8] to settle the first case above, the treatment of the other two requires new operators and techniques. In another paper we shall consider the remaining cases:

- $1 \le p < \infty$, $\gamma_{\bar a}, \gamma_{\bar b} = 1 - r - 1/p, 2 - r - 1/p, \dots, -1/p$ and $\lambda_a, \lambda_b > 1$ or $\lambda_{-\infty}, \lambda_\infty < 1$;
- $p = \infty$, $\gamma_{\bar a}, \gamma_{\bar b} \ne 1 - r, 2 - r, \dots, 0$ and $\lambda_a, \lambda_b > 1$ or $\lambda_{-\infty}, \lambda_\infty < 1$.

Recently, the error of a number of approximation process in $C(w)(I)$ was characterized by K-functionals, as the weight w admits negative exponents at a finite end of the interval I. We shall present these results in detail in the next section, but here we briefly list them:

- The Bernstein and the Goodman-Sharma operators in $C(w)[0,1]$, where $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$ with $\gamma_0, \gamma_1 \in [-1, 0]$ (see Theorem 2.1);
- The Baskakov operator in $C(w)[0,\infty)$, where $w(x) = x^{\gamma_0}(1+x)^{\gamma_\infty - \gamma_0}$ with $\gamma_0 \in [-1, 0]$ and $\gamma_\infty \in \mathbb{R}$ (see Theorem 2.2);
- The modification of the Baskakov operator due to Finta in $C(w)[0,\infty)$, where $w(x) = x^{\gamma_0}(1+x)^{\gamma_\infty - \gamma_0}$ with $\gamma_0, \gamma_\infty \in [-1, 0]$ (see Theorem 2.2);
- The modification of the Meyer-König and Zeller operator due to Cheney and Sharma in $C(w)[0,1)$, where $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$ with $\gamma_0 \in [-1, 0]$ and $\gamma_1 \in \mathbb{R}$ (see Theorem 2.3);
- The modification of the Meyer-König and Zeller operator due to Goodman and Sharma in $C(w)[0,1)$, where $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$ with $\gamma_0, \gamma_1 \in [-1, 0]$ (see Theorem 2.3).

As far as we know, until now, there have not been constructed moduli of smoothness which are equivalent to the K-functionals used to characterize the approximation rate of the above-mentioned operators in the case of a negative weight exponent at a finite end of the interval. In the present paper define such moduli.

The contents of the paper are organized as follows. In the next section we present the above-mentioned applications of the new moduli for characterizing of the approximation error. In Section 3 we briefly formulate the method we use to construct moduli of smoothness. Section 4 contains the auxiliary inequalities for intermediate derivatives we shall use to verify relations between function spaces and to show that the operators we construct possess the properties listed in Definition 3.1. Further, in Sections 5 and 6 we introduce and verify the basic properties of the operators we shall use to build operators that satisfy (1.2) or (1.3). In Section 7 we establish the characterization of the weighted K-functionals in the cases listed above. Finally, in Section 8 we present a way one can simplify the operators $\mathcal{A}$ in the characterizations (1.2) and (1.3) by introducing an operator for treatment of the exponent only at the finite end of a semi-infinite interval.

Comments on the structure of the operators $\mathcal{A}$ and computational examples of the modulus can be found in [7].

2. Applications

In [20] the second author introduces the concept of a *natural weight* for an approximation operator.

Definition 2.1. A weight w is called a *natural weight* for approximation by a sequence Q_n of operators in a specified norm if the norm of the weighted approximation error $w(f - Q_n f)$ allows matching direct and strong inverse estimates for the widest reasonable class of functions f.

The strong inverse estimate in the above definition may be of type A, B, C or D (in the terminology of [3]) in accordance with the structure of the sequence Q_n. The structural characteristic in the direct and strong inverse estimates can be an appropriate K-functional or modulus.

It is meaningful to consider various approximation processes in weighted function spaces with Jacobi-type weights. In [9, 11] the authors characterized the error of the Post-Widder and the Gamma operators in $L_p(w)(0,\infty)$ with arbitrary real exponents γ_0 and γ_∞. To the best of our knowledge, there are no other moduli of smoothness besides those constructed in [8] and here, which are applicable in the case of negative exponents at a finite end of the function domain. Below we shall list a number of other recent results on error characterizations in which such weights naturally appear. In our opinion, that makes treating weighted K-functionals on finite intervals with negative weight exponents important.

2.1. The Bernstein and Goodman-Sharma Operators

The Bernstein operator $B_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, is defined by

$$B_n f(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\, p_{n,k}(x), \qquad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Let $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$, as $\gamma_0, \gamma_1 \in [-1,0]$, and $\varphi(x) = \sqrt{x(1-x)}$. The second author proves in [20] (see also [1]) the following characterization of the error of B_n on $C(w)[0,1]$

$$\|w(f - B_n f)\|_{\infty[0,1]} \sim K\Big(f, \frac{1}{n}; C(w)[0,1], AC^1_{loc}, \varphi^2 D^2\Big).$$

Thus $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$ is a natural weight for the uniform approximation by Bernstein polynomials when $-1 \le \gamma_0, \gamma_1 \le 0$. Moreover, it can be shown that w is not a natural weight when either $\gamma_0 \notin [-1,0]$ or $\gamma_1 \notin [-1,0]$. That is why it seems important to characterize the above K-functional with $\gamma_0, \gamma_1 \in [-1,0]$. Let us mention that this K-functional was characterized in terms of the Ditzian-Totik moduli of smoothness [4] or the moduli introduced by the second author in [19] *only in the case* $\gamma_0 = \gamma_1 = 0$.

The above K-functional is also useful for the characterization of the error of the Goodman-Sharma operator, which is defined by

$$U_n f(x) = f(0)\, p_{n,0}(x) + f(1)\, p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 (n-1) p_{n-2,k-1}(y)\, f(y)\, dy.$$

The second author and Parvanov [21] proved that

$$\|w(f - U_n f)\|_{\infty[0,1]} \sim K\Big(f, \frac{1}{n}; C(w)[0,1], AC^1_{loc}, \varphi^2 D^2\Big)$$

for $f \in C(w)[0,1]$, where as above $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$ with $\gamma_0, \gamma_1 \in [-1,0]$.

Also, Parvanov [25] established an analogous direct estimate of the error of a class of Bernstein-type operators again in weighted spaces with negative exponents.

Lemma 7.1, Theorem 7.2, Remark 7.2 and Theorem 7.8 below imply the equivalence

Theorem 2.1. *Let* $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$, $\gamma_0, \gamma_1 \in [-1,0]$, $\varphi(x) = \sqrt{x(1-x)}$. *For every* $f \in C(w)[0,1]$ *satisfying* $\lim_{x\to 0} w(x)f(x) = 0$ *if* $-1 < \gamma_0 < 0$ *and* $\lim_{x\to 1} w(x)f(x) = 0$ *if* $-1 < \gamma_1 < 0$, *and all* $0 < t \le t_0$ *we have*

$$K\big(f, t^2; C(w)[0,1], AC^1_{loc}, \varphi^2 D^2\big) \sim \omega_2(\mathcal{A}_{\gamma_0} f, t)_{\infty[0,3/4]} + \omega_2(\mathcal{A}_{\gamma_1} \mathcal{S} f, t)_{\infty[0,3/4]},$$

where the operators are given by $\mathcal{S}g(x) = g(1-x)$ *and for* $x \in (0, 3/4]$

$$\mathcal{A}_\gamma F(x) = \begin{cases} F(x^2) - x\displaystyle\int_{3/4}^{x} y^{-2} F(y^2)\,dy, & \textit{for } \gamma = 0;\\ x^{2\gamma}F(x^2) + \dfrac{4\gamma^2-1}{5}x\displaystyle\int_{3/4}^{x} y^{2\gamma-2}F(y^2)\,dy & \\ \quad -\dfrac{4(\gamma+2)(\gamma+3)}{5}x^{-4}\displaystyle\int_{0}^{x} y^{2\gamma+3}F(y^2)\,dy, & \textit{for } -1 < \gamma < 0;\\ x^{-2}F(x^2) + 3x\displaystyle\int_{3/4}^{x} y^{-4}F(y^2)\,dy, & \textit{for } \gamma = -1. \end{cases}$$

Hence

$$\|w(f - B_n f)\|_{\infty[0,1]} \sim \omega_2(\mathcal{A}_{\gamma_0} f, n^{-1/2})_{\infty[0,3/4]} + \omega_2(\mathcal{A}_{\gamma_1}\mathcal{S}f, n^{-1/2})_{\infty[0,3/4]}$$

and

$$\|w(f - U_n f)\|_{\infty[0,1]} \sim \omega_2(\mathcal{A}_{\gamma_0} f, n^{-1/2})_{\infty[0,3/4]} + \omega_2(\mathcal{A}_{\gamma_1}\mathcal{S}f, n^{-1/2})_{\infty[0,3/4]}.$$

The restrictions in Theorem 2.1 on the behavior of f at the ends of the interval are essential (see Remark 7.4). Note that there is no restriction for f at 0 if $\gamma_0 = 0, 1$ or at 1 if $\gamma_1 = 0, 1$.

2.2. Baskakov-type Operators

The Baskakov operator V_n, $n \in \mathbb{N}$, is defined on functions f with domain $[0, \infty)$ by

$$V_n f(x) = \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big)\, v_{n,k}(x), \qquad x \geq 0,$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

Let $w(x) = x^{\gamma_0}(1+x)^{\gamma_\infty - \gamma_0}$, as $\gamma_0 \in [-1, 0]$, $\gamma_\infty \in \mathbb{R}$, and $\phi(x) = \sqrt{x(1+x)}$. Gadjev [15] (see also [14] and [17, Theorem 2.1]) established the following characterization of the error of the Baskakov operator on $C(w)[0,\infty)$

$$\|w(f - V_n f)\|_{\infty[0,\infty)} \sim K\Big(f, \frac{1}{n}; C(w)[0,\infty), AC^1_{loc}, \phi^2 D^2\Big).$$

The second author and Parvanov considered in [22] the following modification of V_n, introduced by Finta [13],

$$\bar{V}_n f(x) = f(0)\, v_{n,0}(x) + \sum_{k=1}^{\infty} v_{n,k}(x) \int_0^{\infty} (n+1) v_{n+2,k-1}(y) f(y)\,dy.$$

They proved the characterization

$$\|w(f-\bar{V}_n f)\|_{\infty[0,\infty)} \sim K\Big(f,\frac{1}{n};C(w)[0,\infty),AC^1_{loc},\phi^2D^2\Big)$$

for $f\in C(w)[0,\infty)$, where $w(x)=x^{\gamma_0}(1+x)^{\gamma_\infty-\gamma_0}$ with $\gamma_0,\gamma_\infty\in[-1,0]$.

Lemma 7.1, Theorem 7.2, Remark 7.2, Theorem 7.8 and [11, Theorem 5.14 and Section 5.4] imply the equivalence

Theorem 2.2. *Let $w(x)=x^{\gamma_0}(1+x)^{\gamma_\infty-\gamma_0}$, $\gamma_0\in[-1,0]$, $\gamma_\infty\in\mathbb{R}$, and $\phi(x)=\sqrt{x(1+x)}$. For every $f\in C(w)[0,\infty)$ satisfying $\lim_{x\to 0}w(x)f(x)=0$ if $-1<\gamma_0<0$, and all $0<t\le t_0$ we have*

$$K\big(f,t^2;C(w)[0,\infty),AC^1_{loc},\phi^2D^2\big)\sim\omega_2(\mathcal{A}_{\gamma_0}f,t)_{\infty[0,3/4]}$$
$$+\,\omega_2(\bar{\mathcal{A}}_{\gamma_\infty}f,t)_{\infty[-1,\infty)}+t^2\|\bar{\mathcal{A}}_{\gamma_\infty}f\|_{\infty[-1,\infty)},\quad \gamma_\infty\neq-1,0,$$

and

$$K\big(f,t^2;C(w)[0,\infty),AC^1_{loc},\phi^2D^2\big)\sim\omega_2(\mathcal{A}_{\gamma_0}f,t)_{\infty[0,3/4]}$$
$$+\,\omega_2(\bar{\mathcal{A}}_{\gamma_\infty}f,t)_{\infty[-1,\infty)}+t\,\omega_1(\bar{\mathcal{A}}_{\gamma_\infty}f,t)_{\infty[-1,\infty)},\quad \gamma_\infty=-1,0,$$

where $\mathcal{A}_\gamma$ is defined in Subsection 2.1, $\bar{\mathcal{A}}_\gamma f(x)=e^{\gamma x}(f(e^x)-(L_\gamma f)(e^x))$, and

$$(L_\gamma f)(y)=\begin{cases}0, & \gamma\ge 0,\\ 2f(1)-f(2), & -1\le\gamma<0,\\ [f(2)-f(1)]y+[2f(1)-f(2)], & \gamma<-1.\end{cases}$$

Hence, for $\gamma_0\in[-1,0]$, $\gamma_\infty\in\mathbb{R}$, we have

$$\|w(f-V_nf)\|_{\infty[0,\infty)}\sim\omega_2(\mathcal{A}_{\gamma_0}f,n^{-1/2})_{\infty[0,3/4]}$$
$$+\,\omega_2(\bar{\mathcal{A}}_{\gamma_\infty}f,n^{-1/2})_{\infty[-1,\infty)}+n^{-1}\|\bar{\mathcal{A}}_{\gamma_\infty}f\|_{\infty[-1,\infty)},\quad \gamma_\infty\neq-1,0,$$

and

$$\|w(f-V_nf)\|_{\infty[0,\infty)}\sim\omega_2(\mathcal{A}_{\gamma_0}f,n^{-1/2})_{\infty[0,3/4]}$$
$$+\,\omega_2(\bar{\mathcal{A}}_{\gamma_\infty}f,n^{-1/2})_{\infty[-1,\infty)}+n^{-1/2}\omega_1(\bar{\mathcal{A}}_{\gamma_\infty}f,n^{-1/2})_{\infty[-1,\infty)},\quad \gamma_\infty=-1,0;$$

and, for $\gamma_0,\gamma_\infty\in[-1,0]$, we have

$$\|w(f-\bar{V}_nf)\|_{\infty[0,\infty)}\sim\omega_2(\mathcal{A}_{\gamma_0}f,n^{-1/2})_{\infty[0,3/4]}$$
$$+\,\omega_2(\bar{\mathcal{A}}_{\gamma_\infty}f,n^{-1/2})_{\infty[-1,\infty)}+n^{-1}\|\bar{\mathcal{A}}_{\gamma_\infty}f\|_{\infty[-1,\infty)},\quad \gamma_\infty\neq-1,0,$$

and

$$\|w(f-\bar{V}_nf)\|_{\infty[0,\infty)}\sim\omega_2(\mathcal{A}_{\gamma_0}f,n^{-1/2})_{\infty[0,3/4]}$$
$$+\,\omega_2(\bar{\mathcal{A}}_{\gamma_\infty}f,n^{-1/2})_{\infty[-1,\infty)}+n^{-1/2}\omega_1(\bar{\mathcal{A}}_{\gamma_\infty}f,n^{-1/2})_{\infty[-1,\infty)},\quad \gamma_\infty=-1,0.$$

2.3. Meyer-König and Zeller-type Operators

The Meyer-König and Zeller operator in the modification of Cheney and Sharma is defined on functions f with domain $[0,1)$ and $n\in\mathbb{N}$ by

$$M_nf(x)=\sum_{k=0}^{\infty}f\Big(\frac{k}{n+k}\Big)\,m_{n,k}(x),\qquad x\in[0,1),$$

where

$$m_{n,k}(x)=\binom{n+k}{k}x^k(1-x)^{n+1}.$$

Let $w(x)=x^{\gamma_0}(1-x)^{\gamma_1}$, as $\gamma_0\in[-1,0]$ and $\gamma_1\in\mathbb{R}$, and $\psi(x)=\sqrt{x}(1-x)$. Gadjev [16] (see also [17, Theorem 1.1]) established the following characterization of the Meyer-König and Zeller operator on $C(w)[0,1)$

$$\|w(f-M_nf)\|_{\infty[0,1)}\sim K\Big(f,\frac{1}{n};C(w)[0,1),AC^1_{loc},\psi^2D^2\Big).$$

Earlier the second author and Parvanov [23] verified a similar characterization of the error of the Goodman-Sharma modification of M_n, given by

$$\bar{M}_nf(x)=f(0)\,m_{n,0}(x)+\sum_{k=1}^{\infty}m_{n,k}(x)\int_0^1 n\,m_{n,k-1}(y)f(y)\frac{1}{(1-y)^2}\,dy.$$

For $f\in C(w)[0,1)$, where $w(x)=x^{\gamma_0}(1-x)^{\gamma_1}$ with $\gamma_0,\gamma_1\in[-1,0]$, they proved the characterization

$$\|w(f-\bar{M}_nf)\|_{\infty[0,1)}\sim K\Big(f,\frac{1}{n};C(w)[0,1),AC^1_{loc},\psi^2D^2\Big).$$

Lemma 7.1, Theorem 7.2, Remark 7.2, Theorem 7.8 and [11, Theorem 5.13 and Section 5.4] imply the equivalence

