CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2016
(K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 175-185
Prof. Marin Drinov Academic Publishing House, Sofia, 2018

Beta and Related Operators Revisited

Margareta Heilmann, Fadel Nasaireh and Ioan Raşa

In this paper we consider the Beta-type operators $\bar{\mathbb{B}}_n$, the corresponding inverse operators $\bar{\mathbb{B}}_n^{-1}$ on the set of polynomials of degree at most n and the operators $F_n := \bar{\mathbb{B}}_n^{-1} \circ B_n$ where B_n are the classical Bernstein operators. We establish Voronovskaya-type formulas and recurrence formulas for their moments. Furthermore the powers of F_n are investigated.

Keywords and Phrases: Beta operators, Voronovskaya-type formulas, moments, iterates.

Mathematics Subject Classification 2010: 41A36.

1. Introduction

Beta-type operators were introduced by Mühlbach in [12] and further investigated by him in [13] and by Lupaş in [11]. For a function $f \in C[0,1]$, $n \in \mathbb{N}$, $x \in [0,1]$ these mappings are given by

$$\bar{\mathbb{B}}_n(f;x) = \begin{cases} f(0), & x = 0, \\ \dfrac{1}{B(nx, n(1-x))} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)\, dt, & x \in (0,1), \\ f(1), & x = 1. \end{cases}$$

with Euler's Beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$. The $\bar{\mathbb{B}}_n$ are positive endomorphisms of $C[0,1]$; they reproduce linear functions and (see [2, 3, 14]) they preserve monotonicity and convexity of arbitrary order.

In [6, Theorem 3.1] it was proved that the mappings $\bar{\mathbb{B}}_n$ are injective. Moreover, the images of the monomials under $\bar{\mathbb{B}}_n$ (see [6, (6)]) show that the restriction to the space $\mathcal{P}_n$ of polynomials of degree at most n, i.e., $\bar{\mathbb{B}}_n : \mathcal{P}_n \longrightarrow \mathcal{P}_n$, is bijective, thus $\bar{\mathbb{B}}_n^{-1}$ exists.

The eigenstructure of $\bar{\mathbb{B}}_n$ is investigated in [6, 8], and the power series constructed with $\bar{\mathbb{B}}_n$ was studied in [1]. The operators $\bar{\mathbb{B}}_n$ are important not only as individual objects, but also in composition with other operators. This aspect is extensively presented by Stanila in [15]. As a significant example we mention the genuine Bernstein-Durrmeyer operators, which can be represented as the composition of classical Bernstein operators and Beta operators.

The classical Bernstein operators $B_n : C[0,1] \longrightarrow \mathcal{P}_n$ are defined by

$$B_n(f;x) = \sum_{j=0}^{n} p_{n,j}(x) f\Big(\frac{j}{n}\Big), \qquad x \in [0,1],$$

where

$$p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}.$$

By composing B_n and $\bar{\mathbb{B}}_n^{-1}$ we obtain the operators

$$F_n : C[0,1] \longrightarrow \mathcal{P}_n, \qquad F_n := \bar{\mathbb{B}}_n^{-1} \circ B_n.$$

These operators were introduced in [6]. Their approximation properties, the Voronovskaya-type formula, and the eigenstructure were investigated in this paper. Other properties of the operators F_n and some related quadrature formulas can be found in [9].

The paper is organized as follows. In Section 2 we obtain Voronovskaya-type results for the operators $\bar{\mathbb{B}}_n^{-1}$. Section 3 is devoted to studying the moments of $\bar{\mathbb{B}}_n^{-1}$ and F_n. In Section 4 the eigenstructure of F_n is used to investigate the asymptotic behavior of the powers of F_n. In Section 5 we present a conjecture and an open problem stated earlier but remained unsolved until now.

