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Approximation of Functions by Some Exponential-type Operators

Ivan Gadjev*

The major goal of this survey is to gather all, or more precisely the best, direct and converse theorems about approximation of functions in L_p-norm by the Kantorovich modifications of the operators of Bernstein, Baskakov, Szász-Mirakjan and Meyer-Konig and Zeller. Also, the best results about the approximation of functions in uniform norm by the classical variants of these operators are summarized.

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1. Introduction

At the beginning of the last century, in order to prove the theorem of Weierstrass about the approximation of continuous functions by algebraic polynomials, Bernstein introduced a new operator, now known as the Bernstein operator. Almost immediately the Bernstein operator became a major tool in approximation of functions by algebraic polynomials and people started to investigate how good the approximation of functions in uniform norm is by using it. In 1930, L. Kantorovich suggested a modification of the Bernstein operator in order to use it to approximate functions in L_p-norm. Some kind of modification is needed because the classical Bernstein operator is not bounded in L_p-norm. Another modifications were introduced later, most important of which is the Durrmeyer's modification.

Later, in order to solve different problems, various operators were introduced: to approximate functions in unbounded interval – the operators of Szász-Mirakjan and Baskakov, and to approximate unbounded functions in finite interval – the Meyer-Konig and Zeller operator. Also, all of these operators

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were modified in one or another way in order to be used for approximation of functions in L_p-norm.

The major goal of this survey is to gather all (more precisely the best) direct and converse theorems about approximation of functions in L_p-norm by the Kantorovich modifications of the operators of Bernstein, Baskakov, Szász-Mirakjan and Meyer-Konig and Zeller. Also, we summarize the best results about the approximation of functions in uniform norm by the classical variants of these operators.

We start with the definitions of the classical operators.

1. The Bernstein operator (see, for instance, [20]) is defined for bounded functions $f(x)$ on $[0,1]$ by the formula

$$B_n f(x) = (B_n f, x) = B_n(f,x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{n,k}(x), \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

2. The Baskakov operator (see [1]) is defined for bounded functions $f(x)$ in $[0,\infty)$ by the formula

$$V_n f(x) = (V_n f, x) = V_n(f,x) = \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) v_{n,k}(x), \tag{2}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

3. The Szász-Mirakjan operator (see [22] and [26]) is defined for bounded functions $f(x)$ in $[0,\infty)$ by the formula

$$S_n f(x) = (S_n f, x) = S_n(f,x) = \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{n,k}(x), \tag{3}$$

where

$$s_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}.$$

4. The Meyer-König and Zeller (MKZ) operator (see [21]) is defined for (not necessarily bounded) functions $f(x)$ in $[0,1)$ by the formula

$$M_n f(x) = (M_n f, x) = M_n(f,x) = \sum_{k=0}^{\infty} f\Big(\frac{k}{n+k}\Big) m_{n,k}(x), \tag{4}$$

where

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

By D we denote the interval $[0,1]$ for the Bernstein operator, the interval $[0,1)$ for the Meyer-König and Zeller operator and the interval $[0,\infty)$ for the Baskakov and Szász-Mirakjan operators.

By $\varphi(x)$ we denote the weight function, naturally connected with every one of the operators mentioned above.

1. Bernstein operator:
$$\varphi(x) = x(1-x). \tag{5}$$
2. Baskakov operator:
$$\varphi(x) = x(1+x). \tag{6}$$
3. Szász-Mirakjan operator:
$$\varphi(x) = x. \tag{7}$$
4. Meyer-König and Zeller operator:
$$\varphi(x) = x(1-x)^2. \tag{8}$$

2. Approximation in Uniform Norm

There is a lot of results during the years about approximation of functions in uniform norm by the the operators (1)–(4). Probably, all of the results before 1987 can be summarized in the next Theorem 1, proved in [7, Theorem 9.3.2]. But before stating it we give some definitions.

By $C(D)$ we denote the space of all continuous on D functions, by $L_\infty(D)$ the space of all Lebesgue measurable and essentially bounded in D functions equipped with the uniform norm $\|\cdot\|$ and by $CB(D) = C(D) \cap L_\infty(D)$ the space of all continuous and bounded in D functions. Also, we define

$$W^2_\infty(\varphi) = \big\{g \,:\, g' \in AC_{loc}(D_+),\ \varphi g'' \in L_\infty(D)\big\},$$

where $AC_{loc}(D_+)$ consists of the functions which are absolutely continuous in D for every interval $[a,b] \subset D$.