Theorem 2.3. *Let* $w(x)=x^{\gamma_0}(1-x)^{\gamma_1}$*, as* $\gamma_0\in[-1,0]$ *and* $\gamma_1\in\mathbb{R}$*, and* $\psi(x)=\sqrt{x}(1-x)$*. For every* $f\in C(w)[0,1)$ *satisfying* $\lim_{x\to 0}w(x)f(x)=0$ *if* $-1<\gamma_0<0$*, and all* $0<t\le t_0$ *we have*

$$K\big(f,t^2;C(w)[0,1),AC^1_{loc},\psi^2D^2\big)\sim\omega_2(\mathcal{A}_{\gamma_0}f,t)_{\infty[0,3/4]}+\omega_2(\hat{\mathcal{A}}_{\gamma_1}\mathcal{S}f,t)_{\infty(-\infty,-1]}+t^2\|\hat{\mathcal{A}}_{\gamma_1}\mathcal{S}f\|_{\infty(-\infty,-1]},\qquad\gamma_1\neq-1,0,$$

and

$$K\big(f,t^2;C(w)[0,1),AC^1_{loc},\psi^2D^2\big)\sim\omega_2(\mathcal{A}_{\gamma_0}f,t)_{\infty[0,3/4]}+\omega_2(\hat{\mathcal{A}}_{\gamma_1}\mathcal{S}f,t)_{\infty(-\infty,-1]}+t\,\omega_1(\hat{\mathcal{A}}_{\gamma_1}\mathcal{S}f,t)_{\infty(-\infty,-1]},\qquad\gamma_1=-1,0,$$

where $\mathcal{A}_\gamma$ *and* $\mathcal{S}$ *are defined in Subsection 2.1,* $\hat{\mathcal{A}}_\gamma F(x) = e^{\gamma x}(F(e^x) - (\mathcal{L}_\gamma F)(e^x))$*, and*

$$(\mathcal{L}_\gamma F)(y) = \begin{cases} 0, & \gamma \le -1, \\ [F(-1) - F(-2)]y, & -1 < \gamma \le 0, \\ [F(-1) - F(-2)]y + [2F(-1) - F(-2)], & \gamma > 0. \end{cases}$$

Hence, for $\gamma_0 \in [-1, 0], \gamma_1 \in \mathbb{R}$*, we have*

$$\begin{aligned} \|w(f - M_n f)\|_{\infty[0,1)} &\sim \omega_2(\mathcal{A}_{\gamma_0} f, n^{-1/2})_{\infty[0,3/4]} \\ &+ \omega_2(\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f, n^{-1/2})_{\infty(-\infty,-1]} + n^{-1} \|\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f\|_{\infty(-\infty,-1]}, \qquad \gamma_1 \ne -1, 0, \end{aligned}$$

and

$$\begin{aligned} \|w(f - M_n f)\|_{\infty[0,1)} \sim \omega_2(\mathcal{A}_{\gamma_0} f, n^{-1/2})_{\infty[0,3/4]} + \omega_2(\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f, n^{-1/2})_{\infty(-\infty,-1]} \\ + n^{-1/2} \omega_1(\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f, n^{-1/2})_{\infty(-\infty,-1]}, \qquad \gamma_1 = -1, 0; \end{aligned}$$

and, for $\gamma_0, \gamma_1 \in [-1, 0]$*, we have*

$$\begin{aligned} \|w(f - \bar{M}_n f)\|_{\infty[0,1)} &\sim \omega_2(\mathcal{A}_{\gamma_0} f, n^{-1/2})_{\infty[0,3/4]} \\ &+ \omega_2(\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f, n^{-1/2})_{\infty(-\infty,-1]} + n^{-1} \|\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f\|_{\infty(-\infty,-1]}, \qquad \gamma_1 \ne -1, 0, \end{aligned}$$

and

$$\begin{aligned} \|w(f - \bar{M}_n f)\|_{\infty[0,1)} \sim \omega_2(\mathcal{A}_{\gamma_0} f, n^{-1/2})_{\infty[0,3/4]} + \omega_2(\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f, n^{-1/2})_{\infty(-\infty,-1]} \\ + n^{-1/2} \omega_1(\hat{\mathcal{A}}_{\gamma_1} \mathcal{S} f, n^{-1/2})_{\infty(-\infty,-1]}, \qquad \gamma_1 = -1, 0. \end{aligned}$$

3. A Method for Characterizing K-functionals

Let us now briefly present the method we used in [5, 8] to establish relations like (1.2) and (1.3). Let X be a Banach space, $\mathcal{D}$ be a differential operator with a domain $\mathcal{D}^{-1}(X) = \{g \in X : \mathcal{D}g \in X\}$, and Y be a linear space such that $Y \cap \mathcal{D}^{-1}(X)$ is dense in X. We consider the Peetre K-functional between the spaces X and Y

$$K(f, t; X, Y, \mathcal{D}) = \inf\bigl\{\|f - g\|_X + t\|\mathcal{D}g\|_X \ : \ g \in Y \cap \mathcal{D}^{-1}(X)\bigr\}.$$

Definition 3.1. We say that the linear operator $\mathcal{A}$ is *a quasi-invertible continuous map of the triplet* $(X_1, Y_1, \mathcal{D}_1)$ *onto the triplet* $(X_2, Y_2, \mathcal{D}_2)$ if and only if there exists a linear operator $\mathcal{B} : X_2 \to X_1$, related to $\mathcal{A} : X_1 \to X_2$, which we call *a quasi-inverse operator to* $\mathcal{A}$, and both operators satisfy the conditions:

(a) $\|\mathcal{A}f\|_{X_2} \le c\|f\|_{X_1}$ for any $f \in X_1$;

(b) $\|\mathcal{D}_2\mathcal{A}f\|_{X_2} \le c\|\mathcal{D}_1 f\|_{X_1}$ for any $f \in Y_1 \cap \mathcal{D}_1^{-1}(X_1)$;

(c) $\|\mathcal{B}F\|_{X_1} \le c\|F\|_{X_2}$ for any $F \in X_2$;

(d) $\|\mathcal{D}_1\mathcal{B}F\|_{X_1} \le c\|\mathcal{D}_2 F\|_{X_2}$ for any $F \in Y_2 \cap \mathcal{D}_2^{-1}(X_2)$;

(e) $\mathcal{A}(Y_1 \cap \mathcal{D}_1^{-1}(X_1)) \subseteq Y_2 \cap \mathcal{D}_2^{-1}(X_2)$;

(f) $\mathcal{B}(Y_2 \cap \mathcal{D}_2^{-1}(X_2)) \subseteq Y_1 \cap \mathcal{D}_1^{-1}(X_1)$;

(g) $f - \mathcal{B}\mathcal{A}f \in Y_1 \cap \ker \mathcal{D}_1$ for any $f \in X_1$;

(h) $F - \mathcal{A}\mathcal{B}F \in Y_2 \cap \ker \mathcal{D}_2$ for any $F \in X_2$.

If $\mathcal{A}$ is a quasi-invertible continuous map of $(X_1, Y_1, \mathcal{D}_1)$ onto $(X_2, Y_2, \mathcal{D}_2)$ and $\mathcal{B}$ is a quasi-inverse operator to $\mathcal{A}$, we write

$$\mathcal{A} : (X_1, Y_1, \mathcal{D}_1) \rightleftharpoons (X_2, Y_2, \mathcal{D}_2) : \mathcal{B}.$$

Above we have set $\ker \mathcal{D} = \{g \in \mathcal{D}^{-1}(X) : \mathcal{D}g = 0\}$. Note that $\ker \mathcal{D} \subset \mathcal{D}^{-1}(X) \subset X$.

If two triplets satisfy the conditions of the definition above, the corresponding K-functionals can be related by means of continuous linear transforms. In [8, Proposition 2.1] we showed that the following assertion holds.

Proposition 3.1. *Let the linear operator $\mathcal{A}$ be a quasi-invertible continuous map of $(X_1, Y_1, \mathcal{D}_1)$ onto $(X_2, Y_2, \mathcal{D}_2)$ and $\mathcal{B}$ be quasi-inverse to $\mathcal{A}$. Then for any $f \in X_1$ and $t > 0$ we have*

$$K(f, t; X_1, Y_1, \mathcal{D}_1) \sim K(\mathcal{A}f, t; X_2, Y_2, \mathcal{D}_2)$$

and for any $F \in X_2$ and $t > 0$ we have

$$K(F, t; X_2, Y_2, \mathcal{D}_2) \sim K(\mathcal{B}F, t; X_1, Y_1, \mathcal{D}_1).$$

Further, as it is known the unweighted K-functional is equivalent to the classical modulus of smoothness (see e.g. [2, Ch. 6, Theorem 2.4])

$$K(F, t^r; L_p(\tilde{I}), AC_{loc}^{r-1}, D^r) \sim \omega_r(F, t)_{p(\tilde{I})}. \tag{3.1}$$

Thus, if

$$\mathcal{A} : (X, Y, \mathcal{D}) \rightleftharpoons (L_p(\tilde{I}), AC_{loc}^{r-1}, D^r) : \mathcal{B},$$

we arrive at the characterization

$$K(f, t; X, Y, \mathcal{D}) \sim \omega_r(\mathcal{A}f, t)_{p(\tilde{I})}.$$

4. Inequalities for Intermediate Derivatives

As is well-known (see e.g. [2, Ch. 2, Theorem 5.6]):

$$(b-a)^k \, \|g^{(k)}\|_{p[a,b]} \le c\big(\|g\|_{p[a,b]} + (b-a)^r \, \|g^{(r)}\|_{p[a,b]}\big) \tag{4.1}$$

for every $g \in W_p^r[a,b]$ and $k = 0, 1, \dots, r$, and also

$$\|g^{(k)}\|_{p(J)} \le c\big(\|g\|_{p(J)} + \|g^{(r)}\|_{p(J)}\big) \tag{4.2}$$

for every $g \in W_p^r(J)$ and $k = 0, 1, \dots, r$, where $J = (-\infty, \infty)$ or $J = (-\infty, a)$ or $J = (a, \infty)$, $a \in \mathbb{R}$. The constant c in (4.1) and (4.2) depends only on r.

Let us set $\chi_\xi(x) = |x - \xi|$ for $\xi \in \mathbb{R}$. Proposition 4.1 in [11] directly implies the following analogues of (4.1) and (4.2) in weighted L_p-norm.

Proposition 4.1. *Let $1 \le p \le \infty$, $r \in \mathbb{N}$, $k \in \{0, 1, \dots, r\}$ and $\alpha, \beta, \gamma \in \mathbb{R}$.*

(a) If $g \in AC_{loc}^{r-1}(a,b)$, $\chi_a^\alpha g, \chi_a^{\beta+r} g^{(r)} \in L_p(a,b)$ and $\gamma \ge \alpha, \beta$, then

$$\|\chi_a^{\gamma+k} g^{(k)}\|_{p(a,b)} \le c\big(\|\chi_a^\alpha g\|_{p(a,b)} + \|\chi_a^{\beta+r} g^{(r)}\|_{p(a,b)}\big).$$

(b) If $g \in AC_{loc}^{r-1}(a,\infty)$, $\chi_{a-1}^\alpha g, \chi_{a-1}^{\beta+r} g^{(r)} \in L_p(a,\infty)$ and $\gamma \le \alpha, \beta$, then

$$\|\chi_{a-1}^{\gamma+k} g^{(k)}\|_{p(a,\infty)} \le c\big(\|\chi_{a-1}^\alpha g\|_{p(a,\infty)} + \|\chi_{a-1}^{\beta+r} g^{(r)}\|_{p(a,\infty)}\big).$$

The constant c is independent of g.

We shall show that in certain cases the first summand on the right-hand side of the inequalities above can be omitted. To prove that we shall use the well-known Hardy's inequalities (see [18, p. 245]), generalized by Muckenhoupt [24].

Proposition 4.2. *Let $\zeta < \eta$ and let F be a measurable function on $[\zeta, \eta]$.*

(a) If $1 \le p \le \infty$, $\beta > 0$, $\gamma \le \beta$ or $p = 1$, $\beta = 0$, $\gamma < 0$, then (with the suitable amendment in the notation for $p = \infty$)

$$\left(\int_\zeta^\eta \Big|(x-\zeta)^{-\gamma-\frac{1}{p}} \int_\zeta^x F(y)\,dy\Big|^p dx\right)^{1/p} \le c\left(\int_\zeta^\eta |(x-\zeta)^{-\beta+1-\frac{1}{p}} F(x)|^p \,dx\right)^{1/p}.$$

(b) If $1 \le p \le \infty$, $\beta \le \gamma$, $\gamma > 0$ or $p = \infty$, $\beta < 0$, $\gamma = 0$, then (with the suitable amendment in the notation for $p = \infty$)

$$\left(\int_\zeta^\eta \Big|(x-\zeta)^{\gamma-\frac{1}{p}} \int_x^\eta F(y)\,dy\Big|^p dx\right)^{1/p} \le c\left(\int_\zeta^\eta |(x-\zeta)^{\beta+1-\frac{1}{p}} F(x)|^p \,dx\right)^{1/p}.$$

Proposition 4.3. *Let $\eta > 0$ and let F be a measurable function on $[\eta, \infty)$.*
(a) If $1 \le p \le \infty$, $\beta \le \gamma$, $\gamma > 0$ or $p = \infty$, $\beta < 0$, $\gamma = 0$, then (with the suitable amendment in the notation for $p = \infty$)

$$\left(\int_\eta^\infty \left| x^{-\gamma-\frac{1}{p}} \int_\eta^x F(y)\,dy \right|^p dx\right)^{1/p} \le c\left(\int_\eta^\infty |x^{-\beta+1-\frac{1}{p}} F(x)|^p\,dx\right)^{1/p}.$$

(b) If $1 \le p \le \infty$, $\beta \ge \gamma$, $\beta > 0$ or $p = 1$, $\beta = 0$, $\gamma < 0$, then (with the suitable amendment in the notation for $p = \infty$)

$$\left(\int_\eta^\infty \left| x^{\gamma-\frac{1}{p}} \int_x^\infty F(y)\,dy \right|^p dx\right)^{1/p} \le c\left(\int_\eta^\infty |x^{\beta+1-\frac{1}{p}} F(x)|^p\,dx\right)^{1/p}.$$

The constant c is independent of F.

We shall also need the following auxiliary result, established in the proof of [11, Lemma 3.1].

Lemma 4.1. *Let $1 \le p \le \infty$ and $\beta \in \mathbb{R}$.*

(a) If $g \in AC_{loc}(a,b)$, $\chi_a^{\beta+1} g' \in L_p(a,b)$ and $\beta < -1/p$, then $g(x)$ has a final limit at a.

(b) If $g \in AC_{loc}(a,\infty)$, $\chi_{a-1}^{\beta+1} g' \in L_p(a,\infty)$ and $\beta > -1/p$, then $g(x)$ has a final limit at ∞.

Remark 4.1. Let us note that the function g is not required to belong to any weighted L_p-space. If in (a) we have in addition $\chi_a^\alpha g \in L_p(a,b)$, with $\alpha \le -1/p$ for $p < \infty$, or $\alpha < 0$ for $p = \infty$, we get $\lim_{x\to a+0} g(x) = 0$. Similarly, if in (b) we have in addition $\chi_{a-1}^\alpha g \in L_p(a,\infty)$ with $\alpha \ge -1/p$ for $p < \infty$, or $\alpha > 0$ for $p = \infty$, we get $\lim_{x\to\infty} g(x) = 0$.

Remark 4.2. Observe that though g itself is not required to belong to any weighted L_p-space, in the hypotheses of (a) we have $\chi_a^\beta(g-\ell) \in L_p(a,b)$, where $\ell = \lim_{x\to a+0} g(x)$, and in the hypotheses of (b) we have $\chi_{a-1}^\beta(g-\ell) \in L_p(a,\infty)$, where $\ell = \lim_{x\to\infty} g(x)$, as it is established by Hardy's inequalities.

By means of the lemma above we can improve the inequalities of Proposition 4.1 in the following sense:

Proposition 4.4. *Let $1 \le p \le \infty$, $r \in \mathbb{N}$, $k \in \{0,1,\dots,r\}$, $\alpha,\beta,\delta \in \mathbb{R}$ and $a,b,\xi \in \mathbb{R}$ as $a < \xi < b$.*

(a) If $g \in AC_{loc}^{r-1}(a,b)$, $\chi_a^\alpha g, \chi_a^{\beta+r} g^{(r)} \in L_p(a,\xi)$ and $\chi_b^\delta g^{(r)} \in L_p(\xi,b)$ as $\beta < 1-r-1/p$, $\delta \in \Gamma_+(p)$ and also $\alpha \le 1-r-1/p$ for $p < \infty$ and $\alpha < 1-r$ for $p = \infty$, then

$$\|\chi_a^{\beta+k}\chi_b^\delta g^{(k)}\|_{p(a,b)} \le c\,\|\chi_a^{\beta+r}\chi_b^\delta g^{(r)}\|_{p(a,b)}.$$

(b) If $g \in AC_{loc}^{r-1}(a,\infty)$, $\chi_{a-1}^{\alpha} g, \chi_{a-1}^{\beta+r} g^{(r)} \in L_p(\xi,\infty)$ and $\chi_a^{\delta} g^{(r)} \in L_p(a,\xi)$ as $\beta > -1/p$, $\delta \in \Gamma_+(p)$ and also $\alpha \geq -1/p$ for $p < \infty$ and $\alpha > 0$ for $p = \infty$, then

$$\|\chi_a^{\delta}\chi_{a-1}^{\beta+k-\delta} g^{(k)}\|_{p(a,\infty)} \leq c\,\|\chi_a^{\delta}\chi_{a-1}^{\beta+r-\delta} g^{(r)}\|_{p(a,\infty)}.$$

The constant c is independent of g.

Proof. It is enough to prove the assertions of the proposition for $k = r-1$. The general case follows from it by iteration.