Throughout this paper we denote by $C[0,1]$ the space of real-valued continuous functions on the interval $[0,1]$, by $\mathcal{P}_n$ the set of polynomials of degree less or equal n and by $\mathcal{P}$ the set of all polynomials. The monomials e_j, $j \in \mathbb{N}_0$, are given by $e_j(x) = x^j$. We will use the Stirling numbers $s(m,l)$ and $S(m,l)$ of first and second kind, defined by

$$\sum_{l=0}^{m} s(m,l) x^l = x^{\overline{m}} \qquad \text{and} \qquad \sum_{l=0}^{m} S(m,l) x^{\underline{l}} = x^m,$$

with the rising and falling factorials

$$a^{\overline{j}} := \prod_{l=0}^{j-1} (a+l), \quad a^{\underline{j}} := \prod_{l=0}^{j-1} (a-l), \quad j \in \mathbb{N}; \qquad a^{\overline{0}} = a^{\underline{0}} := 1.$$

Special values needed for explicit calculations are

$$s(l,l) = 1, \;\; s(l,l-1) = \frac{1}{2} l(l-1), \;\; s(l,l-2) = \frac{1}{24} l(l-1)(l-2)(3l-1), \quad (1)$$

$$S(l,l) = 1, \;\; S(l,l-1) = \frac{1}{2} l(l-1), \;\; S(l,l-2) = \frac{1}{24} l(l-1)(l-2)(3l-5). \quad (2)$$

2. A Voronovskaja-type Formula for $\bar{\mathbb{B}}_n^{-1}$

It is well-known (see, e. g. [8]) that the numbers

$$\eta_k^{(n)} := \frac{(n-1)!}{(n+k-1)!}\, n^k, \qquad k \in \mathbb{N}_0, \tag{3}$$

are the eigenvalues of the restriction $\bar{\mathbb{B}}_n : \mathcal{P} \longrightarrow \mathcal{P}$. Moreover, the associated monic eigenpolynomials $q_k^{(n)} \in \mathcal{P}_k$ satisfy (see [8, (2.12)]

$$\lim_{n\to\infty} q_k^{(n)}(x) = p_k^*(x), \qquad k \in \mathbb{N}_0, \tag{4}$$

uniformly on $[0,1]$, where (see [4, Theorem 4.5])

$$p_0^*(x) = 1, \qquad p_1^*(x) = x - \frac{1}{2}, \qquad \text{and}$$
$$p_k^*(x) = \frac{k!(k-2)!}{(2k-2)!}\, x(x-1) P_{k-2}^{(1,1)}(2x-1), \qquad k \geq 2.$$

Here, $P_m^{(1,1)}$, $m \in \mathbb{N}_0$, denote the Jacobi polynomials, orthogonal with respect to the weight $(1-t)(1+t)$ on $[-1,1]$. In particular (see [4, p. 155]),

$$x(x-1)\big(p_k^*\big)''(x) = k(k-1)p_k^*(x), \qquad k \in \mathbb{N}_0. \tag{5}$$

Theorem 1. *For each $p \in \mathcal{P}$ we have*

$$\lim_{n\to\infty} n\big(\bar{\mathbb{B}}_n^{-1} p(x) - p(x)\big) = -\frac{x(1-x)}{2}\, p''(x), \tag{6}$$

uniformly on $[0,1]$.

Proof. Let $n \geq 1$, $p \in \mathcal{P}_m$, $m \leq n$. Then p can be represented as

$$p = \sum_{k=0}^{m} a_{n,k}(p) q_k^{(n)}, \tag{7}$$

and

$$p = \sum_{k=0}^{m} a_k(p) p_k^*, \tag{8}$$

with suitable real coefficients $a_{n,k}(p)$ and $a_k(p)$. It follows that

$$\lim_{n\to\infty} \sum_{k=0}^{m} a_{n,k}(p) q_k^{(n)} = \sum_{k=0}^{m} a_k(p) p_k^*.$$

This convergence takes place in the finite dimensional space $\mathcal{P}_m$, and with (4) it follows that

$$\lim_{n\to\infty} a_{n,k}(p) = a_k(p), \qquad k = 0, 1, \ldots, m. \tag{9}$$

On the other hand, from (7) we get

$$\bar{\mathbb{B}}_n^{-1} p = \sum_{k=0}^{m} a_{n,k}(p) \frac{1}{\eta_k^{(n)}} \, q_k^{(n)}, \tag{10}$$

and (3) yields

$$\lim_{n\to\infty} n\left(\frac{1}{\eta_k^{(n)}} - 1\right) = \lim_{n\to\infty} n\left(\sum_{l=0}^{k} s(k,l) n^{l-k} - 1\right) = \frac{k(k-1)}{2}, \quad k \in \mathbb{N}_0. \tag{11}$$