The usual tools for getting the right estimations are the modulus of smoothness of second order of Ditzian-Totik, defined by the formula

$$\omega^2_{\sqrt{\varphi}}(f,h) = \sup_{|t|\le h} \big\| f\big(x - \sqrt{\varphi(x)}\,t\big) - 2f(x) + f\big(x + \sqrt{\varphi(x)}\,t\big)\big\|,$$

and the K-functional,

$$K_\varphi(f,t) = \inf\big\{\|f-g\| + t\|\varphi g''\| : g \in W^2_\infty(\varphi),\ f-g \in CB(D)\big\}. \tag{9}$$

It is well known fact that the K-functional $K_\varphi(f,t)$ is equivalent to $\omega^2_{\sqrt{\varphi}}(f,\sqrt{t}\,)$ (see [7]) in a sense that there exists a positive constant C such that

$$C^{-1}K_\varphi(f,t) \le \omega^2_{\sqrt{\varphi}}(f,\sqrt{t}\,) \le CK_\varphi(f,t).$$

Theorem 1. *Let L_n be one of the operators defined by* (1), (2) *or* (3). *Then*

$$\|L_nf - f\| \le C\Big[\omega^2_{\sqrt{\varphi}}\Big(f, \frac{1}{\sqrt{n}}\Big) + \frac{1}{n}\|f\|\Big],$$

$$K_\varphi(f,t) \le \|L_kf - f\| + C\,\frac{k}{n}\,K_\varphi\Big(f, \frac{1}{k}\Big),$$

$$\|L_nf - f\| = O\big(n^{-\alpha/2}\big) \iff \omega^2_{\sqrt{\varphi}}(f,h) = O(h^\alpha), \qquad \alpha < 2.$$

In fact, in [29] Totik proved a more general result. Let L_n be a sequence of positive linear operators mapping $C\{a,b\}$ into $C(a,b)$ satisfying the conditions

$$L_n(1,x) = 1, \qquad L_n(t,x) = 1, \qquad L_n\big((t-x)^2, x\big) \le K\varphi(x)\alpha_n^2,$$

where $\lim_{n\to\infty} \alpha_n = 0 + 0$ and K is a constant.

Theorem 2. *Let L_n satisfy the assumptions above. Then for every function $f \in C\{a,b\}$*

$$\|L_nf - f\| \le K\omega^2_{\sqrt{\varphi}}(f, \alpha_n),$$

with a constant K independent of n.

Also,

Theorem 3. *Let L_n satisfy the assumptions of the above theorem and let us suppose that each $L_n(f,x)$ is twice continuously differentiable and for the numbers α_n we have*

$$\alpha_n/\alpha_{n+1} \le C.$$

If $0 < \alpha < 1$ and $L_nf - f = O\big(\alpha_n^{2\alpha}\big)$ implies

$$|\varphi(x)L_n''(f,x)| \le K_f\alpha_n^{-2}\big(\alpha_n^{2\alpha} + \omega^2_{\sqrt{\varphi}}(f,\alpha_n)\big),$$

then $L_nf - f = O\big(\alpha_n^{2\alpha}\big)$ is equivalent to $\omega^2_{\sqrt{\varphi}}(f,\delta) = O(\delta^{2\alpha})$.

Later, Ditzian by using a new modulus of smoothness (for $\lambda \in [0,1]$)

$$\omega^2_{\varphi^{\lambda/2}}(f,\delta) = \sup_{|h|\le\delta}\ \sup_{t\pm h\varphi^{\lambda/2}(t)\in[0,1]} \Big|f\big(t - \varphi^{\lambda/2}(t)h\big) - 2f(t) + f\big(t + \varphi^{\lambda/2}(t)h\big)\Big|,$$

proved in [5] a direct theorem for the Bernstein operator, i.e.

$$|B_n(f,x) - f(x)| \le C\omega^2_{\phi^{\lambda/2}}\big(f, n^{-1/2}\phi(x)^{(1-\lambda)/2}\big), \qquad x \in [0,1]. \tag{10}$$

This estimation unified the classical estimate for $\lambda = 0$ and the norm estimate for $\lambda = 1$. And the modulus $\omega^2_{\varphi^{\lambda/2}}(f,\delta)$ is equivalent to the K-functional

$$K_{\varphi^{\lambda/2}}(f,t) = \inf\big\{\|f-g\| + t\|\varphi^\lambda g''\| \,:\, g, g' \in AC_{loc}(D_+)\big\}.$$

In the same article Ditzian pointed out that in the cases of Baskakov and Szász-Mirakjan operators similar estimates are true and could be proved in the same way.

For the MKZ operator in [25, Theorem 2.1] the authors proved a similar direct result to (10).