Let us consider (a). Proposition 4.1.a with $k = r-1$, $\gamma = \max\{\alpha,\beta\}$ and b replaced with ξ implies $\chi_a^{\gamma+r-1} g^{(r-1)} \in L_p(a,\xi)$. Next, since $\chi_a^{\beta+r} g^{(r)} \in L_p(a,\xi)$ with $\beta + r - 1 < -1/p$, and also $\gamma + r - 1 \leq -1/p$ for $p < \infty$, and $\gamma + r - 1 < 0$ for $p = \infty$ we get by Lemma 4.1.a and Remark 4.1 that $\lim_{x\to a+0} g^{(r-1)}(x) = 0$. Consequently, we have

$$g^{(r-1)}(x) = \int_a^x g^{(r)}(y)\,dy.$$

Further, we get by Hardy's inequality ($\beta + r - 1 < -1/p$)

$$\|\chi_a^{\beta+r-1} g^{(r-1)}\|_{p(a,\xi)} \leq c\,\|\chi_a^{\beta+r} g^{(r)}\|_{p(a,\xi)}. \tag{4.3}$$

To estimate the L_p-norm of $\chi_b^{\delta} g^{(r-1)}$ on the interval (ξ,b) we proceed as follows (with the appropriate amendment in the notation for $p=\infty$):

$$\begin{aligned}
\|\chi_b^{\delta} g^{(r-1)}\|_{p(\xi,b)} &\leq \left(\int_\xi^b \left|(b-x)^{\delta}\int_a^\xi g^{(r)}(y)\,dy\right|^p dx\right)^{1/p} \\
&\qquad + \left(\int_\xi^b \left|(b-x)^{\delta}\int_\xi^x g^{(r)}(y)\,dy\right|^p dx\right)^{1/p} \\
&\leq c\big(\|g^{(r)}\|_{1(a,\xi)} + \|\chi_b^{\delta} g^{(r)}\|_{p(\xi,b)}\big) \\
&\leq c\big(\|\chi_a^{\beta+r} g^{(r)}\|_{p(a,\xi)} + \|\chi_b^{\delta} g^{(r)}\|_{p(\xi,b)}\big) \\
&\leq c\,\|\chi_a^{\beta+r}\chi_b^{\delta} g^{(r)}\|_{p(a,b)},
\end{aligned} \tag{4.4}$$

as at the second step we have applied Hardy's inequality of Proposition 4.2.b to the second summand, taking into account that $\delta \in \Gamma_+(p)$, and at the third step we have applied Hölder's inequality to the first summand, taking into account that $-\beta - r > 1/p - 1$. Combining (4.3) and (4.4), we get assertion a) of the proposition for $k = r-1$.

Assertion (b) is also verified by this technique as we apply Proposition 4.1.b and Lemma 4.1.b instead of Proposition 4.1.a and Lemma 4.1.a. □

5. Operators That Change Only the Weight w

Here we shall consider bounded linear operators, relating two K-functionals with different w-weight and the same φ-weight. We call them A-operators.

To describe the restrictions on the exponents γ of the weight w, given in Table 1, we use the notations:

$$\begin{aligned}
&\Gamma_+(p) = (-1/p, \infty) \text{ for } 1 \le p < \infty \text{ and } \Gamma_+(\infty) = [0, \infty);\\
&\Gamma_0(p) = (-1/p, \infty);\\
&\Gamma_i(p) = (-i - 1/p, 1 - i - 1/p), \quad i = 1, \dots, r-1;\\
&\Gamma_r(p) = (-\infty, 1 - r - 1/p);\\
&\Gamma_{exc}(p) = \{1 - r - 1/p, 2 - r - 1/p, \dots, -1/p\};\\
&\Gamma_1^*(p) = (-1 - 1/p, \infty);\\
&\Gamma_i^*(p) = (-i - 1/p, 1 - i - 1/p), \quad i = 2, \dots, r-1;\\
&\Gamma_r^*(p) = (-\infty, 1 - r - 1/p);\\
&\Gamma_{exc}^*(p) = \{1 - r - 1/p, 2 - r - 1/p, \dots, -1 - 1/p\}.
\end{aligned}$$

5.1. A-operators for Treatment of Weights w with $\gamma_a \in \Gamma_i(p)$, $\gamma_b \in \Gamma_j(p)$ as at least one of i or j is equal to 0

In [5, 8] we defined the following A-operators.

Definition 5.1. Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $i \in \mathbb{N}_0$ as $i \le r$, and $\xi \in (a, b)$. Let s be one of the ends of the finite interval (a, b) and e the other. For $x \in (a, b)$ and $f \in L_{1,loc}(a, b)$, satisfying the additional requirement $\chi_s^{-i+\rho} f \in L_1(s, (s+e)/2)$ if $i > 0$, we set

$$\begin{aligned}
(A_i(\rho; s, e; \xi) f)(x) = {} & \Big(\frac{x-s}{e-s}\Big)^{\rho} f(x)\\
& + \frac{1}{e-s} \sum_{k=1}^{i} \alpha_{r,k}(\rho) \Big(\frac{x-s}{e-s}\Big)^{k-1} \int_s^x \Big(\frac{y-s}{e-s}\Big)^{-k+\rho} f(y)\, dy\\
& + \frac{1}{e-s} \sum_{k=i+1}^{r} \alpha_{r,k}(\rho) \Big(\frac{x-s}{e-s}\Big)^{k-1} \int_\xi^x \Big(\frac{y-s}{e-s}\Big)^{-k+\rho} f(y)\, dy,
\end{aligned}$$

where $\alpha_{r,k}(\rho)$ are defined by

$$\alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!} \binom{r-1}{k-1} \prod_{\nu=0}^{r-1} (\rho + r - k - \nu), \qquad k = 1, 2, \dots, r. \tag{5.1}$$

These operators relate two K-functionals with the same φ-weight but w-weights with different exponents at the end s and the same exponent at the other end e provided that the exponents of the w-weights at s are not in $\Gamma_{exc}(p)$ and the exponent at e is in $\Gamma_+(p)$ as we showed in [8, Proposition 3.9].

Definition 5.2. Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $j \in \mathbb{N}_0$ as $j \le r$, and $\xi \in (a,\infty)$. For $x \in (a,\infty)$ and $f \in L_{1,loc}(a,\infty)$, satisfying the additional requirement $\chi_a^{-j-1+\rho} f \in L_1(a+1,\infty)$ if $j < r$, we set

$$\begin{aligned}(A_j(\rho;\infty,a;\xi)f)(x) &= (x-a+1)^\rho f(x) \\ &+ \sum_{k=1}^{j} \alpha_{r,k}(\rho)(x-a+1)^{k-1} \int_\xi^x (y-a+1)^{-k+\rho} f(y)\,dy \\ &- \sum_{k=j+1}^{r} \alpha_{r,k}(\rho)(x-a+1)^{k-1} \int_x^\infty (y-a+1)^{-k+\rho} f(y)\,dy,\end{aligned} \tag{5.2}$$

where $\alpha_{r,k}(\rho)$ are defined in (5.1).

These operators relate two K-functionals with the same φ-weight but w-weights with different exponents at infinity and the same exponents at the finite end provided that the exponents of the w-weights at infinity are not in $\Gamma_{exc}(p)$ and the exponent at the finite end is in $\Gamma_+(p)$ as we showed in [8, Proposition 3.8].

In the cases when the operators above do not actually depend on the parameter ξ, we shall write $*$ instead of ξ to underline this fact. For example, we shall write $A_r(\rho; s, e; *)$ and $A_0(\rho; \infty, a; *)$ instead of $A_r(\rho; s, e; \xi)$ and $A_0(\rho; \infty, a; \xi)$, respectively. We shall also use this convention for the A-operators to be defined later in this section.

5.2. A-operators for Treatment of Weights w with $\gamma_s \in \Gamma_i(p)$, $\gamma_e \in \Gamma_j(p)$, $i + j \le r$

By an appropriate modification of $A_i(\rho; s, e; \xi)$ we can extend its application. First, we introduce the following polynomials.

Definition 5.3. For $\mu \in \mathbb{N}$, $\nu \in \mathbb{N}_0$ set

$$q_{\mu,\nu}(x) = (1-x)^\nu \sum_{\ell=0}^{\mu-1} \binom{\nu-1+\ell}{\ell} x^\ell.$$

These polynomials are constructed so that they possess the following properties. Below Π_n denotes the set of all algebraic polynomials of degree at most n.

Lemma 5.1. *We have* $q_{\mu,\nu} \in \Pi_{\mu+\nu-1}$, $q_{\mu,\nu}(0) = 1$, $q_{\mu,\nu}^{(k)}(0) = 0$ *for* $k = 1, 2, \dots, \mu-1$, *and* $q_{\mu,\nu}^{(k)}(1) = 0$ *for* $k = 0, 1, \dots, \nu-1$. *Also for* $x \in [0,1]$ *we have*

$$0 \le 1 - q_{\mu,\nu}(x) \le c\, x^\mu, \qquad 0 \le q_{\mu,\nu}(x) \le c\,(1-x)^\nu.$$

Proof. For $\nu = 0$ we have $q_{\mu,0}(x) \equiv 1$ and the assertion is trivial. For $\nu > 0$ the polynomial of degree $\mu + \nu - 1$ which satisfies the conditions $q_{\mu,\nu}(0) = 1$, $q^{(k)}_{\mu,\nu}(0) = 0$ for $k = 1, 2, \ldots, \mu - 1$, and $q^{(k)}_{\mu,\nu}(1) = 0$ for $k = 0, 1, \ldots, \nu - 1$ is given by

$$q_{\mu,\nu}(x) = \frac{1}{A} \int_x^1 y^{\mu-1}(1-y)^{\nu-1}\, dy,$$

where

$$A = \int_0^1 y^{\mu-1}(1-y)^{\nu-1}\, dy = \frac{(\mu-1)!\,(\nu-1)!}{(\mu+\nu-1)!}.$$

Then the inequalities of the lemma are valid with $c = 1/(\mu A)$ and $c = 1/(\nu A)$ respectively. □

Definition 5.4. Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $i, j \in \mathbb{N}_0$ as $i \le r$. Let s be one of the ends of the finite interval (a, b) and e the other. For $x \in (a, b)$ and $f \in L_1(\chi_s^{-i+\rho})(a, b)$ if $i > 0$ or $f \in L_{1,loc}(a, b) \cap L_1((s + e)/2, e)$ if $i = 0$ we set

$$\begin{aligned}
&(A_{i,j}(\rho; s, e; e)f)(x) = \Big(\frac{x-s}{e-s}\Big)^{\rho} f(x) \\
&+ \frac{1}{e-s} \sum_{k=1}^{r} \alpha_{r,k}(\rho) \Big(\frac{x-s}{e-s}\Big)^{k-1} \int_e^x \Big(\frac{y-s}{e-s}\Big)^{-k+\rho} f(y)\, dy \\
&- \frac{1}{e-s} \sum_{k=1}^{i} \alpha_{r,k}(\rho)\; q_{i-k+1,j}\Big(\frac{x-s}{e-s}\Big) \Big(\frac{x-s}{e-s}\Big)^{k-1} \int_e^s \Big(\frac{y-s}{e-s}\Big)^{-k+\rho} f(y)\, dy,
\end{aligned}$$

where $\alpha_{r,k}(\rho)$ are given in (5.1).

Note that $A_{i,0}(\rho; s, e; e) = A_i(\rho; s, e; e)$. The following proposition shows how $A_{i,j}$ acts between triplets.

Proposition 5.1. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, s be one of the ends of the finite interval (a, b) and e be the other one, and also $w = \chi_s^{\gamma_s}\chi_e^{\gamma_e}$. Let $\rho, \gamma_s, \gamma_e \in \mathbb{R}$ and $i, i', j \in \mathbb{N}_0$ be such that $\gamma_s \in \Gamma_i(p)$, $\gamma_s + \rho \in \Gamma_{i'}(p)$, $\gamma_e \in \Gamma_-(p)$, $\gamma_e + j \in \Gamma_+(p)$, $i + j \le r$ and $i' + j \le r$. Finally, let ϕ be measurable and non-negative on (a, b). Then we have*

$$\begin{aligned}
A_{i,j}(\rho; s, e; e) &: (L_p(w\chi_s^{\rho})(a, b), AC_{loc}^{r-1}, \phi D^r) \\
&\rightleftharpoons (L_p(w)(a, b), AC_{loc}^{r-1}, \phi D^r) : A_{i',j}(-\rho; s, e; e).
\end{aligned}$$

Proof. Let us first note that if $i > 0$, then from Hölder's inequality and the conditions on γ_s, γ_e we get for $k = 1, \ldots, i$

$$\left| \int_e^s \Big(\frac{y-s}{e-s}\Big)^{-k+\rho} f(y)\, dy \right| \le (b-a)^{k-\rho} \|\chi_s^{-k} w^{-1}\|_{p'(a,b)} \|w\chi_s^{\rho} f\|_{p(a,b)}. \tag{5.3}$$

Thus, $A_{i,j}(\rho; s, e; e)f$ is well defined for every $f \in L_p(w\chi_s^\rho)(a, b)$. Similarly, $A_{i',j}(-\rho; s, e; e)F$ is well defined for every $F \in L_p(w)(a, b)$.

We shall show that the operators $\mathcal{A} = A_{i,j}(\rho; s, e; e)$ and $\mathcal{B} = A_{i',j}(-\rho; s, e; e)$ satisfy Definition 3.1 with $X_1 = L_p(w\chi_s^\rho)(a, b)$, $X_2 = L_p(w)(a, b)$, $Y_1 = Y_2 = AC_{loc}^{r-1}(a, b)$ and $\mathcal{D}_1 = \mathcal{D}_2 = \phi D^r$.

To establish condition (a) of Definition 3.1 we proceed as follows.

For $x \in (s, (s + e)/2]$ we use the representation

$$\begin{aligned}(A_{i,j}(\rho; s, e; e)f)(x) &= (A_i(\rho; s, e; e)f)(x) \\ &+ \frac{1}{e-s} \sum_{k=1}^{i} \alpha_{r,k}(\rho) \Big[q_{i-k+1,j}\Big(\frac{x-s}{e-s}\Big) - 1 \Big] \\ &\times \Big(\frac{x-s}{e-s}\Big)^{k-1} \int_s^e \Big(\frac{y-s}{e-s}\Big)^{-k+\rho} f(y)\, dy.\end{aligned} \tag{5.4}$$

By means of Hardy's and Hölder's inequalities, as in the proof of [5, Proposition 5.2] and [8, Proposition 3.1], we establish the estimate

$$\|wA_i(\rho; s, e; e)\|_{p(s,(s+e)/2)} \le c\, \|w\chi_s^\rho\|_{p(a,b)} \tag{5.5}$$

Next, if $i > 0$, then for $x \in (s, (s + e)/2]$ in view of Lemma 5.1 we have

$$w(x) \Big| q_{i-k+1,j}\Big(\frac{x-s}{e-s}\Big) - 1 \Big| \Big(\frac{x-s}{e-s}\Big)^{k-1} \le c\, \chi_s^{\gamma_s + i}(x), \qquad k = 1, \dots, i.$$

If $j > 0$, then $i < r$ and hence, in view of $\gamma_s \in \Gamma_i(p)$, we get $\gamma_s + i \in \Gamma_+(p)$. Consequently, (with the proper modification for $p = \infty$)

$$\Big| \int_s^{(s+e)/2} \Big| w(x) \Big[q_{i-k+1,j}\Big(\frac{x-s}{e-s}\Big) - 1 \Big] \Big(\frac{x-s}{e-s}\Big)^{k-1} \Big|^p dx \Big|^{1/p} \le c, \quad k = 1, \dots, i. \tag{5.6}$$

For $j = 0$, we have $q_{i-k+1,j}(x) \equiv 1$ for $k = 1, \dots, i$ and (5.6) is trivial.

Now, (5.3)–(5.6) imply

$$\|wA_{i,j}(\rho; s, e; e)\|_{p(s,(s+e)/2)} \le c\, \|w\chi_s^\rho\|_{p(a,b)}. \tag{5.7}$$

For $x \in [(s + e)/2, e)$ we use the representation

$$\begin{aligned}(A_{i,j}(\rho; s, e; e)f)(x) &= (A_0(\rho; s, e; e)f)(x) \\ &+ \frac{1}{e-s} \sum_{k=1}^{i} \alpha_{r,k}(\rho)\ q_{i-k+1,j}\Big(\frac{x-s}{e-s}\Big) \Big(\frac{x-s}{e-s}\Big)^{k-1} \int_s^e \Big(\frac{y-s}{e-s}\Big)^{-k+\rho} f(y)\, dy.\end{aligned} \tag{5.8}$$

The inequality

$$\|\chi_e^{\gamma_e} A_0(\rho; s, e; e)\|_{p(s,(s+e)/2)} \le c\, \|\chi_e^{\gamma_e} f\|_{p(s,(s+e)/2)} \tag{5.9}$$

is established in the proof of [5, Proposition 5.2], but it follows directly from Hardy's inequality given in Proposition 4.2.a.

If $i > 0$, then for $x \in [(s+e)/2, e)$ by means of Lemma 5.1 we get

$$w(x)\Big|q_{i-k+1,j}\Big(\frac{x-s}{e-s}\Big)\Big|\Big(\frac{x-s}{e-s}\Big)^{k-1} \leq c\,\chi_e^{\gamma_e+j}(x), \qquad k = 1,\ldots,i,$$

which, in view of $\gamma_e + j \in \Gamma_+(p)$, implies

$$\Big|\int_{(s+e)/2}^{e}\Big|w(x)\Big|q_{i-k+1,j}\Big(\frac{x-s}{e-s}\Big)\Big|\Big(\frac{x-s}{e-s}\Big)^{k-1}\Big|^p dx\Big|^{1/p} \leq c, \quad k = 1,\ldots,i. \tag{5.10}$$

Now, (5.3) and (5.8)–(5.10) yield

$$\|wA_{i,j}(\rho;s,e;e)f\|_{p((s+e)/2,e)} \leq c\|w\chi_s^\rho f\|_{p(a,b)}. \tag{5.11}$$

The estimates (5.7) and (5.11) imply condition (a) of Definition 3.1.

