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Markov L_2-inequality with the Laguerre Weight

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Let $w_\alpha(x) := x^\alpha\, e^{-x}$, where $\alpha > -1$, be the Laguerre weight function, and let $\|\cdot\|_{w_\alpha}$ be the associated L_2-norm,

$$\|f\|_{w_\alpha} = \left(\int_0^\infty |f(x)|^2 w_\alpha(x)\, dx\right)^{1/2}.$$

By $\mathcal{P}_n$ we denote the set of algebraic polynomials of degree $\le n$. We study the best constant $c_n(\alpha)$ in the Markov inequality in this norm

$$\|p_n'\|_{w_\alpha} \le c_n(\alpha)\|p_n\|_{w_\alpha}, \qquad p_n \in \mathcal{P}_n,$$

namely the constant

$$c_n(\alpha) := \sup_{p_n\in\mathcal{P}_n} \frac{\|p_n'\|_{w_\alpha}}{\|p_n\|_{w_\alpha}}.$$

We derive explicit lower and upper bounds for the Markov constant $c_n(\alpha)$, as well as for the asymptotic Markov constant

$$c(\alpha) = \lim_{n\to\infty} \frac{c_n(\alpha)}{n}.$$

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1. Introduction and Statement of the Results

Let $w_\alpha(x) := x^\alpha\, x^{-x}$, where $\alpha > -1$, be the Laguerre weight function, and let $\|\cdot\|_{w_\alpha}$ be the associated L_2-norm,

$$\|f\|_{w_\alpha} = \left(\int_0^\infty |f(x)|^2 w_\alpha(x)\, dx\right)^{1/2}.$$

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By $\mathcal{P}_n$ we denote the set of algebraic polynomials of degree at most n. We study here the best constant $c_n(\alpha)$ in the Markov inequality in this norm

$$\|p_n'\|_{w_\alpha} \le c_n(\alpha)\|p_n\|_{w_\alpha}\,, \qquad p_n \in \mathcal{P}_n\,, \tag{1.1}$$

namely the constant

$$c_n(\alpha) := \sup_{p_n\in\mathcal{P}_n} \frac{\|p_n'\|_{w_\alpha}}{\|p_n\|_{w_\alpha}}\,.$$

Our goal is to obtain *good* and *explicit* lower and upper bounds for $c_n(\alpha)$, i.e., to find constants $\underline{c}(n,\alpha)$ and $\overline{c}(n,\alpha)$ such that

$$\underline{c}(n,\alpha) \le c_n(\alpha) \le \overline{c}(n,\alpha)\,,$$

with a small ratio $\frac{\overline{c}(n,\alpha)}{\underline{c}(n,\alpha)}$. Before formulating our results here, let us give a brief account on the results hitherto known.

It is only the case $\alpha = 0$ where the best Markov constant is known, namely, Turán [6] proved that

$$c_n(0) = \Big(2\sin\frac{\pi}{4n+2}\Big)^{-1}\,.$$

Dörfler [1] showed that $c_n(\alpha) = \mathcal{O}(n)$ for every fixed $\alpha > -1$ by proving the estimates

$$c_n^2(\alpha) \ge \frac{n^2}{(\alpha+1)(\alpha+3)} + \frac{(2\alpha^2+5\alpha+6)\,n}{3(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{\alpha+6}{3(\alpha+2)(\alpha+3)}\,, \tag{1.2}$$

$$c_n^2(\alpha) \le \frac{n(n+1)}{2(\alpha+1)}\,, \tag{1.3}$$

see [2] for a more accessible source. In the same paper, [2], Dörfler proved for the asymptotic constant $c(\alpha) = \lim_{\alpha\to\infty}\frac{c_n(\alpha)}{n}$ that

$$c(\alpha) := \lim_{n\to\infty}\frac{c_n(\alpha)}{n} = \frac{1}{j_{(\alpha-1)/2,1}}\,, \tag{1.4}$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(z)$.

In a recent paper [3] we proved the following statement.

Theorem A ([3, Theorem 1]). *For all $\alpha > -1$ and $n \in \mathbb{N}$, $n \ge 3$, the best constant $c_n(\alpha)$ in the Markov inequality*

$$\|p'\|_{w_\alpha} \le c_n(\alpha)\,\|p\|_{w_\alpha}\,, \qquad p \in \mathcal{P}_n\,,$$

admits the estimates

$$\frac{2\big(n+\frac{2\alpha}{3}\big)\big(n-\frac{\alpha+1}{6}\big)}{(\alpha+1)(\alpha+5)} < c_n^2(\alpha) < \frac{(n+1)\big(n+\frac{2(\alpha+1)}{5}\big)}{(\alpha+1)\big((\alpha+3)(\alpha+5)\big)^{1/3}}\,, \tag{1.5}$$

where for the left-hand side inequality it is additionally assumed that $n > \frac{\alpha+1}{6}$.