Now using (10), (7), (9), (11), (4), (5) and (8), we obtain successively

$$\begin{aligned}
\lim_{n\to\infty} n\left[\bar{\mathbb{B}}_n^{-1} p(x) - p(x)\right] &= \lim_{n\to\infty} \sum_{k=0}^{m} a_{n,k}(p)\, n\left(\frac{1}{\eta_k^{(n)}} - 1\right) q_k^{(n)}(x) \\
&= \sum_{k=0}^{m} a_k(p) \frac{k(k-1)}{2}\, p_k^*(x) \\
&= \sum_{k=0}^{m} a_k(p) \frac{x(x-1)}{2} \left(p_k^*\right)''(x) \\
&= \frac{x(x-1)}{2} \left(\sum_{k=0}^{m} a_k(p) p_k^*(x)\right)'' \\
&= -\frac{x(1-x)}{2}\, p''(x),
\end{aligned}$$

uniformly on $[0,1]$. This completes the proof. □

Remark 1. The Voronovskaja-type formula (6) should be compared with the corresponding one for $\bar{\mathbb{B}}_n$, namely (see, e. g., [7, Corollary 3])

$$\lim_{n\to\infty} n\big[\bar{\mathbb{B}}_n f(x) - f(x)\big] = \frac{x(1-x)}{2}\, f''(x), \qquad f \in C^2[0,1],$$

uniformly on $[0,1]$. Moreover, we also have (see [7, Remark 4]) for each function $f \in C^4[0,1]$

$$\begin{aligned}
&\lim_{n\to\infty} n\Big\{n\big[\bar{\mathbb{B}}_n f(x) - f(x)\big] - \frac{x(1-x)}{2} f''(x)\Big\} \\
&\quad = \frac{x(1-x)}{24}\big[3x(1-x) f^{IV}(x) + 8(1-2x) f'''(x) - 12 f''(x)\big],
\end{aligned}$$

uniformly on $[0,1]$.

In this context we state a corresponding result for the operators $\bar{\mathbb{B}}_n^{-1}$.

Theorem 2. *For each* $p \in \mathcal{P}$,

$$\lim_{n\to\infty} n\Big\{n\big[\bar{\mathbb{B}}_n^{-1}p(x) - p(x)\big] + \frac{x(1-x)}{2}\,p''(x)\Big\}$$
$$= \frac{x(1-x)}{24}\big[3x(1-x)p^{(4)}(x) + 4(1-2x)p^{(3)}(x)\big],$$

uniformly on $[0,1]$.

Proof. We define

$$T(p) := n\Big\{n\big[\bar{\mathbb{B}}_n^{-1}p(x) - p(x)\big] + \frac{x(1-x)}{2}p''(x)\Big\}$$
$$- \frac{x(1-x)}{24}\big[3x(1-x)p^{(4)}(x) + 4(1-2x)p^{(3)}(x)\big],$$

use the images of monomials (see [6, (17)]) given by

$$\bar{\mathbb{B}}_n^{-1}e_j = \frac{1}{n^j}\sum_{k=0}^{j}(-1)^{j-k}\frac{(n+k-1)!}{(n-1)!}\,S(j,k)e_k,$$

and the special values for the Stirling numbers given in (1) and (2).

In order to prove our theorem we show that $\lim_{n\to\infty} T(e_j) = 0$ for each monomial e_j, $j \in \mathbb{N}_0$. For $j = 0$ and $j = 1$ this is obvious. For $j = 2$ and $j = 3$ we have

$$\begin{aligned}
T(e_2) &= n\Big\{n\big[\bar{\mathbb{B}}_n^{-1}e_2(x) - e_2(x)\big] + \frac{x(1-x)}{2}\,2e_0(x)\Big\}\\
&= n\Big\{n\Big[\frac{n+1}{n}\,x^2 - \frac{1}{n}\,x - x^2\Big] + x(1-x)\Big\}\\
&= 0,\\
T(e_3) &= n\Big\{n\big[\bar{\mathbb{B}}_n^{-1}e_3(x) - e_3(x)\big] + \frac{x(1-x)}{2}\,6e_1(x)\Big\}\\
&\quad - \frac{x(1-x)}{24}\,4(1-2x)6e_0(x)\\
&= n\Big\{n\Big[\frac{(n+1)(n+2)}{n^2}\,x^3 - 3\frac{n+1}{n^2}\,x^2 + \frac{1}{n^2}\,x - x^3\Big] + 3x^2(1-x)\Big\}\\
&\quad - x(1-x)(1-2x)\\
&= 0.
\end{aligned}$$