By replacing γ_s with $\gamma_s + \rho$, i with i' and ρ with $-\rho$ in (a) we get (c) of Definition 3.1:

$$\|w\chi_s^\rho A_{i',j}(-\rho;s,e;e)F\|_{p(a,b)} \leq c\|wF\|_{p(a,b)}, \qquad F \in L_p(w)(a,b).$$

As it was shown in [8, Section 3]

$$(A_i(\rho;s,e;e)g)^{(r)}(x) = \Big(\frac{x-s}{e-s}\Big)^\rho g^{(r)}(x), \qquad x \in (a,b),$$

provided that $g \in Y_1 \cap X_1$. Now, since

$$A_{i,j}(\rho;s,e;e)g - A_i(\rho;s,e;e)g \in \Pi_{i+j-1}$$

and $i + j \leq r$, we get for all $g \in Y_1 \cap X_1$

$$(A_{i,j}(\rho;s,e;e)g)^{(r)}(x) = \Big(\frac{x-s}{e-s}\Big)^\rho g^{(r)}(x), \qquad x \in (a,b),$$

which implies condition (b). Just similarly, condition (d) is established. Next, conditions (e) and (f) are verified directly. Finally, conditions (g) and (h) follow from [8, Proposition 2.2] with $\bar{\mathcal{A}} = A_0(\rho;s,e;\xi)$ and [8, (3.4)].

This completes the proof. □

5.3. A-operators for Transfers Between Weights w with Exponents in $\Gamma_{exc}(p)$

Two K-functionals whose w-weights have one exponent in $\Gamma_{exc}(p)$ can be related by means of the A-operators defined in [5, 8]. In the proof of [10, Proposition 4.1] we have actually shown

Proposition 5.2. *Let $1 \le p \le \infty$, $r \in \mathbb{N}$ and $i, i' \in \mathbb{N}_0$ as $i, i' < r$. Let also $\xi, \eta \in (a, b)$ and s be one of the points a or b and e be the other one. We set $w = \chi_s^{-i-1/p}\chi_e^{\gamma}$, where $\gamma \in \Gamma_+(p)$, and $\rho = i - i'$. Finally, let ϕ be measurable and non-negative on (a, b). Then we have*

$$A_i(\rho; s, e; \xi) : (L_p(w\chi_s^{\rho})(a, b), AC_{loc}^{r-1}, \phi D^r) \\ \rightleftharpoons (L_p(w)(a, b), AC_{loc}^{r-1}, \phi D^r) : A_{i'}(-\rho; s, e; \eta).$$

Remark 5.1. Let us mention that the operator $A_{r-1}(r-i-1; s, e; \xi)$ with $i \in \{0, \dots, r-1\}$ does not contain an integral term with a fixed boundary ξ because $\alpha_{r,r}(r-i-1) = 0$ and so, in accordance with our convention, we shall write $A_{r-1}(r-i-1; s, e; *)$.

Just similarly this property can be established for $A_j(\rho; \infty, a; \xi)$.

Proposition 5.3. *Let $1 \le p \le \infty$, $r \in \mathbb{N}$, $j, j' \in \mathbb{N}_0$ as $j, j' < r$, and $\xi, \eta > a$. We set $w = \chi_a^{\gamma}\chi_{a-1}^{-j-1/p-\gamma}$, where $\gamma \in \Gamma_+(p)$, and $\rho = j - j'$. Finally, let ϕ be measurable and non-negative on (a, ∞). Then we have*

$$A_j(\rho; \infty, a; \xi) : (L_p(w\chi_{a-1}^{\rho})(a, \infty), AC_{loc}^{r-1}, \phi D^r) \\ \rightleftharpoons (L_p(w)(a, \infty), AC_{loc}^{r-1}, \phi D^r) : A_{j'}(-\rho; \infty, a; \eta).$$

The operators $A_{i,j}(\rho; s, e; e)$ with $j > 0$ does not possess such a property for $1 \le p < \infty$. In the case $p = \infty$ there holds

Proposition 5.4. *Let $r \in \mathbb{N}$, $i, i', j \in \mathbb{N}_0$ as $i, i' < r$, $i + j \le r$ and $i' + j \le r$. Let also s be one of the points a or b and e be the other one. We set $w = \chi_s^{-i}\chi_e^{\gamma}$, where $\gamma \in [-j, 1)$, and $\rho = i - i'$. Finally, let ϕ be measurable and non-negative on (a, b). Then we have*

$$A_{i,j}(\rho; s, e; e) : (L_\infty(w\chi_s^{\rho})(a, b), AC_{loc}^{r-1}, \phi D^r) \\ \rightleftharpoons (L_\infty(w)(a, b), AC_{loc}^{r-1}, \phi D^r) : A_{i',j}(-\rho; s, e; e).$$

5.4. *A*-operators for Transfers Between *w*-weights with an Exponent in $\Gamma_{exc}(p)$ and not in $\Gamma_{exc}(p)$

Now, we shall construct operators that relate a K-functional with a w-weight whose exponent at a given end of the interval *is* in $\Gamma_{exc}(p)$ to a K-functionals with a w-weight whose exponent at the same end of the interval *is not* in $\Gamma_{exc}(p)$. As we noted in the Introduction, due to the nature of the considered weighted K-functionals we need to study the cases $p < \infty$ and $p = \infty$ separately. First, we give the definitions of the operators we shall use.

Definition 5.5. Let $\rho \in \mathbb{R}$, s denote one of the ends of the finite interval (a,b) and e the other. For $x \in (a,b)$ and $f \in L_{1,loc}(a,b)$ we set

$$(\tilde{A}(\rho; s,e;*)f)(x) = |x-s|^\rho f(x).$$

Definition 5.6. Let $\rho, a \in \mathbb{R}$. For $x \in (a,\infty)$ and $f \in L_{1,loc}(a,\infty)$ we set

$$(\tilde{A}(\rho; \infty,a;*)f)(x) = (x-a+1)^\rho f(x).$$

Let $\bar{a}$ denote either a real number or $-\infty$, and $\bar{b}$ denote either a real number or ∞. We set $C_0(\mathrm{w})[\bar{a},\bar{b}) = \{f \in C(\mathrm{w})(\bar{a},\bar{b}) : \lim_{x\to\bar{a}}(\mathrm{w}f)(x) = 0\}$ with the usual weighted sup-norm. Similarly, the space $C_0(\mathrm{w})(\bar{a},\bar{b}]$ is defined. We also set $C_0(\mathrm{w})[\bar{a},\bar{b}] = C_0(\mathrm{w})[\bar{a},\bar{b}) \cap C_0(\mathrm{w})(\bar{a},\bar{b}]$.

Definition 5.7. Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, s denote one of the ends of the finite interval (a,b) and e the other. For $x \in (a,b)$ and $f \in C(\chi_s^{1-r})[s,e)$ we set

$$(\hat{A}(\rho; s,e;*)f)(x) = |x-s|^\rho \big[f(x) - (x-s)^{r-1}\lim_{y\to s}(y-s)^{1-r}f(y)\big].$$

Definition 5.8. Let $r \in \mathbb{N}$, $\rho, a \in \mathbb{R}$. For $x \in (a,\infty)$ and $f \in C(a,\infty]$ we set

$$(\hat{A}(\rho; \infty,a;*)f)(x) = (x-a+1)^\rho \big[f(x) - \lim_{y\to\infty} f(y)\big].$$

In the propositions below we establish how these operators treat triplets.

The case $1 \le p < \infty$

Proposition 5.5. *Let $1 \le p \le \infty$, $r \in \mathbb{N}$ and $\rho \in \mathbb{R}$. Let s be one of the points a or b and e be the other one. We set $w = \chi_s^{\gamma_s}\chi_e^{\gamma_e}$ and $\varphi = \chi_s^{\lambda_s}\chi_e^{\lambda_e}$ as $\gamma_e + r\lambda_e \in \Gamma_+(p)$, $\gamma_s + r\lambda_s, \gamma_s + \rho + r\lambda_s < 1 - 1/p$ and also $\gamma_s, \gamma_s + \rho \le 1 - r - 1/p$ for $p < \infty$ and $\gamma_s, \gamma_s + \rho < 1 - r$ for $p = \infty$. Then*

$$\begin{aligned}\tilde{A}(\rho; s,e;*) &: (L_p(w\chi_s^\rho)(a,b), AC_{loc}^{r-1}, \varphi^r D^r)\\ &\rightleftharpoons (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) : \tilde{A}(-\rho; s,e;*).\end{aligned}$$

Proof. It is verified directly that the operators $\mathcal{A} = \tilde{A}(\rho; s,e;*)$ and $\mathcal{B} = \mathcal{A}^{-1} = \tilde{A}(-\rho; s,e;*)$ satisfy conditions (a), (c), (g) and (h) of Definition 3.1 with $X_1 = L_p(w\chi_s^\rho)(a,b)$, $X_2 = L_p(w)(a,b)$, $Y_1 = Y_2 = AC_{loc}^{r-1}(a,b)$ and $\mathcal{D}_1 = \mathcal{D}_2 = \varphi^r D^r$. To establish (b) we get by the Leibniz rule for $g \in AC_{loc}^{r-1}(a,b)$ and $x \in (a,b)$

$$w(x)\varphi^r(x)(|x-s|^\rho g(x))^{(r)} = \sum_{k=0}^{r} b_{r,k}(\rho) w(x)\varphi^r(x)|x-s|^{\rho-r+k} g^{(k)}(x),$$

where we have set $b_{r,k}(\rho) = (\operatorname{sgn}(x-s))^{r-k}\binom{r}{k}\rho(\rho-1)\cdots(\rho-r+k+1)$. Hence

$$\|w\varphi^r D^r \mathcal{A}g\|_{p(a,b)} \le c\sum_{k=0}^{r} \|w\varphi^r \chi_s^{\rho-r+k} g^{(k)}\|_{p(a,b)}. \tag{5.12}$$

Proposition 4.4.a with $\alpha = \gamma_s + \rho$, $\beta = \gamma_s + \rho + r(\lambda_s - 1)$ and $\delta = \gamma_e + r\lambda_e$ implies for $g \in AC_{loc}^{r-1}(a,b)$ such that $g, \varphi^r g^{(r)} \in L_p(w\chi_s^\rho)(a,b)$ the inequalities

$$\|w\varphi^r \chi_s^{\rho-r+k} g^{(k)}\|_{p(a,b)} \le c\,\|w\chi_s^\rho \varphi^r g^{(r)}\|_{p(a,b)}, \qquad k = 0,\dots,r. \tag{5.13}$$

Now, (5.12) and (5.13) imply (b) of Definition 3.1. Further, taking into consideration that $\mathcal{A} : Y_1 \to Y_2$, we get condition (e) too.

Just similarly, we get for $G \in AC_{loc}^{r-1}(a,b)$ such that $G, \varphi^r G^{(r)} \in L_p(w)(a,b)$ the estimate

$$\|w\chi_s^\rho \varphi^r D^r \mathcal{B}G\|_{p(a,b)} \le c\sum_{k=0}^{r} \|w\varphi^r \chi_s^{k-r} G^{(k)}\|_{p(a,b)}. \tag{5.14}$$

Again by Proposition 4.4.a but with $\alpha = \gamma_s$, $\beta = \gamma_s + r(\lambda_s - 1)$ and $\delta = \gamma_e + r\lambda_e$ we get

$$\|w\varphi^r \chi_s^{k-r} G^{(k)}\|_{p(a,b)} \le c\,\|w\varphi^r G^{(r)}\|_{p(a,b)}, \qquad k = 0,\dots,r. \tag{5.15}$$

Relations (5.14) and (5.15) imply (d) of Definition 3.1. Finally, taking into consideration that $\mathcal{B} : Y_2 \to Y_1$, we get condition (f). Thus the proof of the proposition is completed. □

Similarly, using Proposition 4.4.b instead of Proposition 4.4.a, we verify the analogue of Proposition 5.5 for a semi-infinite interval.

Proposition 5.6. *Let* $1 \le p \le \infty$, $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty - \gamma_a}$ *and* $\varphi = \chi_a^{\lambda_a}\chi_{a-1}^{\lambda_\infty - \lambda_a}$ *as* $\gamma_a + r\lambda_a \in \Gamma_+(p)$, $\gamma_\infty + r(\lambda_\infty - 1), \gamma_\infty + \rho + r(\lambda_\infty - 1) > -1/p$ *and also* $\gamma_\infty, \gamma_\infty + \rho \ge -1/p$ *for* $p < \infty$ *and* $\gamma_\infty, \gamma_\infty + \rho > 0$ *for* $p = \infty$. *Then*

$$\begin{aligned}\tilde{A}(\rho;\infty,a;*) : (L_p(w\chi_{a-1}^\rho)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r)&\\ \rightleftharpoons (L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r) &: \tilde{A}(-\rho;\infty,a;*).\end{aligned}$$

Remark 5.2. Let us note that the last two propositions are valid for all $1 \le p \le \infty$ but they transform a w-weight with an exponent in $\Gamma_{exc}(p)$ into another with an exponent not in $\Gamma_{exc}(p)$ for $p < \infty$. More precisely,

(a) In Proposition 5.5 for $p < \infty$, the conditions $\gamma_s, \gamma_s + \rho \le 1 - r - 1/p$ and $\lambda_s < 1$ imply the hypotheses imposed on γ_s, λ_s and ρ;

(b) In Proposition 5.6 for $p < \infty$, the conditions $\gamma_\infty, \gamma_\infty + \rho \ge -1/p$ and $\lambda_\infty > 1$ imply the hypotheses imposed on γ_∞, λ_∞ and ρ.

The case $p = \infty$

Proposition 5.7. *Let $r \in \mathbb{N}$ and $\rho \in \mathbb{R}$ as $\rho < 0$. Let s be one of the points a or b and e be the other one. We set $w = \chi_s^{1-r}\chi_e^{\gamma_e}$ and $\varphi = \chi_s^{\lambda_s}\chi_e^{\lambda_e}$ as $\gamma_e, \gamma_e + r\lambda_e \geq 0$ and $\lambda_s < 1$. Then*

$$\tilde{A}(\rho; s, e; *) : (C_0(w\chi_s^\rho)[s,e), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (C(w)[s,e), AC_{loc}^{r-1}, \varphi^r D^r) : \hat{A}(-\rho; s, e; *).$$

Proof. We verify that $\mathcal{A} = \tilde{A}(\rho; s, e; *)$ and $\mathcal{B} = \hat{A}(-\rho; s, e; *)$ satisfy conditions (a) and (c) of Definition 3.1 with $X_1 = C_0(w\chi_s^\rho)[s,e)$, $X_2 = C(w)[s,e)$, $Y_1 = Y_2 = AC_{loc}^{r-1}(a,b)$ and $\mathcal{D}_1 = \mathcal{D}_2 = \varphi^r D^r$. Further, by (5.12) and (5.13) with $\gamma_s = 1 - r$ and $p = \infty$ (note that $1 - r + \rho, 1 - r + \rho + r(\lambda_s - 1) < 1 - r$ and $\gamma_e + r\lambda_e \geq 0$ and Proposition 4.4.a with $p = \infty$ is applicable) we get for any $g \in AC_{loc}^{r-1}(a,b)$ such that $g, \varphi^r g^{(r)} \in C(w\chi_s^\rho)(a,b)$ the estimate

$$\|w\varphi^r D^r \mathcal{A} g\|_{\infty(a,b)} \leq c\, \|w\varphi^r \chi_s^\rho g^{(r)}\|_{\infty(a,b)}.$$

Thus (b) of Definition 3.1 is proved. Hence condition (e) also follows because $\mathcal{A} : Y_1 \to Y_2$.

Next, let us set $\widetilde{G}(x) = G(x) - (x-s)^{r-1}\lim_{y\to s}(y-s)^{1-r}G(y)$ for $G \in AC_{loc}^{r-1}(a,b)$ such that $G, \varphi^r G^{(r)} \in C(w)[s,e)$. Note that $\widetilde{G} \in AC_{loc}^{r-1}(a,b)$ and $\widetilde{G}, \varphi^r \widetilde{G}^{(r)} \in C(w)[s,e)$ as well. Now, property (d) of Definition 3.1 follows from (5.14) with $p = \infty$, $\widetilde{G}$ in the place of G (as $\mathcal{B} = \mathcal{A}^{-1}$ in (5.14)) and the estimates:

$$\|w\varphi^r \chi_s^{-1} \widetilde{G}^{(r-1)}\|_{\infty(a,b)} \leq c\, \|w\varphi^r G^{(r)}\|_{\infty(a,b)}, \tag{5.16}$$

$$\|w\varphi^r \chi_s^{k-r} \widetilde{G}^{(k)}\|_{\infty(a,b)} \leq c\, \|w\varphi^r \chi_s^{-1} \widetilde{G}^{(r-1)}\|_{\infty(a,b)}, \quad k = 0, \dots, r-1. \tag{5.17}$$

To prove (5.16) we note that by Proposition 4.1.a with $p = \infty$, $\alpha = \gamma = 1 - r$ and $\beta = 1 - r + r(\lambda_s - 1) < 1 - r$ we get

$$\chi_s^{1-r+k} G^{(k)} \in L_\infty(s, (s+e)/2), \qquad k = 0, \dots, r-1. \tag{5.18}$$

Now, Lemma 4.1.a with $g = G^{(k-1)}$ and $\beta = k - r < 0$ implies that $G^{(k-1)}$ has a final limit at s for $k = 1, \dots, r-1$, $r \geq 2$, which in view of (5.18) is equal to 0. Thus we have

$$G^{(k)}(s) = 0, \qquad k = 0, \dots, r-2,\ r \geq 2. \tag{5.19}$$

Next, again by Lemma 4.1.a with $g = G^{(r-1)}$ and $\beta = 1 - r + r\lambda_s - 1 = r(\lambda_s - 1) < 0$ we get that $G^{(r-1)}$ has a final limit at s too, which we denote by $G^{(r-1)}(s)$. Hence, using the L'Hospital rule and (5.19) if $r \geq 2$, we get that

$$\lim_{x\to s}(x-s)^{1-r}G(x) = \frac{G^{(r-1)}(s)}{(r-1)!}. \tag{5.20}$$

Consequently, for $x \in (a, b)$ we have

$$\begin{aligned} \widetilde{G}^{(r-1)}(x) &= G^{(r-1)}(x) - (r-1)! \lim_{y \to s} (y-s)^{1-r} G(y) \\ &= G^{(r-1)}(x) - G^{(r-1)}(s) = \int_s^x G^{(r)}(y)\, dy. \end{aligned} \tag{5.21}$$

Now, by means of Hardy's and Hölder's inequalities we get (5.16) as in the proof of Proposition 4.4.a for $k = r - 1$.