Theorem 4. *Let $f \in C[0,1)$ is bounded, $0 \le \lambda \le 1$ and $\varphi(x) = x(1-x)^2$. Then for M_n given by* (4) *the next estimate is true*

$$|M_n(f,x) - f(x)| \le C\omega^2_{\varphi^{\lambda/2}}\big(f, n^{-1/2}\varphi(x)^{(1-\lambda)/2}\big).$$

They also proved a week result similar to the one in Theorem 1.

Theorem 5. *Let $f \in C[0,1)$ is bounded, $0 \le \lambda \le 1$, $0 < \alpha < 2$ and $\varphi(x) = x(1-x)^2$. Then for M_n given by* (4) *the following two statements are equivalent*

$$|M_n(f,x) - f(x)| = O\Big(\Big(\frac{\varphi^{(1-\lambda)/2}(x)}{\sqrt{n}}\Big)^{\alpha}\Big),$$
$$\omega^2_{\varphi^{\lambda/2}}(f,t) = O(t^{\alpha}).$$

In 1993 Ditzian and Ivanov (see [6]) introduced four types of the strong converse inequalities as the strongest are the inequalities of type A.

For a sequence of uniformly bounded operators Q_n and for some sequence $\lambda(n)$ which decreases to zero, they defined the next types of strong converse inequalities:

A $K(f,\lambda(n)) \le C\|f - Q_n f\|$ for all n (or for $n \ge n_0$),

B $K(f,\lambda(n)) \le C\frac{\lambda(n)}{\lambda(k)}\{\|f - Q_n f\| + \|f - Q_k f\|\}$ for all $k \ge rn$ and some fixed $r > 1$,

C $K(f,\lambda(n)) \le C\frac{1}{(r-1)n}\sum_{k=n}^{rn-1}\|f - Q_k f\|$ for all n and some $r > 1$,

D $K(f,\lambda(n)) \le C\sum \sup_{k\ge n}\|f - Q_k f\|$ for all n.

And they also gave a general method for obtaining strong converse inequalities of type B and A by using the iterated operator and proving for it Voronovskaya and Bernstein type of inequalities. In the same article they proved the strong converse inequality of type B for the Bernstein operator, i.e.

Theorem 6. *For $f \in C[0,1]$, $n \ge 12$, $\varphi(x)$ given by* (5) *and $K_\varphi(f,t)$ given by* (9)*, there exists a constant $R \ge 19$ such that for some $k \ge Rn$ the next inequality is true*

$$K_\varphi\Big(f,\frac{1}{n}\Big) \le M(R)\big(\|f - B_n f\| + \|f - B_k f\|\big).$$

In [31] Totik suggested another method for obtaining strong converse inequalities of type B (close in spirit to the classical parabola technique), and extended the Ditzian-Ivanov result to a large family of operators including Baskakov and Szász-Mirakjan operators. The strong converse inequality of type B for Meyer-Konig and Zeller operator is proved in [24].

In [30], by the method suggested by him in [31], Totik proved the strong converse inequality of type A for the Szász-Mirakjan operator, i.e.

Theorem 7. *There exists a constant C, independent of f and n such that for $\varphi(x)$ given by* (7)*, $K(f,t)$ given by* (9) *and $S_n(f,x)$ given by* (3) *the next inequality is true*

$$\omega^2_{\sqrt{\varphi}}\Big(f, \frac{1}{\sqrt{n}}\Big) \sim K_\varphi\Big(f, \frac{1}{n}\Big) \le C\|f - S_n f\|.$$

So, combining this result with the direct theorem, we obtain the exact characterization of the error of approximation, i.e.

$$\|f - S_n f\| \sim K_\varphi\Big(f, \frac{1}{n}\Big) \sim \omega^2_{\sqrt{\varphi}}\Big(f, \frac{1}{\sqrt{n}}\Big).$$

In the same article Totik pointed out that the same method is applyable (with the right modifications) to the Bernstein and Baskakov operators. Another proof (by using the method suggested in [6]) of the strong converse inequality of type A for the Bernstein operator is given in [19]. In the process of proving it, the authors also proved a Bernstein type of inequality for the iterated Bernstein operator which is important of its own.

Theorem 8. *Let us define $B_n^{k+1} = B_n\big(B_n^k\big)$, $B_n^1 = B_n$ where $B_n(f,x)$ is given by* (1)*. The function $\varphi(x)$ is given by* (5)*. Then for all $f \in C^2[0,1]$ and all $1 \le N \le n$ the next inequality is true*

$$\big\|\varphi^{3/2}\big(B_n^N(f)\big)'''\big\| \le C\sqrt{\frac{n\ln(N+1)}{N}}\|\varphi f''\|.$$

In [9] we gave another proof of the strong converse inequality of type A for the Baskakov operator by using the iterated Baskakov operator $V_n(f,x)$ given by (2): $V_n^{k+1} = V_n\big(V_n^k\big)$, $V_n^1 = V_n$. We also proved a Bernstein type of inequality for it.