For $j \geq 4$ we derive

$$
\begin{aligned}
T(e_j) &= n\Big\{n\big[\bar{\mathbb{B}}_n^{-1}e_j(x) - e_j(x)\big] + \frac{x(1-x)}{2}\,\frac{j!}{(j-2)!}\,e_{j-2}(x)\Big\}\\
&\quad - \frac{x(1-x)}{24}\Big[3x(1-x)\frac{j!}{(j-4)!}\,e_{j-4}(x) + 4(1-2x)\frac{j!}{(j-3)!}\,e_{j-3}(x)\Big]\\
&= \frac{1}{n^{j-2}}\sum_{k=0}^{j-3}(-1)^{j-k}\frac{(n-1+k)!}{(n-1)!}\,S(j,k)x^k\\
&\quad + \frac{1}{n^{j-2}}\sum_{k=j-2}^{j}(-1)^{j}\frac{(n-1+k)!}{(n-1)!}\,S(j,k)x^k - n^2x^j + \frac{x(1-x)}{2}\,\frac{j!}{(j-2)!}\,x^{j-2}\\
&\quad - x(1-x)\Big[\frac{x(1-x)}{8}\,\frac{j!}{(j-4)!}\,x^{j-4} + \frac{1-2x}{6}\,\frac{j!}{(j-3)!}x^{j-3}\Big]\\
&= \frac{1}{n^{j-2}}\sum_{k=0}^{j-3}(-1)^{j-k}\frac{(n-1+k)!}{(n-1)!}\,S(j,k)x^k\\
&\quad + \frac{(n+j-3)!}{(n-1)!}\,\frac{j(j-1)(j-2)(3j-5)}{24n^{j-2}}\,x^{j-2} - \frac{(n+j-2)!}{(n-1)!}\,\frac{j(j-1)}{2n^{j-2}}\,x^{j-1}\\
&\quad + \frac{(n+j-1)!}{(n-1)!}\,\frac{1}{2n^{j-2}}\,x^j - n^2x^j + \frac{1}{2}nj(j-1)x^{j-1} - \frac{1}{2}\,nj(j-1)x^j\\
&\quad - \frac{1}{8}\,\frac{j!}{(j-4)!}\,x^j + \frac{1}{4}\,\frac{j!}{(j-4)!}\,x^{j-1} - \frac{1}{8}\,\frac{j!}{(j-4)!}\,x^{j-2}\\
&\quad + \frac{1}{2}\,\frac{j!}{(j-3)!}\,x^{j-1} - \frac{1}{6}\,\frac{j!}{(j-3)!}\,x^{j-2}\\
&= \frac{1}{n^{j-2}}\sum_{k=0}^{j-3}(-1)^{j-k}\frac{(n-1+k)!}{(n-1)!}\,S(j,k)x^k\\
&\quad + x^{j-2}\Big\{\frac{1}{24n^{j-2}}\,\frac{j!(3j-5)}{(j-3)!}\sum_{k=0}^{j-2}s(j-2,k)n^k - \frac{1}{8}\,\frac{j!}{(j-4)!} - \frac{1}{6}\,\frac{j!}{(j-3)!}\Big\}\\
&\quad + x^{j-1}\Big\{-\frac{1}{2n^{j-2}}\,\frac{j!}{(j-2)!}\sum_{k=0}^{j-1}s(j-1,k)n^k\\
&\qquad\qquad + \frac{1}{2}n\,\frac{j!}{(j-2)!} + \frac{1}{4}\,\frac{j!}{(j-4)!} + \frac{1}{2}\,\frac{j!}{(j-3)!}\Big\}\\
&+ x^j\Big\{\frac{1}{n^{j-2}}\sum_{k=0}^{j}s(j,k)n^k - n^2 - \frac{1}{2}n\,\frac{j!}{(j-2)!} - \frac{1}{8}\,\frac{j!}{(j-4)!} - \frac{1}{3}\,\frac{j!}{(j-3)!}\Big\}\\
&=: T_0 + T_1 + T_2 + T_3.
\end{aligned}
$$