Assertion (5.17) is trivial for $r = 1$, whereas for $r \geq 2$ it follows from Proposition 4.4.a with $p = \infty$, $r - 1$ in the place of r, $\alpha = 1 - r < 1 - (r - 1)$, $\beta = 1-r+r\lambda_s-1-(r-1) = 1-r+r(\lambda_s-1) < 1-(r-1)$ and $\delta = \gamma_e+r\lambda_e \geq 0$. Thus condition (d) of Definition 3.1 is verified. Hence, in view of the fact that $\mathcal{B} : Y_2 \cap X_2 \to Y_1 \cap X_1$, we get (f) too.

In order to verify (g) of Definition 3.1 we take into consideration that we have $\lim_{x\to s} (\chi_s^{1-r+\rho} f)(x) = 0$ for $f \in C_0(w\chi_s^{\rho})[s, e)$ and, consequently, $\mathcal{B}\mathcal{A}f = f$. Finally, we have for every function $F \in C(w)[s, e)$ that $\mathcal{A}\mathcal{B}F = F - \chi_s^{r-1} \lim_{x\to s} (\chi_s^{1-r} F)(x)$, which yields (h) of Definition 3.1. The proof of the proposition is completed. □

Remark 5.3. Let us explicitly note that in the proof above we did not have to assume $\lim_{x\to s} (\chi_s^{1-r+\rho} g)(x) = 0$. This follows from $g \in AC_{loc}^{r-1}(a, b)$ and $g, \chi_s^{r\lambda_s} g^{(r)} \in L_\infty(\chi_s^{1-r+\rho})(s, (s + e)/2)$ with $\rho < 0$ and $\lambda_s < 1$. In fact, $g \in AC_{loc}^{r-1}(a, b)$ and $g, \chi_s^{r\lambda_s} g^{(r)} \in L_\infty(\chi_s^{\gamma_s})(s, (s + e)/2)$ with $\gamma_s \notin \Gamma_{exc}(\infty)$ and $\lambda_s < 1$ imply $\lim_{x\to s} (\chi_s^{\gamma_s} g)(x) = 0$. Indeed, if $\kappa_s \in \Gamma_r(\infty)$, i.e. $\kappa_s < 1 - r$, then Proposition 4.4.a with $p = \infty$, $\alpha = \kappa_s < 1 - r$, $\beta = \kappa_s + r(\lambda_s - 1) < 1 - r$ and $k = 0$ implies $\chi_s^{\beta} g \in L_\infty(s, (s + e)/2)$, which, in view of $\beta < \kappa_s$, yields $\lim_{x\to s}(\chi_s^{\kappa_s} g)(x) = 0$. Further, if $\kappa_s \in \Gamma_i(\infty)$ with $i < r$, we set $G = A_r(\rho; s, e; *)g$ with $\rho > \kappa_s + r - 1$ and get by [8, Proposition 3.9] that $G \in AC_{loc}^{r-1}(a, b)$ and $G, \chi_s^{r\lambda_s} G^{(r)} \in L_\infty(\chi_s^{\kappa_s-\rho})(s, \xi)$. Consequently, according to what we have already proved in the case $\kappa_s \in \Gamma_r(\infty)$, we have $\lim_{x\to s} (\chi_s^{\kappa_s-\rho} G)(x) = 0$. Then $\lim_{x\to s} (\chi_s^{\kappa_s} \bar{g})(x) = 0$, where $\bar{g} = A_i(-\rho; s, e; \xi)G$, as [8, Lemma 8.1.a] with $\sigma = 1$ and $a = s$ implies. But by [8, Proposition 3.9] we have $g = \bar{g}+c_i\chi_s^i+\cdots+c_{r-1}\chi_s^{r-1}$, where $c_i, \ldots, c_{r-1} \in \mathbb{R}$. Hence, in view of $\kappa_s + i > 0$, we have $\lim_{x\to s}(\chi_s^{\kappa_s} g)(x) = 0$ too.

Similarly, as we saw in the proof of Proposition 5.7, if $G \in AC_{loc}^{r-1}(a, b)$ is such that $G, \chi_s^{r\lambda_s} G^{(r)} \in L_\infty(\chi_s^{1-r})(s, (s + e)/2)$ with $\lambda_s < 1$, then the limit $\lim_{x\to s} (\chi_s^{1-r} G)(x)$ exists. Further, by means of Proposition 5.2 and [8, Lemma 8.1.a] we extend this to: if $G \in AC_{loc}^{r-1}(a, b)$ is such that $G, \chi_s^{r\lambda_s} G^{(r)} \in L_\infty(\chi_s^{-i})(s, (s + e)/2)$ with $i \in \{0, \ldots, r - 1\}$ and $\lambda_s < 1$, then the limit $\lim_{x\to s}(\chi_s^{-i} G)(x)$ exists.

Similarly, using Propositions 4.1.b and 4.4.b, and also Lemma 4.1.b, we establish

Proposition 5.8. *Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$ as $\rho > 0$, $w = \chi_a^{\gamma_a}\chi_{a-1}^{-\gamma_a}$ and $\varphi = \chi_a^{\lambda_a}\chi_{a-1}^{\lambda_\infty-\lambda_a}$ as $\gamma_a, \gamma_a + r\lambda_a \geq 0$ and $\lambda_\infty > 1$. Then*

$$\tilde{A}(\rho;\infty,a;*) : (C_0(w\chi_{a-1}^{\rho})(a,\infty], AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (C(w)(a,\infty], AC_{loc}^{r-1}, \varphi^r D^r) : \hat{A}(-\rho;\infty,a;*).$$

Proof. Let $\mathcal{A} = \tilde{A}(\rho;\infty,a;*)$, $\mathcal{B} = \hat{A}(-\rho;\infty,a;*)$, $X_1 = C_0(w\chi_{a-1}^{\rho})(a,\infty]$, $X_2 = C(w)(a,\infty]$, $Y_1 = Y_2 = AC_{loc}^{r-1}(a,\infty)$ and $\mathcal{D}_1 = \mathcal{D}_2 = \varphi^r D^r$. It is verified directly that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy conditions (a) and (c) of Definition 3.1. We establish properties (g) and (h) of Definition 3.1 just as in the proof of Proposition 5.7. We prove property (b) analogously as in Proposition 5.7 by means of Proposition 4.4.b instead of Proposition 4.4.a. Hence (e) directly follows.

Next, similarly to the case considered in the proposition above, (d) follows from the Leibniz rule and the inequalities

$$\|w\varphi^r\chi_{a-1}^{k-r}G^{(k)}\|_{\infty(a,\infty)} \leq c\,\|w\varphi^r G^{(r)}\|_{\infty(a,\infty)}, \qquad k = 1,\ldots,r, \tag{5.22}$$

$$\|w\varphi^r\chi_{a-1}^{-r}\widetilde{G}\|_{\infty(a,\infty)} \leq c\,\|w\varphi^r\chi_{a-1}^{1-r}G'\|_{\infty(a,\infty)}, \tag{5.23}$$

where we have set $\widetilde{G}(x) = G(x) - \lim_{y\to\infty} G(y)$ for $G \in AC_{loc}^{r-1}(a,\infty)$ and $G, \varphi^r G^{(r)} \in C(w)(a,\infty]$. Proposition 4.1.b with $k = 1$, $\alpha = \gamma = 0$ and $\beta = r(\lambda_\infty - 1) > 0$ implies $\chi_{a-1}G' \in L_\infty(a+1,\infty)$. Now, the inequalities (5.22) follow from Proposition 4.4.b with $p = \infty$, $r-1$ in the place of r, $g = G'$, $\alpha = 1$, $\beta = r\lambda_\infty - (r-1) > 0$ and $\delta = \gamma_a + r\lambda_a \geq 0$.

The inequality (5.23) follows from

$$\widetilde{G}(x) = -\int_x^\infty G'(y)\,dy,$$

Hardy's and Höder's inequalities.

Finally, condition (f) follows from (d) and $\mathcal{B} : Y_2 \cap X_2 \to Y_1 \cap X_1$. □

Remark 5.4. Let us mention that if $g \in AC_{loc}^{r-1}(a,\infty)$ and $g, \chi_{a-1}^{r\lambda_\infty}g^{(r)} \in L_\infty(\chi_{a-1}^{\gamma_\infty})(a+1,\infty)$ as $\gamma_\infty \notin \Gamma_{exc}(\infty)$ and $\lambda_\infty > 1$, then $\lim_{x\to\infty}(\chi_{a-1}^{\gamma_\infty}g)(x) = 0$; and also if $G \in AC_{loc}^{r-1}(a,\infty)$ and $G, \chi_{a-1}^{r\lambda_\infty}G^{(r)} \in L_\infty(\chi_{a-1}^{-i})(a+1,\infty)$ as $i \in \{0,\ldots,r-1\}$ and $\lambda_\infty > 1$, then the limit $\lim_{x\to\infty}(\chi_{a-1}^{-i}g)(x)$ exists.

6. Operators That Change Both Weights w and φ

In [5, 8] we defined operators, which enable us to relate K-functionals with different exponents in the φ-weight. We call them B-operators. Below we give the definitions of the B-operators we need here.

Definition 6.1. Let $r \in \mathbb{N}$, $\sigma > 0$, $i \in \mathbb{N}$ as $i \le r$, and $\xi \in (a,b)$. Let s be one of the ends of the interval (a,b) and e the other. For $x \in (a,b)$ and $f \in L_{1,loc}(a,b)$, satisfying the additional requirement $\chi_s^{(1-i)/\sigma-1} f \in L_1(s,(s+e)/2)$ if $i > 1$, we set

$$
\begin{aligned}
(B_i(\sigma; s, e; \xi) f)(x) &= f\Big(s + (e-s)\Big(\frac{x-s}{e-s}\Big)^{\sigma}\Big) \\
&+ \frac{1}{e-s}\sum_{k=2}^{i} \beta_{r,k}(\sigma)\Big(\frac{x-s}{e-s}\Big)^{k-1} \int_s^x \Big(\frac{y-s}{e-s}\Big)^{-k} f\Big(s + (e-s)\Big(\frac{y-s}{e-s}\Big)^{\sigma}\Big)\,dy, \\
&+ \frac{1}{e-s}\sum_{k=i+1}^{r} \beta_{r,k}(\sigma)\Big(\frac{x-s}{e-s}\Big)^{k-1} \int_\xi^x \Big(\frac{y-s}{e-s}\Big)^{-k} f\Big(s + (e-s)\Big(\frac{y-s}{e-s}\Big)^{\sigma}\Big)\,dy,
\end{aligned}
$$

where

$$
\beta_{r,k}(\sigma) = \frac{(-1)^{r-k}}{(r-2)!}\binom{r-2}{k-2}\prod_{i=1}^{r-1}(k-1-i\sigma), \qquad k = 2,3,\dots,r. \tag{6.1}
$$

Definition 6.2. Let $r \in \mathbb{N}$, $\sigma > 0$, $j \in \mathbb{N}$ as $j \le r$, and $\xi \in (a,\infty)$. For $x \in (a,\infty)$ and $f \in L_{1,loc}(a,\infty)$, satisfying the additional requirement $\chi_a^{-j/\sigma-1} f \in L_1(a+1,\infty)$ if $j < r$, we set

$$
\begin{aligned}
(B_j(\sigma;\infty,a;\xi)f)(x) &= f(a-1+(x-a+1)^{\sigma}) \\
&+ \sum_{k=2}^{j} \beta_{r,k}(\sigma)(x-a+1)^{k-1}\int_\xi^x (y-a+1)^{-k} f(a-1+(y-a+1)^{\sigma})\,dy \\
&- \sum_{k=j+1}^{r} \beta_{r,k}(\sigma)(x-a+1)^{k-1}\int_x^\infty (y-a+1)^{-k} f(a-1+(y-a+1)^{\sigma})\,dy,
\end{aligned}
$$

where $\beta_{r,k}(\sigma)$ are defined in (6.1).

Definition 6.3. Let $r \in \mathbb{N}$, $\sigma < 0$, $j \in \mathbb{N}$ as $j \le r$ and (a,b) be an interval. Let s be one of the endpoints of the interval (a,b), e the other and $\xi \in (e,\infty)$. For $x \in (e,\infty)$ and $f \in L_{1,loc}(a,b)$, satisfying the additional requirement $\chi_s^{-j/\sigma-1} f \in L_1(s,(s+e)/2)$ if $j < r$, we set

$$
\begin{aligned}
B_j(\sigma; s, e; \infty, e; \xi) f)(x) &= f(s+(e-s)(x-e+1)^{\sigma}) \\
&+ \sum_{k=2}^{j} \beta_{r,k}(\sigma)(x-e+1)^{k-1}\int_\xi^x y^{-k} f(s+(e-s)(y-e+1)^{\sigma})\,dy \\
&- \sum_{k=j+1}^{r} \beta_{r,k}(\sigma)(x-e+1)^{k-1}\int_x^\infty y^{-k} f(s+(e-s)(y-e+1)^{\sigma})\,dy,
\end{aligned}
$$

where $\beta_{r,k}(\sigma)$ are defined in (6.1).

Definition 6.4. Let $r \in \mathbb{N}$, $\sigma < 0$, $i \in \mathbb{N}$ as $i \le r$ and (a,b) be an interval. Let s be one of the endpoints of the interval (a,b), e the other and $\eta \in (a,b)$. For $x \in (a,b)$ and $f \in L_{1,loc}(e,\infty)$, satisfying the additional requirement $\chi_e^{(1-i)/\sigma-1} f \in L_1(e+1,\infty)$ if $i > 1$, we set

$$\begin{aligned}(B_i(\sigma;\infty,e;s,e;\eta)f)(x) &= f\Big(\Big(\frac{x-s}{e-s}\Big)^{\sigma} + e - 1\Big)\\ &+ \frac{1}{e-s}\sum_{k=2}^{i} \beta_{r,k}(\sigma)\Big(\frac{x-s}{e-s}\Big)^{k-1}\int_s^x \Big(\frac{y-s}{e-s}\Big)^{-k} f\Big(\Big(\frac{y-s}{e-s}\Big)^{\sigma} + e - 1\Big)\,dy\\ &+ \frac{1}{e-s}\sum_{k=i+1}^{r} \beta_{r,k}(\sigma)\Big(\frac{x-s}{e-s}\Big)^{k-1}\int_\eta^x \Big(\frac{y-s}{e-s}\Big)^{-k} f\Big(\Big(\frac{y-s}{e-s}\Big)^{\sigma} + e - 1\Big)\,dy,\end{aligned}$$

where $\beta_{r,k}(\sigma)$ are defined in (6.1).

For $r = 1$ all the assertions about the operators of type B, given in [5, 8], hold without any restrictions on the exponents of the w-weights [8, Remark 4.2]. For $r \ge 2$ these B-operators relate two K-functionals with w-weights with one exponent in $\Gamma_{exc}(p)$ each only when their σ-parameter belongs to a certain finite set of *positive* numbers. Thus, generally, these operators cannot be used to clear the φ-weight in the second term of the K-functional, nor for transfers between spaces of functions defined respectively on a finite and semi-infinite interval (see [8, Propositions 4.17 and 4.18]). This also means that we cannot change the exponent of the φ-weight with one at the opposite side of 1 at an end where the exponent of the w-weight is in $\Gamma_{exc}(p)$, as this requires B-operators with $\sigma < 0$.

The following propositions are verified similarly to Propositions 5.2 and 5.3. We shall not use them here, but state them to compare them with those about the A-operators.

Proposition 6.1. *Let $1 \le p \le \infty$, $i, i', r \in \mathbb{N}$ as $r \ge 2$ and $i, i' < r$. Let also $\xi, \eta \in (a,b)$ and s be one of the points a or b and e be the other one. We set $w = \chi_s^{-i-1/p}\chi_e^{\gamma}$, where $\gamma \in \Gamma_+(p)$, and $\sigma = i/i'$. Finally, let $\phi = \chi_s^{\tau_s}\chi_e^{\tau_e}$, $\tau_s, \tau_e \in \mathbb{R}$. Then*

$$\begin{aligned}B_i(\sigma;s,e;\xi) : (L_p(w\chi_s^{i-i'})(a,b), AC_{loc}^{r-1}, \phi\chi_s^{(r-\tau_s)(1-i'/i)}D^r)\\ \rightleftharpoons (L_p(w)(a,b), AC_{loc}^{r-1}, \phi D^r) : B_{i'}(\sigma^{-1};s,e;\eta).\end{aligned}$$

Proposition 6.2. *Let $1 \le p \le \infty$, $j, j', r \in \mathbb{N}$ as $r \ge 2$, $j, j' < r$, and $\xi, \eta > a$. We set $w = \chi_a^{\gamma}\chi_{a-1}^{-j-1/p-\gamma}$, where $\gamma \in \Gamma_+(p)$, and $\sigma = j/j'$. Finally, let $\phi = \chi_a^{\tau_a}\chi_{a-1}^{\tau_\infty-\tau_a}$, $\tau_a, \tau_\infty \in \mathbb{R}$. Then*

$$\begin{aligned}B_j(\sigma;\infty,a;\xi) : (L_p(w\chi_{a-1}^{j-j'})(a,\infty), AC_{loc}^{r-1}, \phi\chi_{a-1}^{(r-\tau_\infty)(1-j'/j)}D^r)\\ \rightleftharpoons (L_p(w)(a,\infty), AC_{loc}^{r-1}, \phi D^r) : B_{j'}(\sigma^{-1};\infty,a;\eta).\end{aligned}$$

7. Characterization of K-functionals

Now, we shall derive characterizations of the weighted K-functionals (1.1) by means of the unweighted fixed-step moduli of smoothness.