Clearly, Theorem A implies some inequalities for the asymptotic Markov constant $c(\alpha)$ and, through (1.4), inequalities for $j_{\nu,1}$, the first positive zero of the Bessel function J_ν (see [3, Corollaries 1, 3]).

We also proved in [3, Theorem 2] that $c(\alpha) = \mathcal{O}(\alpha^{-1})$, which shows that the upper estimate for $c_n(\alpha)$ in (1.5), though rather good for moderate α, is not optimal.

Our approach here is based on the norm estimates of a related positive definite matrix $\mathbf{A}_n$. In [4] the same approach has been applied for derivation of bounds for the best Markov constant in the L_2 Markov inequality with the Gegenbauer weight.

Our main result is an upper bound for $c_n(\alpha)$ which is of the right order with respect to both n and α as they grow to infinity.

Theorem 1.1. *For all $n \in \mathbb{N}$, $n \geq 3$, the best constant $c_n(\alpha)$ in the Markov inequality* (1.1) *satisfies the inequality*

$$c_n^2(\alpha) \leq \frac{4n\big(n+2+\frac{3(\alpha+1)}{4}\big)}{\alpha^2+10\alpha+8}\,, \qquad \alpha \geq 2\,.$$

As a consequence of Theorem 1.1 and Dörfler's lower bound (1.2) for $c_n(\alpha)$ we show that

$$c_n^2(\alpha) \asymp \frac{n(n+\alpha+3)}{(\alpha+1)(\alpha+8)}\,, \qquad n \geq 3,\ \alpha \geq 2\,.$$

Corollary 1.1. *For all $\alpha \geq 2$ and $n \geq 3$ the best constant $c_n(\alpha)$ in the Markov inequality* (1.1) *satisfies*

$$\frac{2n(n+\alpha+3)}{3(\alpha+1)(\alpha+8)} \leq c_n^2(\alpha) \leq \frac{4n(n+\alpha+3)}{(\alpha+1)(\alpha+8)}\,.$$

As another consequence, we find the limit value of $(\alpha+1)c_n^2(\alpha)$ as α tends to -1, and obtain asymptotic estimates for $\alpha\, c_n^2(\alpha)$ as α tends to infinity.

Corollary 1.2. *The best constant $c_n(\alpha)$ in the Markov inequality* (1.1) *satisfies:*

(i) $\displaystyle \lim_{\alpha\to-1}(\alpha+1)c_n^2(\alpha) = \frac{n(n+1)}{2}$;

(ii) $\displaystyle \frac{2n}{3} \leq \lim_{\alpha\to\infty} \alpha\, c_n^2(\alpha) \leq 3\,n\,.$

Finally, Theorem 1.1 provides an upper bound for the asymptotic Markov constant $c(\alpha)$ which is of the correct order $O(\alpha^{-1})$ as α tends to infinity. As a consequence of Theorem A and Theorem 1.1 we have the following

Corollary 1.3. *The asymptotic Markov constant* $c(\alpha) = \lim_{n\to\infty} n^{-1}c_n(\alpha)$ *satisfies the inequalities*

$$\frac{2}{(\alpha+1)(\alpha+5)} < c^2(\alpha) < \begin{cases} \dfrac{1}{(\alpha+1)\sqrt[3]{(\alpha+3)(\alpha+5)}}\,, & -1<\alpha\le\alpha^*\,, \\ \dfrac{4}{\alpha^2+10\alpha+8}\,, & \alpha>\alpha^*\,, \end{cases}$$

where $\alpha^* \approx 43.4$ *.*

It is worth noticing here that, for all $\alpha > -1$, the ratio of the upper and the lower bound for $c(\alpha)$ in Corollary 1.3 is less than $\sqrt{2}$.

The rest of the paper is organized as follows. Sect. 2 contains a brief characterization of the squared best Markov constant $c_n^2(\alpha)$ as the largest eigenvalue of a specific matrix $\mathbf{A}_n$. In Sect. 3 we prove some estimates for ratios of Gamma functions needed for the proof of Theorem 1.1. The proof of Theorem 1.1 is given in Sect. 4. Sect. 5 is concerned with the evaluation of $\|\mathbf{A}_n\|_F$, the Frobenius norm of $\mathbf{A}_n$, and the bounds for $c_n(\alpha)$ implied thereby; in particular, we reproduce Dörfler's lower bound (1.2). The proof of Corollaries 1.1–1.3 is given in Sect. 6.