Obviously $\lim_{n\to\infty} T_0 = 0$. After some easy calculations we get

$$T_1 = \frac{1}{24}\,\frac{j!(3j-5)}{(j-3)!}\sum_{k=0}^{j-3} s(j-2,k)n^{k-j+2},$$

$$T_2 = \frac{1}{4}\,\frac{j!}{(j-2)!}\sum_{k=0}^{j-3} s(j-1,k)n^{k-j+2},$$

$$T_3 = \frac{1}{24}\sum_{k=0}^{j-3} s(j,k)n^{k-j+2}.$$

So, altogether, $\lim_{n\to\infty} T(e_j) = 0$ for every $j \in \mathbb{N}_0$, which proves the theorem. □

3. The Moments of $\bar{\mathbb{B}}_n^{-1}$ and F_n

Consider two linear operators $P, Q : \mathcal{P} \longrightarrow \mathcal{P}$, such that $Q(\mathcal{P}_m) \subset \mathcal{P}_m$, $m \in \mathbb{N}_0$. Let

$$U_j(x) := P(e_1 - xe_0)^j(x), \qquad j \geq 0,\ x \in [0,1],$$

and

$$V_i(x) := Q(e_1 - xe_0)^i(x), \qquad i \geq 0,\ x \in [0,1],$$

be the moments of P and Q. Denote by

$$\mathbb{W}_m(x) := (PQ)(e_1 - xe_0)^m(x), \qquad m \in \mathbb{N}_0,\ x \in [0,1],$$

the moments of PQ. From [5, Theorem 4] we know that

$$\mathbb{W}_m = m! \sum_{\substack{i,k\geq 0\\ i+k=m}} \sum_{j=k}^{m} \binom{j}{k} \frac{1}{j!i!}\, U_j V_i^{(j-k)} = \sum_{j=0}^{m} U_j \sum_{i=0}^{j} \binom{m}{i} \frac{1}{(j-i)!}\, V_{m-i}^{(j-i)}. \tag{12}$$

For a fixed $n \geq 1$, let $P := \bar{\mathbb{B}}_n^{-1}$, $Q := \bar{\mathbb{B}}_n$. Then U_j are the moments of $\bar{\mathbb{B}}_n^{-1}$, V_i those of $\bar{\mathbb{B}}_n$, and $\mathbb{W}_m$ the moments of the identity operator $I = PQ$. Clearly $\mathbb{W}_0(x) = 1$ and $\mathbb{W}_m = 0$, $m \geq 1$. Consequently we have from (12) for $m \geq 1$

$$U_m \sum_{i=0}^{m} \binom{m}{i} \frac{1}{(m-i)!} V_{m-i}^{(m-i)} = -\sum_{j=0}^{m-1} U_j \sum_{i=0}^{j} \binom{m}{i} \frac{1}{(j-i)!} V_{m-i}^{(j-i)}. \tag{13}$$

With the same fixed n, the moments V_k of $\bar{\mathbb{B}}_n$ satisfy the recursion relation (see [7, Corollary 1])

$$\begin{aligned} V_0(x) = 1, \qquad V_1(x) = 0,\\ (k+n)V_{k+1}(x) = k[x(1-x)V_{k-1}(x) + (1-2x)V_k(x)], \qquad k \geq 1. \end{aligned} \tag{14}$$

Then $V_2(x) = \frac{x(1-x)}{n+1}$ and, generally, $V_k \in \mathcal{P}_k$. This shows that $V_k^{(k)}$ is a constant, and so $\sum_{i=0}^{m} \binom{m}{i} \frac{1}{(m-i)!} V_{m-i}^{(m-i)}$ is a constant.

To resume, we can state

Theorem 3. *The moments U_m of $\bar{\mathbb{B}}_n^{-1}$ satisfy the recursion relation* (13), *with $U_0(x) = 1$, $U_1(x) = 0$, where the coefficients V_k are given by* (14).