7.1. The Cases $\kappa_a \in \Gamma_i(p)$, $\kappa_b \in \Gamma_j(p)$, $i+j \le r$, $1 \le p < \infty$, and $\kappa_a = -i$, $\kappa_b = -j$, $0 \le i, j < r$, $i+j \le r$, $p = \infty$

The operators $A_{i,j}(\rho; s, e; e)$ constructed in Section 5.2 allow us to characterize the K-functional on a finite interval with power-type weights under milder restrictions on the weight w than in [8, Subsection 6.1]. More precisely, [8, Theorem 6.1] and Proposition 5.1 imply

Theorem 7.1. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $(1-\lambda_a)(1-\nu_a) > 0$ and $(1-\lambda_b)(1-\nu_b) > 0$. Let also $\kappa_a \in \Gamma_i(p), \kappa_b \in \Gamma_j(p), \mu_a \in \Gamma_{i'}(p)$ and $\mu_b \in \Gamma_{j'}(p)$ as $i+j \le r$ and $i'+j' \le r$. Then there exist bounded linear operators $\mathcal{A}$ and $\mathcal{B}$ such that*

$$\mathcal{A} : (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,b), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

Remark 7.1. Explicit constructions of the operators $\mathcal{A}$ and $\mathcal{B}$, whose existence is stated in the theorem, are got by combining the operators given in the proof of [8, Theorem 6.1] with operator of type $A_{i,j}(\rho; s, e; e)$ if necessary.

In particular, [8, Theorem 6.2] is extended to

Theorem 7.2. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\lambda_a, \lambda_b \in (-\infty, 1)$. Let also $i, j \in \mathbb{N}_0$ as $i+j \le r$. For $p < \infty$ we assume that $\kappa_a \in \Gamma_i(p), \kappa_b \in \Gamma_j(p)$, and for $p = \infty$ we assume that $\kappa_a = -i, \kappa_b = -j$ as also $i, j < r$. Set $w = \chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$ and*

$$\mathcal{A} = B_1(\sigma_b; b, a; \xi) B_1(\sigma_a; a, b; \xi) A_0(\rho_b; b, a; \xi) A_0(\rho_a; a, b; \xi),$$

where $\xi \in (a,b)$ and

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_b = \frac{1}{1-\lambda_b}, \quad \rho_a = \kappa_a + \frac{\lambda_a}{p}, \quad \rho_b = \kappa_b + \frac{\lambda_b}{p}.$$

Then for $t > 0$ and $f \in L_p(w)(a,b)$ we have

$$K(f, t^r; L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p(a,b)}.$$

Proof. Let $p < \infty$. For $i = 0$ or $j = 0$ the assertion is contained in [8, Theorem 6.2]. So, let $i, j > 0$. By Proposition 5.1 with $\rho = \rho_a$, $s = a$, $e = b$, $\gamma_s = -\lambda_a/p \in \Gamma_0(p)$, and $\gamma_e = \kappa_b \in \Gamma_-(p)$ we get

$$\begin{aligned} A_{0,j} : (\rho_a; a, b; b) &: (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \\ &\rightleftharpoons (L_p(\chi_a^{-\lambda_a/p}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \varphi^r D^r) : A_{i,j}(-\rho_a; a, b; b). \quad (7.1)\end{aligned}$$

Note that since $i+j \le r$ and $i > 0$ we have $j < r$ and then in view of $\kappa_b \in \Gamma_j(p)$ we get $\gamma_e + j = \kappa_b + j \in \Gamma_+(p)$. On the other hand, in the proof of [8, Theorem 6.2] we established

$$\begin{aligned} \mathcal{A}^{\#} : (L_p(\chi_a^{-\lambda_a/p}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \\ \rightleftharpoons (L_p(a,b), AC_{loc}^{r-1}, D^r) : \mathcal{B}^{\#}, \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} \mathcal{A}^{\#} &= B_1(\sigma_b; b, a; \xi) B_1(\sigma_a; a, b; \xi) A_0(\rho_b; b, a; \xi), \\ \mathcal{B}^{\#} &= A_j(-\rho_b; b, a; \eta) B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_b^{-1}; b, a; \eta), \qquad \eta \in (a,b). \end{aligned}$$

Now, as we combine (7.1) and (7.2) and then apply Proposition 3.1 and relation (3.1), we get the assertion of the theorem for $p < \infty$ with $\mathcal{A}' = \mathcal{A}^{\#} A_{0,j}(\rho_a; a, b; b)$ in the place of $\mathcal{A}$. In the case of $p = \infty$ this follows from the scheme:

$$\begin{array}{cr} \boxed{(L_\infty(a,b), AC_{loc}^{r-1}, D^r)} & \\ \mathcal{A}_1 \quad \upharpoonleft\downharpoonright \quad \mathcal{B}_1 & \text{Step 1} \\ \boxed{(L_\infty(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b} D^r)} & \\ A_0(-j; b, a; \xi) \quad \upharpoonleft\downharpoonright \quad A_j(j; b, a; \eta) & \text{Step 2} \\ \boxed{(L_\infty(\chi_b^{-j})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b} D^r)} & \\ A_{0,j}(-i; a, b; b) \quad \upharpoonleft\downharpoonright \quad A_{i,j}(i; a, b; b) & \text{Step 3} \\ \boxed{(L_\infty(\chi_a^{-i}\chi_b^{-j})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b} D^r)}\,, & \end{array}$$

where we have set $\mathcal{A}_1 = B_1(\sigma_b; b, a; \xi) B_1(\sigma_a; a, b; \xi)$ and $\mathcal{B}_1 = B_1(\sigma_a^{-1}; a, b; \eta)$ $B_1(\sigma_b^{-1}; b, a; \eta)$.

Step 1 is contained in [8, Theorem 6.2] or alternatively in [5, Theorem 5.4]. Step 2 follows from Proposition 5.2 with $p = \infty$, $s = b$, $e = a$, $i = 0$, $i' = j$, $\gamma = 0 \in \Gamma_+(\infty)$ and $\phi = \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}$. Step 3 follows from Proposition 5.4 with $s = a$, $e = b$, i and i' interchanged, $i' = 0$, $\gamma = -j$ and $\phi = \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}$. Thus the assertion of the theorem with $\mathcal{A}'$ in the place of $\mathcal{A}$ is established for $p = \infty$ too.

To complete the proof of the theorem we only need to observe that $A_{0,j}(\rho_a; a, b; b)f - A_0(\rho_a; a, b; \xi)f \in \Pi_{r-1}$ for every $f \in L_p(w)(a,b)$ and by [8, (3.5) and (4.3)] we have $\mathcal{A}^{\#} : \Pi_{r-1} \to \Pi_{r-1}$. Hence $\mathcal{A}'f - \mathcal{A}f \in \Pi_{r-1}$ for every $f \in L_p(w)(a,b)$. □

Remark 7.2. The assertions of Theorem 7.1 and Theorem 7.2 remain valid if L_∞ is replaced by C with one or two weighted limit conditions at the ends of the domains (cf. [8, Section 8]).

7.2. The Cases $\kappa_s, \kappa_\infty \in \Gamma_{exc}(\infty)$, $\lambda_s, \lambda_\infty \neq 1$ as $p = \infty$

In the remaining subsections we shall consider weights with singularities only at one end of the interval to keep the operators simpler and cover a wilder range of the weight exponents. Let us recall that generally the singularities can be separated in the following sense (cf. [2, Ch. 6, Lemma 2.3] or [8, Section 7.1]).

Lemma 7.1. *Let $I_1 = (\bar{a}, b_1)$ and $I_2 = (a_1, \bar{b})$ be two intervals on the real line such that $\bar{a} < a_1 < b_1 < \bar{b}$, where $\bar{a}$ is finite or $-\infty$ and $\bar{b}$ is finite or ∞. Let $I = (\bar{a}, \bar{b}) = I_1 \cup I_2$ and let w and φ be non-negative measurable on I weights such that $w \sim 1$ and $\varphi \sim 1$ on $[a_1, b_1]$. Then for $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le b_1 - a_1$ and $f \in L_p(w)(I)$ we have*

$$K(f, t^r; L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$$
$$\sim K(f, t^r; L_p(w)(I_1), AC_{loc}^{r-1}, \varphi^r D^r) + K(f, t^r; L_p(w)(I_2), AC_{loc}^{r-1}, \varphi^r D^r).$$

Thus we can derive characterizations of K-functionals under milder restrictions on the w-weight and simpler operators but with a sum of *two* moduli of order r (cf. [8, Section 7]). Thus, it is sufficient to consider only function spaces of the type $L_p(\chi_s^{\kappa_s})(a, b)$ and $L_p(\chi_{a-1}^{\kappa_\infty})(a, \infty)$ (see [8, Subsection 2.4]).

Also, let us note that in [8, Section 6.4] we have constructed operators my means of which we can relate K-functionals with $\lambda_s < 1$ and $\lambda_\infty > 1$, and respectively $\lambda_s > 1$ and $\lambda_\infty < 1$ for γ's not in $\Gamma_{exc}(p)$. Hence a characterization of the K-functional on a finite interval with $\lambda_s < 1$ (resp. $\lambda_s > 1$) implies a characterization of the K-functional on a semi-infinite interval with $\lambda_\infty > 1$ (resp. $\lambda_\infty < 1$). These relations can be extended to γ's in $\Gamma_{exc}(p)$ by means of the propositions in Subsection 5.3.

After these preliminaries let us proceed to the construction of $\mathcal{A}$-operators for characterizing the K-functionals (1.1) in the case $p = \infty$ with weights with singularities only at one of the ends of the interval when κ_s or $\kappa_\infty \in \Gamma_{exc}(\infty)$ and respectively λ_s or $\lambda_\infty \neq 1$. The case of a finite interval (a, b) with weights w and φ with singularities only at the end s of the interval as $\lambda_s < 1$ was settled in the previous subsection (see Theorem 7.2). Also the cases when $\kappa_s = \kappa_\infty = 0$ were considered in [5, 8]. Let us consider the case of a semi-infinite interval and $\lambda_\infty < 1$.

Theorem 7.3. *Let $r, \ell \in \mathbb{N}$ as $\ell < r$ and $\lambda_\infty \in (-\infty, 1)$. Set*

$$\mathcal{A} = B_1(\sigma_\infty; \infty, a; \xi) A_0(-\ell; \infty, a; \xi),$$

where $\xi > a$ and

$$\sigma_\infty = \frac{1}{1 - \lambda_\infty}.$$

Then for $t > 0$ and $f \in L_\infty(\chi_{a-1}^{-\ell})(a, \infty)$ we have

$$K(f, t^r; L_\infty(\chi_{a-1}^{-\ell})(a, \infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \sim \omega_r(\mathcal{A}f, t)_{\infty(a,\infty)}.$$

Proof. We set

$$\mathcal{B} = A_\ell(\ell;\infty,a;\eta)B_1(\sigma_\infty^{-1};\infty,a;\eta),$$

where $\eta > a$. We shall show that

$$\mathcal{A} : (L_\infty(\chi_{a-1}^{-\ell})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \rightleftharpoons (L_\infty(a,\infty), AC_{loc}^{r-1}, D^r) : \mathcal{B}. \quad (7.3)$$

Then by Proposition 3.1 we have

$$K(f,t^r; L_\infty(\chi_{a-1}^{-\ell})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \sim K(\mathcal{A}f,t^r; L_\infty(a,\infty), AC_{loc}^{r-1}, D^r),$$

which, in view of (3.1), implies the assertion of the theorem.

Relation (7.3) follows from

$$\boxed{(L_\infty(a,\infty), AC_{loc}^{r-1}, D^r)}$$

$$B_1(\sigma_\infty;\infty,a;\xi) \quad \upharpoonleft\downharpoonleft \quad B_1(\sigma_\infty^{-1};\infty,a;\eta) \qquad \text{Step 1}$$

$$\boxed{(L_\infty(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}$$

$$A_0(-\ell;\infty,a;\xi) \quad \upharpoonleft\downharpoonleft \quad A_\ell(\ell;\infty,a;\eta) \qquad \text{Step 2}$$

$$\boxed{(L_\infty(\chi_{a-1}^{-\ell})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}\,.$$

Step 1. We use [8, Propositions 4.14] with $p=\infty$, $j=j'=1$, $\sigma=\sigma_\infty>0$, $\gamma_a = 0 \in \Gamma_+(\infty)$, $\gamma_\infty = 0 \in \Gamma_1^*(\infty)$ and $\tau_a = \tau_\infty = 0$ as we take into consideration that $\gamma_\infty/\sigma = 0 \in \Gamma_1^*(\infty)$.

Step 2. We apply Proposition 5.3 with $p=\infty$, $j=0$, $j'=\ell$, $\gamma=0\in\Gamma_+(\infty)$ and $\phi = \chi_{a-1}^{r\lambda_\infty}$. ◻

We proceed to the case $\lambda_s > 1$.

Theorem 7.4. *Let $r \in \mathbb{N}$, (a,b) be a finite interval, s denote one of its ends and e the other. Let also $\ell \in \mathbb{N}$ as $\ell < r$ and $\lambda_s \in (1,\infty)$. Set*

$$\mathcal{A} = B_1(\sigma_s; s,e;\infty,e;\xi_1)A_0(-\ell;s,e;\xi_2),$$

where $\xi_1 > e$, $\xi_2 \in (a,b)$ and

$$\sigma_s = \frac{1}{1-\lambda_s}.$$

Then for $t>0$ and $f \in L_\infty(\chi_s^{-\ell})(a,b)$ we have

$$K(f,t^r; L_\infty(\chi_s^{-\ell})(a,b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \sim \omega_r(\mathcal{A}f,t)_{\infty(e,\infty)}.$$

Proof. We set

$$\mathcal{B} = A_\ell(\ell; s, e; \eta) B_1(\sigma_s^{-1}; \infty, e; s, e; \eta),$$

where $\eta \in (a, b)$. We shall show that

$$\mathcal{A} : (L_\infty(\chi_s^{-\ell})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \rightleftharpoons (L_\infty(e, \infty), AC_{loc}^{r-1}, D^r) : \mathcal{B}. \tag{7.4}$$

Then by Proposition 3.1 we have

$$K(f, t^r; L_\infty(\chi_s^{-\ell})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \sim K(\mathcal{A}f, t^r; L_\infty(e, \infty), AC_{loc}^{r-1}, D^r),$$

which, in view of (3.1), implies the assertion of the theorem.

Relation (7.4) follows from

$$\boxed{(L_\infty(e, \infty), AC_{loc}^{r-1}, D^r)}$$

$$B_1(\sigma_s; s, e; \infty, e; \xi_1) \quad \upharpoonleft\downharpoonright \quad B_1(\sigma_s^{-1}; \infty, e; s, e; \eta) \qquad \text{Step 1}$$

$$\boxed{(L_\infty(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}$$

$$A_0(-\ell; s, e; \xi_2) \quad \upharpoonleft\downharpoonright \quad A_\ell(\ell; s, e; \eta) \qquad \text{Step 2}$$

$$\boxed{(L_\infty(\chi_s^{-\ell})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}\,.$$

Step 1. We use [8, Propositions 4.17] with $p = \infty$, $j = i' = 1$, $\sigma = \sigma_s < 0$, $\gamma_e = 0 \in \Gamma_+(\infty)$, $\gamma_\infty = 0 \in \Gamma_1^*(\infty)$ and $\tau_e = \tau_\infty = 0$ as we take into consideration that $\gamma_\infty / \sigma = 0 \in \Gamma_1^*(\infty)$.

Step 2. We apply Proposition 5.2 with $p = \infty$, $i = 0$, $i' = \ell$, $\gamma = 0 \in \Gamma_+(\infty)$ and $\phi = \chi_s^{r\lambda_s}$. □

Finally, we establish the characterization in the case $\lambda_\infty > 1$.

Theorem 7.5. *Let $r, \ell \in \mathbb{N}$ as $\ell < r$ and $\lambda_\infty \in (1, \infty)$. Set*

$$\mathcal{A} = B_1(\sigma_\infty; \infty, a; b, a; \xi_1) A_0(-\ell; \infty, a; \xi_2),$$

where $\xi_1 \in (a, b)$, $\xi_2 > a$ and

$$\sigma_\infty = \frac{1}{1 - \lambda_\infty}.$$

Then for $t > 0$ and $f \in L_\infty(\chi_{a-1}^{-\ell})(a, \infty)$ we have

$$K(f, t^r; L_\infty(\chi_{a-1}^{-\ell})(a, \infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \sim \omega_r(\mathcal{A}f, t)_{\infty(a,b)}.$$

Proof. We set

$$\mathcal{B} = A_\ell(\ell; \infty, a; \eta) B_1(\sigma_\infty^{-1}; b, a; \infty, a; \eta),$$

where $\eta > a$. We shall show that

$$\mathcal{A} : (L_\infty(\chi_{a-1}^{-\ell})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \rightleftharpoons (L_\infty(a,b), AC_{loc}^{r-1}, D^r) : \mathcal{B}. \quad (7.5)$$

Then by Proposition 3.1 we have

$$K(f,t^r; L_\infty(\chi_{a-1}^{-\ell})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \sim K(\mathcal{A}f,t^r; L_\infty(a,b), AC_{loc}^{r-1}, D^r),$$

which, in view of (3.1), implies the assertion of the theorem.