Theorem 9. *Let $2 \le N \le \frac{n-2}{2}$, $n \ge 10$. The function $\varphi(x)$ is given by* (6)*. Then, there exists an absolute constant C such that for all functions $f \in L_\infty[0,\infty)$ and $\varphi f'' \in L_\infty[0,\infty)$ the next inequality holds*

$$\big\|\varphi^{3/2}\big(V_n^N(f)\big)'''\big\| \le K(N)\sqrt{n}\|\varphi f''\|, \quad \textit{where} \;\; K(N) \le CN^{-1/4}\ln N.$$

The strong converse inequality of type A for the Meyer-König and Zeller operator is proved in [10] by using the close connection between the Meyer-König and Zeller operator and the weighted approximation by the Baskakov operator.

Summarizing the results above, for all of the operators (1)–(4) we have the exact characterization of the error of approximation by the appropriate modulus of smoothness of Ditzian-Totik, i.e. for the operators L_n given by (1)–(4) the next equivalency is true

$$\|f - L_n f\| \sim K_\varphi\Big(f, \frac{1}{n}\Big) \sim \omega^2_{\sqrt{\varphi}}\Big(f, \frac{1}{\sqrt{n}}\Big).$$

3. Approximation in L_p-norm by Kantorovich Modifications

In order to approximate functions in L_p-norm by the operators (1)–(4) we need to modify them in an appropriate way because they are not bounded operators in L_p. There are many modifications but the most important are the Durrmeyer and Kantorovich modifications. The Durrmeyer modification has a lot of advantages over Kantorovich modification but the Kantorovich modification has one big advantage – it is easier to compute. Of course, there is a price to pay for that – it is very difficult to work with these operators.

In this paper we consider only Kantorovich modification because the Durrmeyer modifications is a big topic and needs a separate survey.

Considering the approximation of functions in L_1-norm, Kantorovich suggested in [18] a modification of the classical Bernstein operator:

$$B_n^*(f,x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(u)\,du. \tag{11}$$

Analogously to the Bernstein case, the Kantorovich modifications of the other operators were introduced (see, for instance, [2, 7, 15, 28, 27, 32, 33]).

For the Baskakov operator:

$$V_n^*(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x)\, n\int_{k/n}^{(k+1)/n} f(u)\,du. \tag{12}$$

For the Szász-Mirakjan operator:

$$S_n^*(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x)\, n\int_{k/n}^{(k+1)/n} f(u)\,du. \tag{13}$$

For the Meyer-König and Zeller operator:

$$M_n^*(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x)\frac{(n+k)(n+k+1)}{n}\int_{k/(n+k)}^{(k+1)/(n+k+1)} f(u)\,du. \tag{14}$$

Here, again like for the classical variants of these operators, the Theorem 1 summarizes the best results for the operators (11), (12) and (13) up to 1987. For the operator (14) Totik [27] proved:

Theorem 10. *For the function $\varphi(x)$ given by* (8) *and $M_n^*(f,x)$ given by* (14)*, we have: except the case $p=1$, $\alpha=1$, for all $1 \le p < \infty$ and $0 < \alpha \le 1$ the following two statements are equivalent for $f \in L_p(0,1)$:*

$$\|M_n^* f - f\|_p \le Cn^{-\alpha}, \tag{15}$$

$$\|\Delta^2_{h\sqrt{\varphi}}\|_{L_p(h^2,1)} \le Ch^{2\alpha}, \qquad h \ge 0. \tag{16}$$

On the other hand, for $p=\alpha=1$, (15) *is equivalent to the condition: f is absolutely continuous and $\varphi f'$ is of bounded variation on* $(0,1)$*.*

In [8] Ditzian and Zhou proved for the operator (11) the next two theorems.

Theorem 11. *Let $\psi(\delta)$ is an increasing function such that*

$$\psi(\delta) \sim \delta^3 \int_\delta^1 \frac{\psi(u)}{u^4}\, du.$$

Then for $1 < p < \infty$ the next is true

$$\|B_n^* f - f\|_p = O\Big(\psi\Big(\frac{1}{\sqrt{n}}\Big)\Big) \quad \Longrightarrow \quad \omega^2_{\sqrt{\varphi}}(f,u)_p = O(\psi(u)).$$

In the above $\omega^2_\phi(f,t)_p$ is the modulus of smoothness of Ditzian-Totik of second order:

$$\omega^2_\phi(f,h)_p = \sup_{|t|\le h} \big\|f(x-\phi(x)t) - 2f(x) + f(x+\phi(x)t)\big\|_p. \tag{17}$$

In order to deal with the case $p = 1$ they used a new K-functional:

$$K^*_\varphi(f,t)_p = \inf\big\{\|f-g\|_p + t^2\|(\varphi g')'\|_p + t^3\|\varphi^{3/2} g^{(3)}\|_p\big\}$$

and proved the next theorem.