Let us return to (12) and set, for a fixed $n \geq 1$, $P := \bar{\mathbb{B}}_n^{-1}$, $Q := B_n$. Then U_j are the moments of $\bar{\mathbb{B}}_n^{-1}$, V_i those of B_n, and $\mathbb{W}_m$ the moments of $F_n = PQ$. The recursion formula for V_i is well-known (see, e. g., [10, Theorem 1.5.1]):

$$\begin{aligned} V_0(x) = 1, \qquad V_1(x) = 0, \\ nV_{i+1}(x) = x(1-x)\big[V_i'(x) + iV_{i-1}(x)\big], \qquad i \geq 1. \end{aligned} \tag{15}$$

Consequently, according to (12) we can state

Theorem 4. *The moments $\mathbb{W}_m$ of F_n are given by*

$$\mathbb{W}_m = \sum_{j=0}^{m} U_j \sum_{i=0}^{j} \binom{m}{i} \frac{1}{(j-i)!} V_{m-i}^{(j-i)},$$

where V_i satisfy (15) *and U_m satisfy* (13).

In particular, the moments $\mathbb{W}_m$, $m = 0, \ldots, 6$, are calculated in [6, Section 6].

4. The Powers of F_n

The eigenvalues of the restriction $F_n : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ are the numbers (see [6, Section 5])

$$\nu_0^{(n)} = \nu_1^{(n)} = 1, \quad \nu_k^{(n)} = \frac{(n-1+k)!}{(n-k)!} \frac{1}{n^{2k-1}} < 1, \quad 2 \leq k \leq n.$$

For $2 \leq k \leq n$ the values of $\nu_k^{(n)}$ can be expressed also as

$$\nu_k^{(n)} = \frac{(n^2-1)(n^2-4)\cdots(n^2-(k-1)^2)}{n^{2k-2}}$$

and this shows that they are less than 1 if $k \geq 2$.

Let $p_0^{(n)}(x) = 1$, $p_1^{(n)}(x) = x$, $p_2^{(n)}(x), \ldots, p_n^{(n)}(x)$ be the associated monic eigenpolynomials, $p_j^{(n)} \in \mathcal{P}_j$, $j \geq 0$.

Theorem 5. *For each $n \geq 1$ we have*

$$\lim_{k\to\infty} F_n^k f(x) = f(0)(1-x) + f(1)x, \qquad x \in [0,1], \ f \in C[0,1], \tag{16}$$

uniformly on $[0,1]$.

Proof. Let $f \in C[0,1]$. Then $F_n f \in \mathcal{P}_n$, hence

$$F_n f = \sum_{j=0}^{n} a_j(f) p_j^{(n)}, \tag{17}$$

for suitable real coefficients $a_j(f)$. For $k \geq 1$, (17) implies

$$F_n^k f = \sum_{j=0}^{n} a_j(f) \big(\nu_j^{(n)}\big)^{k-1} p_j^{(n)}.$$

It follows that

$$\lim_{k\to\infty} F_n^k f = a_0(f) p_0^{(n)} + a_1(f) p_1^{(n)},$$

and therefore

$$\lim_{k\to\infty} F_n^k f = a_0(f) + a_1(f)x, \qquad x \in [0,1]. \tag{18}$$

Since $F_n f(0) = f(0)$ and $F_n f(1) = f(1)$, we get from (18) $f(0) = a_0(f)$, $f(1) = a_0(f) + a_1(f)$, i. e., $a_0(f) = f(0)$, $a_1(f) = f(1) - f(0)$. Now (18) yields

$$\lim_{k\to\infty} F_n^k f(x) = f(0) + [f(1) - f(0)]x,$$

which implies (16). This concludes the proof. □

5. Open Problems

In this section we recall a conjecture and an open problem already stated earlier in [6, 9].

Conjecture 1. *For $f \in C^3[0,1]$ we have*

$$\lim_{n\to\infty} n^2 (F_n f - f)(x) = \frac{x(1-x)}{2} f''(x) - \frac{x(1-x)(1-2x)}{6} f'''(x),$$

uniformly on $[0,1]$.

In [6, Section 10] it was proved that the statement of this conjecture holds true for all polynomials.