Relation (7.5) follows from

$$\boxed{(L_\infty(a,b), AC_{loc}^{r-1}, D^r)}$$
$$B_1(\sigma_\infty;\infty,a;b,a;\xi_1) \quad \uparrow\downarrow \quad B_1(\sigma_\infty^{-1};b,a;\infty,a;\eta) \qquad \text{Step 1}$$
$$\boxed{(L_\infty(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}$$
$$A_0(-\ell;\infty,a;\xi_2) \quad \uparrow\downarrow \quad A_\ell(\ell;\infty,a;\eta) \qquad \text{Step 2}$$
$$\boxed{(L_\infty(\chi_{a-1}^{-\ell})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}\ .$$

Step 1. We use [8, Propositions 4.18] with $p = \infty$, $s = b$, $e = a$, $i = j' = 1$, $\sigma = \sigma_\infty < 0$, $\gamma_e = 0 \in \Gamma_+(\infty)$, $\gamma_s = 0 \in \Gamma_1^*(\infty)$ and $\tau_s = \tau_e = 0$ as we take into consideration that $\gamma_s/\sigma = 0 \in \Gamma_1^*(\infty)$.

Step 2. Just as Step 2 in the proof of Theorem 7.3. □

Here we have considered only L_∞-spaces, but these results can be extended to spaces of bounded continuous functions with or without restrictions on their behaviour at the ends of the interval just as in [8, Section 8].

7.3. The Case $\kappa_s \in \Gamma_{exc}(p), \lambda_s < 1$ or $\kappa_\infty \in \Gamma_{exc}(p)$, $\lambda_\infty > 1$ as $1 \le p < \infty$

Theorem 7.6. *Let $1 \le p < \infty$, $r \in \mathbb{N}$, (a,b) be a finite interval, s denote one of its ends and e the other. Let also $\ell \in \mathbb{N}_0$ as $\ell < r$ and $\lambda_s \in (-\infty, 1)$. Set*

$$\mathcal{A} = B_1(\sigma_s; s,e;\xi) A_0(\rho_s; s,e;\xi) \tilde{A}(1;s,e;*) A_{r-1}(r-\ell-1; s,e;*),$$

where $\xi \in (a,b)$ and

$$\sigma_s = \frac{1}{1-\lambda_s}, \qquad \rho_s = \frac{\lambda_s - 1}{p} - r.$$

Then for $t > 0$ and $f \in L_p(\chi_s^{-\ell-1/p})(a,b)$ we have

$$K(f,t^r; L_p(\chi_s^{-\ell-1/p})(a,b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \sim \omega_r(\mathcal{A}f,t)_{p(a,b)}.$$

Proof. We set

$$\mathcal{B} = A_\ell(\ell - r + 1; s, e; \eta)\tilde{A}(-1; s, e; *)A_r(-\rho_s; s, e; *)B_1(\sigma_s^{-1}; s, e; \eta),$$

where $\eta \in (a, b)$. We shall show that

$$\mathcal{A} : (L_p(\chi_s^{-\ell-1/p})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \rightleftharpoons (L_p(a, b), AC_{loc}^{r-1}, D^r) : \mathcal{B}. \quad (7.6)$$

Then by Proposition 3.1 we have

$$K(f, t^r; L_p(\chi_s^{-\ell-1/p})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \sim K(\mathcal{A}f, t^r; L_p(a, b), AC_{loc}^{r-1}, D^r),$$

which, in view of (3.1), implies the assertion of the theorem.

Relation (7.6) follows from

$$\boxed{(L_p(a, b), AC_{loc}^{r-1}, D^r)}$$

$$B_1(\sigma_s; s, e; \xi) \quad \uparrow\downarrow \quad B_1(\sigma_s^{-1}; s, e; \eta) \qquad \text{Step 1}$$

$$\boxed{(L_p(\chi_s^{-\lambda_s/p})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}$$

$$A_0(\rho_s; s, e; \xi) \quad \uparrow\downarrow \quad A_r(-\rho_s; s, e; *) \qquad \text{Step 2}$$

$$\boxed{(L_p(\chi_s^{-r-1/p})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}$$

$$\tilde{A}(1; s, e; *) \quad \uparrow\downarrow \quad \tilde{A}(-1; s, e; *) \qquad \text{Step 3}$$

$$\boxed{(L_p(\chi_s^{1-r-1/p})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}$$

$$A_{r-1}(r - \ell - 1; s, e; *) \quad \uparrow\downarrow \quad A_\ell(\ell - r + 1; s, e; \eta) \qquad \text{Step 4}$$

$$\boxed{(L_p(\chi_s^{-\ell-1/p})(a, b), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}\,.$$

Step 1. Here we use [8, Proposition 4.15] with $i = i' = 1$, $\sigma = \sigma_s > 0$, $\gamma_e = 0 \in \Gamma_+(p)$, $\gamma_s = 0 \in \Gamma_1^*(p)$ and $\tau_s = \tau_e = 0$ as we take into consideration that $(\gamma_s + 1/p)/\sigma - 1/p = -\lambda_s/p \in \Gamma_1^*(p)$ since $\lambda_s < 1$.

Step 2. Now we apply [8, Proposition 3.9] with $i = 0$, $i' = r$, $\rho = \rho_s$, $\gamma_s = -\lambda_s/p \in \Gamma_0(p)$ (since $\lambda_s < 1$), $\gamma_e = 0 \in \Gamma_+(p)$ and $\phi = \chi_s^{r\lambda_s}$ as we take into consideration that $\gamma_s + \rho = -r - 1/p \in \Gamma_r(p)$.

Step 3. We use Proposition 5.5 with $\rho = 1$, $\gamma_s = -r - 1/p$ and $\gamma_e = \lambda_e = 0$.

Step 4. We apply Proposition 5.2 with $i = r - 1$, $i' = \ell$, $\gamma_e = 0 \in \Gamma_+(p)$ and $\phi = \chi_s^{r\lambda_s}$. □

Just similarly we verify the following analogue of the last theorem in the case of semi-infinite interval.

Theorem 7.7. *Let $1 \le p < \infty$ and $r \in \mathbb{N}$. Let also $\ell \in \mathbb{N}_0$ as $\ell < r$ and $\lambda_\infty \in (1, \infty)$. Set*

$$\mathcal{A} = A_0(-r - 1; b, a; \xi)B_r(\sigma_\infty; \infty, a; b, a; *)\tilde{A}(\rho_\infty; \infty, a; *)A_0(-\ell; \infty, a; *),$$

where $\xi \in (a,b)$ *and*

$$\sigma_\infty = \frac{1}{1-\lambda_\infty}, \qquad \rho_\infty = (r+1-1/p)(1-\lambda_\infty).$$

Then for $t > 0$ *and* $f \in L_p(\chi_{a-1}^{-\ell-1/p})(a,\infty)$ *we have*

$$K(f,t^r; L_p(\chi_{a-1}^{-\ell-1/p})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_s} D^r) \sim \omega_r(\mathcal{A}f,t)_{p(a,b)}.$$

Proof. We set

$$\mathcal{B} = A_\ell(\ell;\infty,a;\eta)\tilde{A}(-\rho_\infty;\infty,a;*)B_1(\sigma_\infty^{-1};b,a;\infty,a;*)A_r(r+1;b,a;*),$$

where $\eta > a$. We shall show that

$$\mathcal{A} : (L_p(\chi_{a-1}^{-\ell-1/p})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \rightleftharpoons (L_p(a,b), AC_{loc}^{r-1}, D^r) : \mathcal{B}. \quad (7.7)$$

Then by Proposition 3.1 we have

$$K(f,t^r; L_p(\chi_{a-1}^{-\ell-1/p})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \sim K(\mathcal{A}f,t^r; L_p(a,b), AC_{loc}^{r-1}, D^r),$$

which, in view of (3.1), implies the assertion of the theorem.

Relation (7.7) follows from

$$\boxed{(L_p(a,b), AC_{loc}^{r-1}, D^r)}$$

$$A_0(-r-1;b,a;\xi) \quad \upharpoonleft\downharpoonright \quad A_r(r+1;b,a;*) \qquad \text{Step 1}$$

$$\boxed{(L_p(\chi_b^{-r-1})(a,b), AC_{loc}^{r-1}, D^r)}$$

$$B_r(\sigma_\infty;\infty,a;b,a;*) \quad \upharpoonleft\downharpoonright \quad B_1(\sigma_\infty^{-1};b,a;\infty,a;*) \qquad \text{Step 2}$$

$$\boxed{(L_p(\chi_{a-1}^{-\rho_\infty-1/p})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}$$

$$\tilde{A}(\rho_\infty;\infty,a;*) \quad \upharpoonleft\downharpoonright \quad \tilde{A}(-\rho_\infty;\infty,a;*) \qquad \text{Step 3}$$

$$\boxed{(L_p(\chi_{a-1}^{-1/p})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}$$

$$A_0(-\ell;\infty,a;*) \quad \upharpoonleft\downharpoonright \quad A_\ell(\ell;\infty,a;\eta) \qquad \text{Step 4}$$

$$\boxed{(L_p(\chi_{a-1}^{-\ell-1/p})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}\,.$$

Step 1 is accomplished by [8, Proposition 3.9], Step 2 by [8, Proposition 4.18], Step 3 by Proposition 5.6, and Step 4 by Proposition 5.3. □

7.4. The Case $\kappa_s \notin \Gamma_{exc}(\infty)$, $\lambda_s < 1$ or $\kappa_\infty \notin \Gamma_{exc}(\infty)$, $\lambda_\infty > 1$ as $p = \infty$

Theorem 7.8. *Let* $r \in \mathbb{N}$, (a,b) *be a finite interval, s denote one of its ends and e the other. Let* $\kappa_s, \lambda_s \in \mathbb{R}$ *be such that* $\kappa_s \notin \Gamma_{exc}(\infty)$ *and* $\lambda_s \in (-\infty,1)$. *Set*

$$\mathcal{A} = B_1(\sigma_s;s,e;\xi)A_0(1-r;s,e;\xi)\tilde{A}(\tilde{\rho};s,e;*)A_r(\rho_s;s,e;*),$$

where $\xi \in (a,b)$, $\tilde{\rho} < 0$ *and*

$$\sigma_s = \frac{1}{1-\lambda_s}, \qquad \rho_s = \kappa_s - \tilde{\rho} + r - 1.$$

Then for $t > 0$ *and* $f \in C(\chi_s^{\kappa_s})(s,e)$ *such that* $\lim_{x\to s} |x-s|^{\kappa_s} f(x) = 0$ *we have*

$$K(f,t^r; C(\chi_s^{\kappa_s})(s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \sim \omega_r(\mathcal{A}f,t)_{\infty(a,b)}.$$

Proof. Let the integer i' be determined by the condition $\Gamma_{i'}(\infty) \ni \kappa_s$ and $\eta \in (a,b)$. We set

$$\mathcal{B} = A_{i'}(-\rho_s; s,e;\eta)\hat{A}(-\tilde{\rho}; s,e;*)A_{r-1}(r-1;s,e;*)B_1(\sigma_s^{-1};s,e;\eta).$$

We shall show that

$$\mathcal{A} : (C_0(\chi_s^{\kappa_s})[s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \rightleftharpoons (C[s,e), AC_{loc}^{r-1}, D^r) : \mathcal{B}. \tag{7.8}$$

Then by Proposition 3.1 we have

$$K(f,t^r; C_0(\chi_s^{\kappa_s})[s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \sim K(\mathcal{A}f,t^r; C[s,e), AC_{loc}^{r-1}, D^r). \tag{7.9}$$

On the other hand, we established in the first part of Remark 5.3 that if $g \in AC_{loc}^{r-1}(a,b)$ and $g, \chi_s^{r\lambda_s} g^{(r)} \in L_\infty(\chi_s^{\gamma_s})(s,e)$ with $\gamma_s \notin \Gamma_{exc}(\infty)$ and $\lambda_s < 1$, then $\lim_{x\to s}(\chi_s^{\gamma_s} g)(x) = 0$. Hence

$$\begin{aligned} K(f,t^r; C_0(\chi_s^{\kappa_s})[s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \\ = K(f,t^r; C(\chi_s^{\kappa_s})(s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r). \end{aligned} \tag{7.10}$$

Now, (7.9), (7.10) and (3.1) imply the assertion of the theorem.

It remains to prove relation (7.8). It follows from

$$\boxed{(C[s,e), AC_{loc}^{r-1}, D^r)}$$

$$B_1(\sigma_s; s,e;\xi) \quad \upharpoonleft\downharpoonright \quad B_1(\sigma_s^{-1}; s,e;\eta) \qquad \text{Step 1}$$

$$\boxed{(C[s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}$$

$$A_0(1-r; s,e;\xi) \quad \upharpoonleft\downharpoonright \quad A_{r-1}(r-1; s,e;*) \qquad \text{Step 2}$$

$$\boxed{(C(\chi_s^{1-r})[s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}$$

$$\tilde{A}(\tilde{\rho}; s,e;*) \quad \upharpoonleft\downharpoonright \quad \hat{A}(-\tilde{\rho}; s,e;*) \qquad \text{Step 3}$$

$$\boxed{(C_0(\chi_s^{1-r+\tilde{\rho}})[s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}$$

$$A_r(\rho_s; s,e;*) \quad \upharpoonleft\downharpoonright \quad A_{i'}(-\rho_s; s,e;\eta) \qquad \text{Step 4}$$

$$\boxed{(C_0(\chi_s^{\kappa_s})[s,e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r)}\,.$$

Step 1. We use [8, Propositions 4.15 and 8.3] with $p = \infty$, $i = i' = 1$, $\sigma = \sigma_s > 0$, $\gamma_e = 0 \in \Gamma_+(\infty)$, $\gamma_s = 0 \in \Gamma_1^*(\infty)$ and $\tau_s = \tau_e = 0$ as we take into consideration that $\gamma_s/\sigma = 0 \in \Gamma_1^*(\infty)$.

Step 2. We apply Proposition 5.2 and [8, Proposition 8.3] with $p = \infty$, $i = 0$, $i' = r-1$, $\gamma = 0 \in \Gamma_+(\infty)$ and $\phi = \chi_s^{r\lambda_s}$.

Step 3. We apply Proposition 5.7 with $\rho = \tilde{\rho} < 0$ and $\gamma_e = \lambda_e = 0$.

Step 4. We use [8, Propositions 3.9 and 8.3] with $p = \infty$, $i = r$, $\rho = \rho_s$, $\gamma_s = 1 - r + \tilde{\rho} \in \Gamma_r(\infty)$ (because $\tilde{\rho} < 0$), $\gamma_e = 0 \in \Gamma_+(\infty)$ and $\phi = \chi_s^{r\lambda_s}$ as we take into consideration that $\gamma_s + \rho = \kappa_s \in \Gamma_{i'}(\infty)$. Finally, we need to note that $\lim_{x\to s}(\chi_s^{\gamma} A_i(\rho; s, e; \xi) f)(x) = 0$ if $\lim_{x\to s}(\chi_s^{\gamma+\rho} f)(x) = 0$, where $\rho, \gamma \in \mathbb{R}$ and $\gamma \in \Gamma_i(\infty)$, as [8, Lemma 8.1.a] with $\sigma = 1$ and $a = s$ implies. □

Remark 7.3. Let us note that if $\kappa_s < 1-r$, then we may set $\tilde{\rho} = \kappa_s + r - 1$ in Theorem 7.8. Hence $\rho_s = 0$ and the components of $\mathcal{A}$ reduce with one operator.

Remark 7.4. The assertion of the theorem above is not valid in the case $\lim_{x\to s}(\chi_s^{\kappa_s} f)(x) \neq 0$. Actually, if $f \in C(\chi_s^{\kappa_s})(s, e)$, $\kappa_s \notin \Gamma_{exc}(\infty)$, is such that $\lim_{x\to s}(\chi_s^{\kappa_s} f)(x) \neq 0$ or the limit does not exist and also $\lambda_s < 1$, then $K(f, t^r; C(\chi_s^{\kappa_s})[s, e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \not\to 0$ when $t \to 0$. This follows from the fact that $\lim_{x\to s}(\chi_s^{\kappa_s} g)(x) = 0$ for any $g \in AC_{loc}^{r-1}(a, b)$ such that $g, \chi_s^{r\lambda_s} g^{(r)} \in L_\infty(\chi_s^{\kappa_s})(s, e)$ provided that $\kappa_s \notin \Gamma_{exc}(\infty)$ and $\lambda_s < 1$ (cf. Remark 5.3). However, for the space $C(\chi_s^{\kappa_s})[s, e)$ it can be shown that

$$K(f, t^r; C(\chi_s^{\kappa_s})[s, e), AC_{loc}^{r-1}, \chi_s^{r\lambda_s} D^r) \sim \omega_r(\mathcal{A}f, t)_{\infty(a,b)} + |\lim_{x\to s}(\chi_s^{\kappa_s} f)(x)|,$$

where $\mathcal{A}$ is defined in Theorem 7.8.

Here is the corresponding result in the case of a semi-infinite interval. The operator is even simpler.