Theorem 12. *For the operator* (11)*, the function $\varphi(x)$ given by* (5) *and the K-functional $K^*_\varphi(f,t)_p$, the next inequality is true*

$$\|B_n^* f - f\|_1 \le C\Big(K^*_\varphi\Big(f, \frac{1}{\sqrt{n}}\Big)_1 + n^{-1} E_0(f)_1\Big)$$

where $E_0(f)_1$ is the best approximation constant in L_1-norm.
Also, if

$$\|B_n^* f - f\|_1 = O\Big(\psi\Big(\frac{1}{\sqrt{n}}\Big)\Big),$$

then

$$K^*_\varphi\Big(f, \frac{1}{\sqrt{n}}\Big)_1 = O\Big(\psi\Big(\frac{1}{\sqrt{n}}\Big)\Big).$$

Later, in [4] Berens and Xu introduced a new K-functional

$$\tilde{K}_\varphi(f,t)_p = \inf\big\{\|f-g\|_p + t\|P(D)g\|_p \,:\, g\in C^2[0,1], \lim_{x\to 0_+,1_-} \varphi(x)g'(x) = 0\big\},$$

where

$$P(D) = \frac{d}{dx}\Big(\varphi\frac{d}{dx}\Big),$$

and proved the direct and a week converse inequality for B_n^* in terms of $\tilde{K}_\varphi(f,t)_p$, i.e.,

Theorem 13. *Let $f \in L_p(0,1)$, $1 \le p < \infty$ and $f \in C[0,1]$, $p = \infty$, respectively ($\varphi(x)$ is given by (5)). Then*

$$\|f - B_n^* f\|_p \le C\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p, \qquad n \in \mathbb{N}_0.$$

Conversely, for all $t > 0$ and $n \in \mathbb{N}_0$,

$$\tilde{K}_\varphi(f,t)_p \le \|f - B_n^* f\|_p + C(n+1)t\tilde{K}_\varphi\Big(f, \frac{1}{n+1}\Big)_p.$$

The constants only depend on p.

Later, Chen and Ditzian [34], by using the K-functional $\tilde{K}_\varphi(f,t)_p$ proved the strong converse inequality of type B:

Theorem 14. *For $f \in L_p[0,1]$, $\varphi(x)$ given by (5), there exists a constant $R > 1$ such that, for $l \ge Rn$*

$$\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p \le C\frac{l}{n}\left(\|f - B_n^* f\|_p + \|f - B_l^* f\|_p\right), \qquad 1 < p \le \infty.$$

Combining the direct and converse result, they also established the next equivalency valid only for $p > 1$.

Theorem 15. *For $f \in L_p[0,1]$, $\varphi(x)$ given by (5), there exists an integer m such that, for $l < p \le \infty$,*

$$\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p \sim \|f - B_n^* f\|_p + \|f - B_{mn}^* f\|_p,$$

and hence for $1 < p < \infty$

$$\|f - B_n^* f\|_p + \|f - B_{mn}^* f\|_p \sim \omega^2_{\sqrt{\varphi}}\Big(f, \frac{1}{\sqrt{n}}\Big)_p + \omega\Big(f, \frac{1}{n}\Big)_p,$$

where $\omega(f,t)_p$ is the classical modulus of continuity and $\omega^2_\phi(f,t)_p$ is the modulus of smoothness of Ditzian-Totik of second order (17).

Chen and Ditzian also pointed out that for $p = 1$ such result cannot be true.

Almost at the same time Gonska and Zhou [16] proved the strong converse inequality of type A.

Theorem 16. *There exists an absolute positive constant C such that for $f \in L_p[0,1]$, $1 \le p \le \infty$, $\varphi(x)$ given by (5), there holds*

$$\|f - B_n^* f\|_p \sim \tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p.$$

They also characterized (for $p > 1$) the K-functional $\tilde{K}_\varphi(f,t)_p$ by appropriate moduli of smoothness.