In [9, Theorem 1] we proved that if $f(x) = \sum_{j=0}^{\infty} c_j x^j$, $x \in [0,1]$, with $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\sum_{j=0}^{\infty} \frac{M_j}{j!} \|f^{(j)}\|_\infty < \infty$, then

$$\lim_{n\to\infty} \|F_n(f) - f\|_\infty = 0.$$

Here $M_j := \sup_{n\geq 1} \|\bar{\mathbb{B}}_n^{-1} e_j\|_\infty$ with (see [9, (21)]) $M_j < \infty$, $j \in \mathbb{N}_0$.

The open problem can be stated as follows.

Open Problem. *Is there a constant $M \geq 1$ such that $M_j \leq M^j$, $j \geq 0$?*

If the question can be answered in the affirmative, then the above mentioned result can be applied to each function $f \in C^\infty[0,1]$ having the property that $\|f^{(j)}\|_\infty \leq m$, $j \in \mathbb{N}_0$, for some constant $m > 0$ depending only on f. Thus, a solution of this problem would enlarge the class of functions f for which $F_n f$ converges uniformly to f on the interval $[0,1]$.

References

[1] T. Acar, A. Aral and I. Raşa, Power series of Beta operators, *Appl. Math. Comput.* **247** (2014), 815–823.

[2] J. A. Adell, F. German Badia and J. de la Cal, Beta-type operators preserve shape properties, *Stochastic Process. Appl.* **48** (1993), 1–8.

[3] A. Attalienti and I. Raşa, Total positivity: an application to positive linear operators and to their limiting semigroups, *Rev. Anal. Numér. Théor. Approx.* **36** (2007), 51–66.

[4] Sh. Cooper and Sh. Waldron, The eigenstructure of the Bernstein operator, *J. Approx. Theory* **105** (2000), 133–165.

[5] H. Gonska and I. Raşa, On the composition and decomposition of positive linear operators (II), *Studia Sci. Math. Hungar.* **47** (2010), no. 4, 448–461.

[6] H. Gonska, M. Heilmann, A. Lupaş and I. Raşa, On the composition and decomposition of positive linear operators III: A non-trivial decomposition of the Bernstein operator, `http://arxiv.org/abs/1204.2723` August 30, 2012, 1–28.

[7] H. Gonska, I. Raşa and E. Stanila, Beta operators with Jacobi weights, *in* "Constructive Theory of Functions, Sozopol 2013" (K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 99-112, Prof. Marin Drinov Academic Publishing House, Sofia, 2014.

[8] H. Gonska, M. Heilmann and I. Raşa, Eigenstructure of the genuine Beta operators of Lupaş and Mühlbach, submitted.

[9] M. Heilmann and I. Raşa, On the decomposition of Bernstein operators, *Numer. Funct. Anal. Optim.* **36** (2015), no. 1, 72–85.

[10] G. G. Lorentz, "Bernstein Polynomials", 2nd edition, Chelsea Publishing Company, New York, N. Y., 1986.

[11] A. Lupaş, "Die Folge der Betaoperatoren", Dissertation, Universität Stuttgart, 1972.

[12] G. Mühlbach, Verallgemeinerungen der Bernstein- und der Lagrangepolynome. Bemerkungen zu einer Klasse linearer Polynomoperatoren von D. D. Stancu, *Rev. Roumaine Math. Pures Appl.* **15** (1970), 1235–1252.

[13] G. Mühlbach, Rekursionsformeln für die zentralen Momente der Pólya- und der Beta-Verteilung, *Metrika* **19** (1972), 171–177.

[14] I. Raşa and E. Stanila, On some operators linking the Bernstein and the genuine Bernstein-Durrmeyer operators, *J. Appl. Funct. Anal.* **9** (2014), no. 3–4, 369–378.

[15] E. Stanila, "On Bernstein-Euler-Jacobi Operators", PhD thesis, Universität Duisburg-Essen, 2014.

Margareta Heilmann

School of Mathematics and Natural Sciences
University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal
GERMANY
E-mail: heilmann@math.uni-wuppertal.de

Fadel Nasaireh

Department of Mathematics
Technical University
Str. Memorandumului 28
RO-440114 Cluj-Napoca
ROMANIA
E-mail: vfadelnasierh@gmail.com

Ioan Raşa

Department of Mathematics
Technical University
Str. Memorandumului 28
RO-440114 Cluj-Napoca
ROMANIA
E-mail: Ioan.Rasa@math.utcluj.ro