2. Preliminaries

It is well-known that the squared best Markov constant $c_n^2(\alpha)$ equals to the largest eigenvalue of a certain positive definite matrix $\mathbf{A}_n$. Here we derive the explicit form of $\mathbf{A}_n$.

The orthogonal polynomials with respect to the Laguerre weight function $w_\alpha(x) = x^\alpha e^{-x}$, $x \in \mathbb{R}_+$, are Laguerre polynomials $\{L_m^{(\alpha)}\}_{m\in\mathbb{N}_0}$, with the standard normalization

$$\|L_m^\alpha\|_{w_\alpha} = \Big(\frac{\Gamma(m+\alpha+1)}{\Gamma(m+1)}\Big)^{1/2} =: \beta_{m+1}\,, \qquad m\in\mathbb{N}_0 \tag{2.1}$$

(for simplicity sake, we suppress the dependance of the β's on α). Further specific properties of the Laguerre polynomials are (see, e.g., [5, eqs. (5.1.13), (5.1.14)])

$$\frac{d}{dx}L_m^{(\alpha)}(x) = -L_{m-1}^{(\alpha+1)}(x)\,, \qquad m\in\mathbb{N}\,, \tag{2.2}$$

$$L_m^{(\alpha+1)}(x) = \sum_{\nu=0}^{m} L_\nu^{(\alpha)}(x)\,. \tag{2.3}$$

Assume that $\hat{p}_n \in \mathcal{P}_n$, $\|\hat{p}_n\|_{w_\alpha} = 1$, is an extreme polynomial in the L_2 Markov inequality (1.1), i.e.,

$$\sup\{\|p'\|_{w_\alpha}^2 \,:\, p\in\mathcal{P}_n\,,\ \|p\|_{w_\alpha}=1\} = c_n^2(\alpha) = \|\hat{p}_n'\|_{w_\alpha}^2\,. \tag{2.4}$$

Without loss of generality, $\hat{p}_n$ can be represented in the form

$$\hat{p}_n = \sum_{\nu=1}^{n} a_\nu \, L_\nu^{(\alpha)} \,, \qquad a_\nu \in \mathbb{R}\,, \quad 1 \le \nu \le n\,,$$

then

$$\|\hat{p}_n\|_{w_\alpha}^2 = \sum_{\nu=1}^{n} a_\nu^2 \beta_{\nu+1}^2 =: \sum_{\nu=1}^{n} t_\nu^2 =: \|\mathbf{t}\|^2 = 1\,,$$

where $\mathbf{t} = (t_1, \ldots, t_n)^\top \in \mathbb{R}^n$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$, i.e., $\|\mathbf{t}\|^2 = \mathbf{t}^\top \mathbf{t}$.

By (2.1), (2.2) and (2.3), we get

$$\begin{aligned}\|\hat{p}_n'\|_{w_\alpha}^2 &= \Big\| \sum_{\nu=1}^{n} a_\nu \Big(\sum_{\mu=0}^{\nu-1} L_\mu^{(\alpha)} \Big) \Big\|_{w_\alpha}^2 = \Big\| \sum_{\mu=1}^{n} \Big(\sum_{\nu=\mu}^{n} a_\nu \Big) L_{\mu-1} \Big\|_{w_\alpha}^2 \\ &= \sum_{\mu=1}^{n} \Big(\sum_{\nu=\mu}^{n} \frac{\beta_\mu}{\beta_{\nu+1}} \, t_\nu \Big)^2 = \|\mathbf{C}_n \mathbf{t}\|^2,\end{aligned}$$

where $\mathbf{C}_n$ is the upper triangular $n \times n$ matrix

$$\mathbf{C}_n = \begin{pmatrix} \frac{\beta_1}{\beta_2} & \frac{\beta_1}{\beta_3} & \cdots & \frac{\beta_1}{\beta_{n+1}} \\ 0 & \frac{\beta_2}{\beta_3} & \cdots & \frac{\beta_2}{\beta_{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\beta_n}{\beta_{n+1}} \end{pmatrix}.$$