Theorem 17. *Let $E_0(f)_p$ denotes the best approximation constant of f defined by*

$$E_0(f)_p = \inf_c \|f - c\|_p.$$

Then, for a weight $\varphi(x)$ given by (5)*,*

$$\tilde{K}_\varphi(f,t)_p \sim \omega^2_{\sqrt{\varphi}}\Big(f, \sqrt{t}\Big)_p + tE_0(f)_p, \qquad 1 < p < \infty,$$

and

$$\tilde{K}_\varphi(f,t)_\infty \sim \omega^2_{\sqrt{\varphi}}(f, \sqrt{t})_\infty + \omega(f,t)_\infty.$$

Of course, in the above one can replace $tE_0(f)_\infty$ in the first relation of the Theorem 17 by $\omega(f,t)_\infty$ and the two estimations can be combined in one. Thus, the Theorem 17 can be reformulated as

$$\tilde{K}_\varphi(f,t)_p \sim \omega^2_{\sqrt{\varphi}}(f, \sqrt{t})_p + \omega(f,t)_p, \qquad 1 < p \le \infty.$$

But this equivalency cannot be extended for $p = 1$ because of functions like $f(x) = \ln x$ for which we have $\tilde{K}_\varphi(f,t)_1 \sim t$ and, at the same time, $\omega^2_{\sqrt{\varphi}}(f, \sqrt{t})_1 \sim t|\ln t|$. Later, in [17] Ivanov defined a new modulus of smoothness in order to characterize $\tilde{K}_\varphi(f,t)_1$.

For $f \in L_1[0,1]$ and $0 < x < 1$ he defined the operator A by

$$(Af)(x) = f(x) + \int_{1/2}^{x} \Big(\frac{x}{y^2} - \frac{1-x}{(1-y)^2}\Big) f(y)\, dy.$$

The values of Af at $x = 0$ and $x = 1$ are defined by continuity when possible. He proved the next equivalency for $p = 1$.

Theorem 18. *For every $t \in (0,1]$, $f \in L_1[0,1]$, and $\varphi(x)$ given by* (5)*,we have*

$$\tilde{K}_\varphi(f,t)_1 \sim K_\varphi(Af, \sqrt{t})_1 + tE_0(f)_1.$$

As a consequence of this theorem we have that for every $t \in (0,1]$ and $f \in L_1[0,1]$:

$$\tilde{K}_\varphi(f,t)_1 \sim \omega^2_{\sqrt{\varphi}}(Af, \sqrt{t})_1 + t\omega(f,1)_1.$$

But, for $1 < p < \infty$ the connection between the K-functionals $\tilde{K}_\varphi(f,t)_p$ and $K_\varphi(Af,t)_p$ is not so satisfactory, as Ivanov pointed out in [17]. He proved:

Theorem 19. *For every $t \in (0,1]$ and $f \in L_p[0,1]$, $1 < p < \infty$ the next inequalities are true:*

$$\tilde{K}_\varphi(f,t)_p \le C\big(K_\varphi\big(Af, t^{1/2p}\big)_p + t^{1/p}E_0(f)_p\big),$$

$$K_\varphi(Af, \sqrt{t})_p + tE_0(f)_p \le Cp\tilde{K}_\varphi(f,t)_p.$$

Although the right-hand side of the first inequality can be improved, the example of $f(x) = \ln x - \ln(1-x)$ shows that the power $1/p$ of t in $t^{1/p}E_0(f)_p$ is exact.

The definitions (12) and (14) look as the most natural way to define Kantorovich modifications of the Baskakov and Meyer-König and Zeller operators, but they are not the best possible ones because the operators (12) and (14) are not contractions, and hence they are not very suitable for approximating functions in L_p-norm for $p < \infty$. That is why in [7] Ditzian and Totik introduced another Kantorovich modification of the Baskakov operator which is a contraction:

$$\tilde{V}_n(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x)\,(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} f(u)\,du. \tag{18}$$

It is not difficult to see that the Theorem 1 is still true for the operator (18) and the proof is the same as in [7]. In [11] we investigated the approximation of functions by the operator (18). Analogously to the Bernstein case we defined a new K-functional $\tilde{K}_\varphi(f,t)_p$, where $\varphi(x)$ is given by (6) – actually introduced for the first time in slightly different way by Berdisheva [3] in order to approximate functions by Durrmeyer modification of the Baskakov operator:

$$\tilde{K}_\varphi(f,t)_p = \inf\left\{\|f-g\|_p + t\|P(D)g\|_p : f-g\in L_p[0,\infty),\ g\in \tilde{W}_p[0,\infty)\right\}, \tag{19}$$

where

$$\tilde{W}_p[0,\infty) = \left\{f : f, f' \in AC_{loc}(0,\infty),\ P(D)f \in L_p[0,\infty),\ \lim_{x\to 0_+}\varphi(x)f'(x) = 0\right\}.$$

We proved a direct theorem and strong converse inequality of type B for the operator (18) by using the K-functional $\tilde{K}_\varphi(f,t)_p$.