Theorem 7.9. *Let $r \in \mathbb{N}$. Let $\kappa_\infty, \lambda_\infty \in \mathbb{R}$ be such that $\kappa_\infty \notin \Gamma_{exc}(\infty)$ and $\lambda_\infty \in (1, \infty)$. Set*

$$\mathcal{A} = B_1(\sigma_\infty; \infty, a; b, a; \xi)\tilde{A}(\bar{\rho}; \infty, a; *)A_0(\kappa_\infty - \bar{\rho}; \infty, a; *),$$

where $\xi \in (a, b)$, $\bar{\rho} > 0$ and

$$\sigma_\infty = \frac{1}{1 - \lambda_\infty}.$$

Then for $t > 0$ and $f \in C(\chi_{a-1}^{\kappa_\infty})(a, \infty)$ such that $\lim_{x\to\infty} x^{\kappa_\infty} f(x) = 0$ we have

$$K(f, t^r; C(\chi_{a-1}^{\kappa_\infty})(a, \infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \sim \omega_r(\mathcal{A}f, t)_{\infty(a,b)}.$$

Proof. Let the integer j' be determined by the condition $\Gamma_{j'}(\infty) \ni \kappa_\infty$ and $\eta > a$. We set

$$\mathcal{B} = A_{j'}(\bar{\rho} - \kappa_\infty; \infty, a; \eta)\hat{A}(-\bar{\rho}; \infty, a; *)B_1(\sigma_\infty^{-1}; b, a; \infty, a; *).$$

We shall show that

$$\mathcal{A} : (C_0(\chi_{a-1}^{\kappa_\infty})(a,\infty], AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \rightleftharpoons (C(a,b], AC_{loc}^{r-1}, D^r) : \mathcal{B}. \qquad (7.11)$$

Then by Proposition 3.1 we have

$$K(f,t^r; C_0(\chi_{a-1}^{\kappa_\infty})(a,\infty], AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \sim K(\mathcal{A}f, t^r; C(a,b], AC_{loc}^{r-1}, D^r). \qquad (7.12)$$

On the other hand, as in the proof of the previous theorem, we establish by means of the first assertion in Remark 5.4 that

$$\begin{aligned} K(f,t^r; C_0(\chi_{a-1}^{\kappa_\infty})(a,\infty], AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r) \\ = K(f,t^r; C(\chi_{a-1}^{\kappa_\infty})(a,\infty), AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r). \end{aligned} \qquad (7.13)$$

Now, (7.12), (7.13) and (3.1) imply the assertion of the theorem.

It remains to prove relation (7.11). It follows from

$$\boxed{(C(a,b], AC_{loc}^{r-1}, D^r)}$$

$$B_1(\sigma_\infty; \infty, a; b, a; \xi) \quad \uparrow\downharpoonleft \quad B_1(\sigma_\infty^{-1}; b, a; \infty, a; *) \qquad \text{Step 1}$$

$$\boxed{(C(a,\infty], AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}$$

$$\tilde{A}(\bar\rho; \infty, a; *) \quad \uparrow\downharpoonleft \quad \hat{A}(-\bar\rho; \infty, a; *) \qquad \text{Step 2}$$

$$\boxed{(C_0(\chi_{a-1}^{\bar\rho})(a,\infty], AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}$$

$$A_0(\kappa_\infty - \bar\rho; \infty, a; *) \quad \uparrow\downharpoonleft \quad A_{j'}(\bar\rho - \kappa_\infty; \infty, a; \eta) \qquad \text{Step 3}$$

$$\boxed{(C_0(\chi_{a-1}^{\kappa_\infty})(a,\infty], AC_{loc}^{r-1}, \chi_{a-1}^{r\lambda_\infty} D^r)}\,.$$

Step 1. We apply [8, Propositions 4.18 and 8.3] with $p = \infty$, $s = b$, $e = a$, $i = 1$, $j' = 1$, $\sigma = \sigma_\infty < 0$, $\gamma_e = 0 \in \Gamma_+(\infty)$, $\gamma_s = 0 \in \Gamma_1^*(\infty)$ and $\tau_a = \tau_b = 0$ as we take into consideration that $\gamma_s/\sigma = 0 \in \Gamma_1^*(\infty)$.

Step 2. We apply Proposition 5.8 with $\gamma_a = \lambda_a = 0$ and $\rho = \bar\rho > 0$.

Step 3. Now we apply [8, Propositions 3.8 and 8.3] with $p = \infty$, $j = 0$, $\rho = \kappa_\infty - \bar\rho$, $\gamma_a = 0 \in \Gamma_+(\infty)$, $\gamma_\infty = \bar\rho \in \Gamma_0(\infty)$ and $\phi = \chi_{a-1}^{r\lambda_\infty}$. We take into account that $\gamma_\infty + \rho = \kappa_\infty \in \Gamma_{j'}(\infty)$. We also use [8, Lemma 8.1.b] with $\sigma = 1$ and $a - 1$ in the place of a to show that the operators preserve the zero weighted limit at infinity. □

Remark 7.5. In Theorem 7.9, if $\kappa_\infty > 0$, then we may set $\bar\rho = \kappa_\infty$ and the components of $\mathcal{A}$ reduce with one.

8. An A-operator for Treatment of the Exponent only at the Finite End of a Semi-infinite Interval

The A-operators $A_{i,j}(\rho; a, \infty; \xi)$ we used in [8] to treat the singularity of w at the finite end a of the semi-infinite interval (a, ∞) have the disadvantage of changing the exponent of w at infinity too. Thus generally we need to use also an operator of the type $A_j(\rho; \infty, a; \xi)$ to correct this. In this section we shall give the definition of an operator through which one can change only the exponent of w at the point a without affecting the one at infinity. However, this operator more complicated than $A_{i,j}(\rho; a, \infty; \xi)$. We start with a more general setting (cf. [5, (6.1)]). We set for $r \in \mathbb{N}$ and an interval

$$W^r_{\infty,loc}(I) = \left\{ f \in AC^{r-1}_{loc}(I) : f^{(r)} \in L_{\infty,loc}(I) \right\}.$$

Definition 8.1. Let $I \subseteq \mathbb{R}$ be an open interval, $r \in \mathbb{N}$, $\xi \in I$ and $\mathrm{w} \in W^r_{\infty,loc}(I)$. For $x \in I$ and $f \in L_{1,loc}(I)$ we set

$$(A(\mathrm{w};\xi)f)(x) = \mathrm{w}(x)f(x) + \sum_{i=1}^{r} (-1)^i \binom{r}{i} \int_\xi^x \frac{(x-y)^{i-1}}{(i-1)!} \mathrm{w}^{(i)}(y) f(y)\, dy.$$

Proposition 8.1. *If $g \in AC^{r-1}_{loc}(I)$, then $A(\mathrm{w};\xi)g \in AC^{r-1}_{loc}(I)$ as well and for $k = 0, 1, \ldots, r$ and $x \in I$ there holds*

$$(A(\mathrm{w};\xi)g)^{(k)}(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{r-i-1}{k-i} \mathrm{w}^{(k-i)}(x) g^{(i)}(x) + \sum_{i=k+1}^{r} (-1)^i \binom{r}{i} \int_\xi^x \frac{(x-y)^{i-k-1}}{(i-k-1)!} \mathrm{w}^{(i)}(y) g(y)\, dy.$$

In particular, $(A(\mathrm{w};\xi)g)^{(r)}(x) = \mathrm{w}(x) g^{(r)}(x)$, $x \in I$.

Proof. The first assertion follows from the definition of $A(\mathrm{w};\xi)$ and the conditions imposed on g and w. The second one is verified by direct computations. We set for $i = 1, \ldots, r$

$$\Psi_i(x) = (-1)^i \binom{r}{i} \int_\xi^x \frac{(x-y)^{i-1}}{(i-1)!} \mathrm{w}^{(i)}(y) g(y)\, dy.$$

We have for $k = 0, 1, \ldots, r$ and $x \in I$

$$\begin{aligned}(A(\mathrm{w};\xi)g)^{(k)}(x) &= \sum_{i=0}^{k} (-1)^i \binom{r}{i} \left(\mathrm{w}^{(i)} g\right)^{(k-i)}(x) + \sum_{i=k+1}^{r} \Psi_i^{(k)}(x) \\ &= \sum_{i=0}^{k} (-1)^i \binom{r}{i} \sum_{j=0}^{k-i} \binom{k-i}{j} \mathrm{w}^{(k-j)}(x)\, g^{(j)}(x) + \sum_{i=k+1}^{r} \Psi_i^{(k)}(x)\end{aligned}$$

$$= \sum_{j=0}^{k} \left[\sum_{i=0}^{k-j} (-1)^i \binom{r}{i} \binom{k-i}{j} \right] \mathrm{w}^{(k-j)}(x)\, g^{(j)}(x) + \sum_{i=k+1}^{r} \Psi_i^{(k)}(x)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{r-j-1}{k-j} \mathrm{w}^{(k-j)}(x)\, g^{(j)}(x) + \sum_{i=k+1}^{r} \Psi_i^{(k)}(x)$$
$$= \sum_{i=0}^{k} (-1)^{k-i} \binom{r-i-1}{k-i} \mathrm{w}^{(k-i)}(x)\, g^{(i)}(x) + \sum_{i=k+1}^{r} \Psi_i^{(k)}(x),$$

as by [26, Ch. 1, (5a)] (which follows from Vandermonde's identity) we have

$$\sum_{i=0}^{k-j} (-1)^i \binom{r}{i} \binom{k-i}{j} = (-1)^{k-j} \binom{r-j-1}{k-j}. \qquad \square$$

Proposition 8.2. *Let* $\mathrm{w}_1, \mathrm{w}_2 \in W^r_{\infty,loc}(I)$ *and* $\xi \in I$*, where* $I \subseteq \mathbb{R}$ *is an open interval. Then* $A(\mathrm{w}_1;\xi)A(\mathrm{w}_2;\xi) = A(\mathrm{w}_1\mathrm{w}_2;\xi)$*. Hence* $A(\mathrm{w}_1;\xi)$ *and* $A(\mathrm{w}_2;\xi)$ *commute and if* $\mathrm{w} \in W^r_{\infty,loc}(I)$ *is such that* $\mathrm{w}(x) \neq 0$ *on* I*, then* $A(\mathrm{w};\xi)$ *is invertible and* $A^{-1}(\mathrm{w};\xi) = A(1/\mathrm{w};\xi)$*.*

Proof. Using Proposition 8.1 we get for every $g \in AC^{r-1}_{loc}(I)$

$$(A(\mathrm{w}_1;\xi)A(\mathrm{w}_2;\xi)g)^{(r)}(x) = \mathrm{w}_1(x)(A(\mathrm{w}_2;\xi)g)^{(r)}(x)$$
$$= \mathrm{w}_1(x)\mathrm{w}_2(x)g^{(r)}(x) = (A(\mathrm{w}_1\mathrm{w}_2;\xi)g)^{(r)}(x).$$

Next, again by Proposition 8.1, we get for every $k = 0, 1, \dots, r-1$ the relations

$$(A(\mathrm{w}_1;\xi)A(\mathrm{w}_2;\xi)g)^{(k)}(\xi) = \sum_{i=0}^{k} (-1)^{k-i} \binom{r-i-1}{k-i} \mathrm{w}_1^{(k-i)}(\xi)(A(\mathrm{w}_2;\xi)g)^{(i)}(\xi)$$
$$= \sum_{i=0}^{k} (-1)^{k-i} \binom{r-i-1}{k-i} \mathrm{w}_1^{(k-i)}(\xi) \sum_{j=0}^{i} (-1)^{i-j} \binom{r-j-1}{i-j} \mathrm{w}_2^{(i-j)}(\xi) g^{(j)}(\xi)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \left[\sum_{i=j}^{k} \binom{r-i-1}{k-i} \binom{r-j-1}{i-j} \mathrm{w}_1^{(k-i)}(\xi)\mathrm{w}_2^{(i-j)}(\xi) \right] g^{(j)}(\xi)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \left[\sum_{\ell=0}^{k-j} \binom{r-j-\ell-1}{k-j-\ell} \binom{r-j-1}{\ell} \mathrm{w}_1^{(k-j-\ell)}(\xi)\mathrm{w}_2^{(\ell)}(\xi) \right] g^{(j)}(\xi)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{r-j-1}{k-j} \left[\sum_{\ell=0}^{k-j} \binom{k-j}{\ell} \mathrm{w}_1^{(k-j-\ell)}(\xi)\mathrm{w}_2^{(\ell)}(\xi) \right] g^{(j)}(\xi)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{r-j-1}{k-j} (\mathrm{w}_1\mathrm{w}_2)^{(k-j)}(\xi) g^{(j)}(\xi)$$
$$= (A(\mathrm{w}_1\mathrm{w}_2;\xi)g)^{(k)}(\xi).$$

Now the Taylor formula gives $A(\mathrm{w}_1;\xi)A(\mathrm{w}_2;\xi)g = A(\mathrm{w}_1\mathrm{w}_2;\xi)g$ for every $g \in AC_{loc}^{r-1}(I)$. Since for any $[a,b] \subset I$ such that $\xi \in [a,b]$ the linear operators $A(\mathrm{w}_1;\xi)A(\mathrm{w}_2;\xi)$ and $A(\mathrm{w}_1\mathrm{w}_2;\xi)$, considered as maps on $L_1[a,b]$ into itself, are bounded and $W_p^r[a,b]$ is dense in $L_1[a,b]$, we get that $A(\mathrm{w}_1;\xi)A(\mathrm{w}_2;\xi)f = A(\mathrm{w}_1\mathrm{w}_2;\xi)f$ for every $f \in L_{1,loc}(I)$. □

To get an A-operator which treats only the singularity of w at the finite end of the interval (a,∞), we can set

$$\mathrm{w}(x) = \arctan^\rho(x-a) \sim \begin{cases} (x-a)^\rho, & x \in (a, a+1], \\ 1, & x \in [a+1,\infty), \end{cases}$$

where $\rho \in \mathbb{R}$. Thus, on the basis of Definition 8.1 we get

Definition 8.2. Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $i,j \in \mathbb{N}_0$ as $i \le j \le r$, and $\xi \in (a,\infty)$. For $x \in (a,\infty)$ and $f \in L_{1,loc}(a,\infty)$, satisfying the additional requirements $\chi_a^{-i+\rho} f \in L_1(a,a+1)$ if $i > 0$ and $\chi_a^{-j-2} f \in L_1(a+1,\infty)$ if $j < r$, we set

$$\begin{aligned}(\bar{A}_{i,j}(\rho;a,\infty;\xi)f)(x) &= \arctan^\rho(x-a)f(x) \\ &\quad + \sum_{k=1}^{i}(x-a)^{k-1}\int_a^x \alpha_{r,k}(\rho;y-a)f(y)\,dy \\ &\quad + \sum_{k=i+1}^{j}(x-a)^{k-1}\int_\xi^x \alpha_{r,k}(\rho;y-a)f(y)\,dy \\ &\quad - \sum_{k=j+1}^{r}(x-a)^{k-1}\int_x^\infty \alpha_{r,k}(\rho;y-a)f(y)\,dy,\end{aligned}$$

where

$$\begin{aligned}\alpha_{r,k}(\rho;x) = k\binom{r}{k}\sum_{\ell=1}^{r}\binom{\rho}{\ell}\arctan^{\rho-\ell}x \sum_{m=\max\{0,\ell-k\}}^{r-k}(-1)^{m-\ell}\frac{x^m}{(1+x^2)^{k+m}} \\ \times \sum_{n_1+\cdots+n_\ell=k+m-\ell} p_{n_1}(x)\cdots p_{n_\ell}(x)\end{aligned}$$

and

$$p_n(x) = \sum_{s=0}^{[n/2]}(-1)^s\binom{n}{2s}\frac{x^{n-2s}}{2s+1}.$$

The proposition below shows how these operators act on triplets.

Proposition 8.3. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi,\eta > a$ and $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ with $\gamma_a, \gamma_\infty, \gamma_a+\rho \notin \Gamma_{exc}(p)$. Assume that $i \le j$ and $i' \le j$, where*

i,j,i' are determined by $\Gamma_i(p) \ni \gamma_a$, $\Gamma_j(p) \ni \gamma_\infty$ and $\Gamma_{i'}(p) \ni \gamma_a + \rho$. Finally, let ϕ be measurable and non-negative on (a,∞). Then we have

$$\bar{A}_{i,j}(\rho;a,\infty;\xi) : (L_p(w\chi_a^\rho\chi_{a-1}^{-\rho})(a,\infty), AC_{loc}^{r-1}, \phi D^r) \rightleftharpoons (L_p(w)(a,\infty), AC_{loc}^{r-1}, \phi D^r) : \bar{A}_{i',j}(-\rho;a,\infty;\eta).$$

Proof. We need to show that $\mathcal{A} = \bar{A}_{i,j}(\rho;a,\infty;\xi)$ and $\mathcal{B} = \bar{A}_{i',j}(-\rho;a,\infty;\eta)$ satisfy the conditions of Definition 3.1 with $X_1 = L_p(w\chi_a^\rho\chi_{a-1}^{-\rho})(a,\infty)$, $X_2 = L_p(w)(a,\infty)$, $Y_1 = Y_2 = AC_{loc}^{r-1}(a,\infty)$ and $\mathcal{D}_1 = \mathcal{D}_1 = \phi D^r$. Conditions (a) and (c) are verified by means of Hardy's and Hölder's inequalities; conditions (b) and (d)–(f) follow from Proposition 8.1; finally conditions (g) and (h) follow from [8, Proposition 2.2] with $\bar{\mathcal{A}} = \bar{A}_{0,r}(\rho;a,\infty;\xi)$ and Proposition 8.2. □

The operator $A(\mathrm{w};\xi) : L_p(\mathrm{w})(I) \to L_p(I)$ is bounded under proper restrictions on the function w and proper values of the ξ's, $a \le \xi \le \infty$, (not necessarily the same in each integral summand). Thus, in view of Propositions 8.1, 8.2 and [8, Proposition 2.2], we shall get

$$A(\mathrm{w};\xi) : (L_p(\mathrm{w})(I), AC_{loc}^{r-1}, \phi D^r) \rightleftharpoons (L_p(I), AC_{loc}^{r-1}, \phi D^r) : A(1/\mathrm{w};\eta),$$

and hence

$$K(f,t;L_p(\mathrm{w})(I), AC_{loc}^{r-1}, \phi D^r) \sim K(A(\mathrm{w};\xi)f,t;L_p(I), AC_{loc}^{r-1}, \phi D^r).$$

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Borislav R. Draganov
Department of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
and
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia
BULGARIA
E-mail: bdraganov@fmi.uni-sofia.bg

Kamen G. Ivanov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia
BULGARIA
E-mail: kamen@math.bas.bg