Hence, (2.4) admits the equivalent formulation

$$c_n^2(\alpha) = \sup_{\substack{\mathbf{t} \in \mathbb{R}^n \\ \|\mathbf{t}\|=1}} \|\mathbf{C}_n \mathbf{t}\|^2 = \sup_{\substack{\mathbf{t} \in \mathbb{R}^n \\ \|\mathbf{t}\|=1}} \mathbf{t}^\top \mathbf{C}_n^\top \mathbf{C}_n \mathbf{t} = \mu_{\max}(\mathbf{A}_n)\,,$$

where $\mu_{\max}(\mathbf{A}_n)$ is the largest eigenvalue of the positive definite matrix

$$\mathbf{A}_n := \mathbf{C}_n^\top \mathbf{C}_n\,.$$

After a straightforward calculation we find

$$\mathbf{A}_n = \begin{pmatrix} \frac{\beta_1^2}{\beta_2^2} & \frac{\beta_1^2}{\beta_2\beta_3} & \frac{\beta_1^2}{\beta_2\beta_4} & \cdots & \frac{\beta_1^2}{\beta_2\beta_{n+1}} \\ \frac{\beta_1^2}{\beta_2\beta_3} & \frac{1}{\beta_3^2}(\sum_{j=1}^2 \beta_j^2) & \frac{1}{\beta_3\beta_4}(\sum_{j=1}^2 \beta_j^2) & \cdots & \frac{1}{\beta_3\beta_{n+1}}(\sum_{j=1}^2 \beta_j^2) \\ \frac{\beta_1^2}{\beta_2\beta_4} & \frac{1}{\beta_3\beta_4}(\sum_{j=1}^2 \beta_j^2) & \frac{1}{\beta_4^2}(\sum_{j=1}^3 \beta_j^2) & \cdots & \frac{1}{\beta_4\beta_{n+1}}(\sum_{j=1}^3 \beta_j^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_1^2}{\beta_2\beta_{n+1}} & \frac{1}{\beta_3\beta_{n+1}}(\sum_{j=1}^2 \beta_j^2) & \frac{1}{\beta_4\beta_{n+1}}(\sum_{j=1}^3 \beta_j^2) & \cdots & \frac{1}{\beta_{n+1}^2}(\sum_{j=1}^n \beta_j^2) \end{pmatrix}.$$

We observe that the elements $a_{k,i}$ of the matrix $\mathbf{A}_n$ are given by

$$a_{k,i} = \frac{1}{\beta_{i+1}\beta_{k+1}} \sum_{j=1}^{\min\{k,i\}} \beta_j^2 = \begin{cases} \frac{\beta_{i+1}}{\beta_{k+1}}\Big(\frac{1}{\beta_{i+1}^2}\sum_{j=1}^{i}\beta_j^2\Big), & i \le k\,, \\ \frac{\beta_{k+1}}{\beta_{i+1}}\Big(\frac{1}{\beta_{k+1}^2}\sum_{j=1}^{k}\beta_j^2\Big), & i \ge k\,, \end{cases}$$

so that

$$a_{k,k} = \frac{1}{\beta_{k+1}^2}\sum_{j=1}^{k}\beta_j^2\,, \qquad a_{k,i} = \begin{cases} \frac{\beta_{i+1}}{\beta_{k+1}}\, a_{i,i}\,, & i \le k\,, \\ \frac{\beta_{k+1}}{\beta_{i+1}}\, a_{k,k}\,, & i \ge k\,. \end{cases} \tag{2.5}$$

Hence, $\mathbf{A}_n$ can be written in the following simplified form

$$\mathbf{A}_n = \begin{pmatrix} a_{11} & \frac{\beta_2}{\beta_3}\, a_{11} & \frac{\beta_2}{\beta_4}\, a_{11} & \cdots & \frac{\beta_2}{\beta_{n+1}}\, a_{11} \\ \frac{\beta_2}{\beta_3}\, a_{11} & a_{22} & \frac{\beta_3}{\beta_4}\, a_{22} & \cdots & \frac{\beta_3}{\beta_{n+1}}\, a_{22} \\ \frac{\beta_2}{\beta_4}\, a_{11} & \frac{\beta_3}{\beta_4}\, a_{22} & a_{33} & \cdots & \frac{\beta_4}{\beta_{n+1}}\, a_{33} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_2}{\beta_{n+1}}\, a_{11} & \frac{\beta_3}{\beta_{n+1}}\, a_{22} & \frac{\beta_4}{\beta_{n+1}}\, a_{33} & \cdots & a_{nn} \end{pmatrix}. \tag{2.6}$$

We complete this section with giving explicit formulae for $a_{k,k}$ and the trace of $\mathbf{A}_n$.