Theorem 20. *For $1 < p \le \infty$ and for every $f \in L_p[0,\infty) + \tilde{W}_p[0,\infty)$, $\varphi(x)$ is given by (6), there exist absolute constants $R, C > 0$ such that for every natural $l \ge Rn$,*

$$C^{-1}\|\tilde{V}_n f - f\|_p \le \tilde{K}_\varphi(f,t)_p \le C\frac{l}{n}\left(\|\tilde{V}_n f - f\|_p + \|\tilde{V}_l f - f\|_p\right).$$

The left inequality is true for $p = 1$ as well.

Remark 1. Another way to state Theorem 20 is: there exists an integer k such that

$$\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p \sim \|\tilde{V}_n f - f\|_p + \|\tilde{V}_{kn} f - f\|_p, \qquad p > 1.$$

Remark 2. Both inequalities in Theorem 20 are stronger than the results mentioned above. Indeed, if $\varphi(x)$ is given by (6), from the simple inequality

$$\|P(D)f\|_p \le \|\varphi' f'\|_p + \|\varphi f''\|_p,$$

and from [7, Theorem 9.5.3-a),c)] it follows that

$$\|P(D)f\|_p \le C\big(\|\varphi f''\|_p + \|f\|_p\big),$$

and consequently

$$\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p \le C\Big[w^2_{\sqrt{\varphi}}\Big(f, n^{-1/2}\Big)_p + n^{-1}\|f\|_p\Big].$$

Observe that $\tilde{K}_\varphi\big(f, \frac{1}{n}\big)_p$ is not equivalent to $w^2_{\sqrt{\varphi}}\big(f, n^{-1/2}\big)_p + n^{-1}\|f\|_p$. For instance, for $p = 1$ and $f(x) = (1+x)^{-1}\ln^{-1}(1+x)$ we have $\tilde{K}_\varphi\big(f, \frac{1}{n}\big)_1 \le n^{-1}$ and $\|f\|_1 = \infty$.

In [12] we characterized the K-functional $\tilde{K}_\varphi(f,t)_p$, $p > 1$, by appropriate moduli of smoothness.

Theorem 21. *There exist absolute constants C_1, C_2 and t_0 such that for every $f \in L_p[0,\infty) + \tilde{W}_p[0,\infty)$ and $0 < t \le t_0$ the next inequalities are true ($\varphi(x)$ is given by* (6)*):*

$$\tilde{K}_\varphi(f,t)_p \le C_1\big(\omega^2_{\sqrt{\varphi}}\big(f, \sqrt{t}\big)_p + tE_0(f)\big), \qquad 1 \le p < \infty,$$

and

$$\omega^2_{\sqrt{\varphi}}\Big(f, \sqrt{t}\Big)_p + tE_0(f) \le C_2\tilde{K}_\varphi(f,t)_p, \qquad 1 < p < \infty.$$

Remark 3. The above means that for $1 < p < \infty$ we have the equivalency

$$\tilde{K}_\varphi(f,t)_p \sim \omega^2_{\sqrt{\varphi}}(f, \sqrt{t})_p + tE_0(f).$$

Like in the case for the Baskakov operator, the definition (14) of the Kantorovich modification of the Meyer-König and Zeller operator is not the best possible one because it is not a contraction. In [23] Müller defined another Kantorovich modification of the Meyer-König and Zeller operator $\tilde{M}_n(f,x)$ which is a contraction by the formula:

$$\tilde{M}_n(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x)\frac{(n+k+1)(n+k+2)}{n+1}\int_{k/(n+k+1)}^{(k+1)/(n+k+2)} f(u)\,du. \quad (20)$$

The operator $\tilde{M}_n(f,x)$ could be written more compactly as

$$\tilde{M}_n(f,x) = \frac{n+2}{(1-x)^2}\sum_{k=0}^{\infty} m_{n+2,k}(x)\int_{k/(n+k+1)}^{(k+1)/(n+k+2)} f(u)\,du.$$

In [14] we investigated the approximation of functions in L_p norm by $\tilde{M}_n(f,x)$ and by defining a new K-functional we proved the direct inequality for it.

For $\varphi(x)$ given by (8) we defined the space

$$\tilde{W}_p[0,1) = \big\{f \,:\, f, f' \in AC_{loc}(0,1),\ P(D)f \in L_p[0,1),\ \lim_{x\to 0_+} \varphi(x)f'(x) = 0\big\},$$

and the K-functional

$$\tilde{K}_\varphi(f,t)_p = \inf\big\{\|f-g\|_p + t\|P(D)g\|_p \,:\, f-g \in L_p[0,1),\ g \in \tilde{W}_p[0,1)\big\}.$$

We proved a direct theorem for the operator (20) by using the K-functional $\tilde{K}_\varphi(f,t)_p$.

Theorem 22. *There exists an absolute constant $C > 0$ such that for every natural n, $\tilde{M}_n$ defined by* (14)*, $\varphi(x)$ by* (8)*, the K-functional by* (19) *and for every $f \in L_p[0,1) + \tilde{W}_p[0,1)$*

$$\|\tilde{M}_n f - f\|_p \le C\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p, \qquad 1 \le p \le \infty.$$

Remark 4. The set of functions that can be approximated by $\tilde{M}_n$ is essentially bigger than $L_p[0,1)$ for which we can use the Kantorovich modification of Bernstein operator. For instance for $p = 1$ and $f(x) = (1-x)^{-1}(1+\ln(1-x))^{-1}$ we have $f(x) \notin L_1[0,1)$ and $f(x) \in L_1[0,1) + \tilde{W}_1[0,1)$.

We also characterized the K-functional $K_\varphi(f,t)_p$, $p > 1$, by appropriate moduli of smoothness.

Theorem 23. *There exist absolute constants C_1, C_2 and t_0 such that for every $f \in L_p[0,1) + \tilde{W}_p[0,1)$ and $0 < t \le t_0$ the next inequalities are true ($\varphi(x)$ given by* (8)*)*

$$K_\varphi(f,t)_p \le C_1\Big(\omega^2_{\sqrt{\varphi}}(f,\sqrt{t})_p + tE_0(f)\Big), \qquad 1 \le p < \infty,$$

and

$$\omega^2_{\sqrt{\varphi}}(f,\sqrt{t})_p + tE_0(f) \le C_2 K_\varphi(f,t)_p, \qquad 1 < p < \infty,$$

where $\omega^2_\phi(f,t)_p$ is the modulus of smoothness of Ditzian-Totik of second order.

In [13] we investigated the approximation of functions in L_p norm by $\tilde{S}_n(f,x)$ and by defining a new K-functional we proved the direct inequality and strong converse inequality of type B for it.

For $\varphi(x)$ given by (7) we defined the spaces

$$\tilde{W}_p[0,\infty) = \big\{f \,:\, f, f' \in AC_{loc}(0,\infty),\ P(D)f \in L_p[0,\infty),\ \lim_{x\to 0_+} xf'(x) = 0\big\},$$

$$L_p[0,\infty) + \tilde{W}_p[0,\infty) = \big\{f \,:\, f = f_1 + f_2,\ f_1 \in L_p[0,\infty),\ f_2 \in \tilde{W}_p[0,\infty)\big\}.$$

Also, we defined the K-functional $\tilde{K}_\varphi(f,t)_p$ by the formula

$$\tilde{K}_\varphi(f,t)_p = \inf\big\{\|f-g\|_p + t\|\tilde{D}g\|_p \,:\, f-g \in L_p[0,\infty),\ g \in \tilde{W}_p[0,\infty)\big\}. \quad (21)$$

Theorem 24. *For $\tilde{S}_n$ defined by (13), $\varphi(x)$ by (7), the K-functional given by (21), $1 < p \le \infty$ and for every $f \in L_p[0,\infty) + \tilde{W}_p[0,\infty)$, there exist absolute constants $L, C > 0$ such that for every natural $l \ge Ln$,*

$$C^{-1}\|\tilde{S}_n f - f\|_p \le \tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p \le C\frac{l}{n}\big(\|\tilde{S}_n f - f\|_p + \|\tilde{S}_l f - f\|_p\big).$$

The left inequality is true for $p = 1$ as well.

Remark 5. Another way to state Theorem 24 is: there exists an integer k such that

$$\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p \sim \|\tilde{S}_n f - f\|_p + \|\tilde{S}_{kn} f - f\|_p, \qquad p > 1.$$

Remark 6. Both inequalities in Theorem 24 are stronger than the results mentioned above. Indeed, using the arguments similar to Remark 2, it is not difficult to see that for $\varphi(x)$ given by (7) and the K-functional by (21),

$$\tilde{K}_\varphi\Big(f, \frac{1}{n}\Big)_p \le C\big[w^2_{\sqrt{\varphi}}(f, n^{-1/2})_p + n^{-1}\|f\|_p\big].$$

At the same time, $\tilde{K}_\varphi\left(f, \frac{1}{n}\right)_p$ is not equivalent to $w^2_{\sqrt{\varphi}}(f, n^{-1/2})_p + n^{-1}\|f\|_p$. For instance, for $p = 1$ and $f(x) = (1+x)^{-1}$ we have $\tilde{K}_\varphi\left(f, \frac{1}{n}\right)_1 \le n^{-1}$ and $\|f\|_1 = \infty$.

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Ivan Gadjev

Department of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
E-mail: gadjevivan@hotmail.com