Proposition 2.1. *For every* $k \in \mathbb{N}$ *and* $\alpha > -1$,

$$a_{k,k} = \frac{k}{\alpha+1}$$

and consequently

$$\mathrm{tr}\,(\mathbf{A}_n) = \frac{n(n+1)}{2(\alpha+1)}\,.$$

Proof. In view of (2.5), we need to show that

$$\frac{1}{\beta_{k+1}^2}\sum_{j=1}^{k}\beta_j^2 = \frac{k}{\alpha+1}\,. \tag{2.7}$$

The proof is by induction with respect to k. Since

$$\frac{\beta_k^2}{\beta_{k+1}^2} = \frac{\frac{\Gamma(k+\alpha)}{\Gamma(k)}}{\frac{\Gamma(k+1+\alpha)}{\Gamma(k+1)}} = \frac{k}{k+\alpha}\,,$$

(2.7) is true for $k = 1$. Assuming that (2.7) is true for $k-1 \in \mathbb{N}$, we obtain

$$\frac{1}{\beta_{k+1}^2}\sum_{j=1}^{k}\beta_j^2 = \frac{\beta_k^2}{\beta_{k+1}^2} + \frac{\beta_k^2}{\beta_{k+1}^2}\Big(\frac{1}{\beta_k^2}\sum_{j=1}^{k-1}\beta_j^2\Big) = \frac{k}{k+\alpha}\Big(1+\frac{k-1}{\alpha+1}\Big) = \frac{k}{\alpha+1}\,.$$

Hence, the induction step is performed, and the proof of (2.7) is complete. $\square$

Remark 2.1. Dörfler's upper estimate (1.3) is simply the inequality

$$c_n^2(\alpha) = \mu_{\max}(\mathbf{A}_n) \le \operatorname{tr}(\mathbf{A}_n) = \frac{n(n+1)}{2(\alpha+1)}\,.$$

3. Estimates for $\frac{\beta_i}{\beta_k}$

We shall need estimates for the elements $a_{k,i}$, $k \ne i$, of the matrix $\mathbf{A}_n$ in (2.6), and this requires estimates for the ratios of the β's. We prove the following lemma.

Lemma 3.1. *For every* $\alpha \ge 1$ *and* $i, k \in \mathbb{N}$, $i < k$*, there holds*

$$\frac{\frac{\Gamma(i+\alpha)}{\Gamma(i)}}{\frac{\Gamma(k+\alpha)}{\Gamma(k)}} \le \left(\frac{i+\frac{\alpha-1}{2}}{k+\frac{\alpha-1}{2}}\right)^{\alpha}.$$

Proof. It suffices to prove only the case $k = i+1$, for then the general case will follow from

$$\frac{\frac{\Gamma(i+\alpha)}{\Gamma(i)}}{\frac{\Gamma(k+\alpha)}{\Gamma(k)}} = \prod_{\nu=i}^{k-1} \frac{\frac{\Gamma(\nu+\alpha)}{\Gamma(\nu)}}{\frac{\Gamma(\nu+1+\alpha)}{\Gamma(\nu+1)}}\,, \qquad \frac{i+\frac{\alpha-1}{2}}{k+\frac{\alpha-1}{2}} = \prod_{\nu=i}^{k-1} \frac{\nu+\frac{\alpha-1}{2}}{\nu+1+\frac{\alpha-1}{2}}\,.$$

Thus, we need to show that

$$\frac{i}{i+\alpha} \le \left(\frac{i+\frac{\alpha-1}{2}}{i+1+\frac{\alpha-1}{2}}\right)^{\alpha}, \qquad i \ge 1,\ \alpha \ge 1,$$

or, equivalently,

$$\left(1 + \frac{1}{i+\frac{\alpha-1}{2}}\right)^{\alpha} \le 1 + \frac{\alpha}{i}\,. \tag{3.1}$$

Clearly, (3.1) turns into identity when $\alpha = 1$, so we assume further that $\alpha > 1$. Set

$$z = \frac{1}{i+\frac{\alpha-1}{2}}\,, \qquad 0 < z \le \frac{2}{\alpha+1} < 1\,,$$

then

$$i = \frac{2-(\alpha-1)z}{2z}\,,$$

and inequality (3.1) becomes

$$(1+z)^{\alpha} \le 1 + \frac{2\alpha z}{2-(\alpha-1)z}\,, \qquad 0 < z \le \frac{2}{\alpha+1} < 1\,, \quad \alpha > 1\,.$$

Assume that $m-1<\alpha\le m$, where $m\in\mathbb{N}$, $m\ge 2$. By Maclaurin's formula, we have

$$(1+z)^{\alpha}\le 1+\sum_{\nu=1}^{m}\frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}\,z^{\nu}$$

and it suffices to show that

$$\sum_{\nu=1}^{m}\frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}\,z^{\nu}\le\frac{2\alpha z}{2-(\alpha-1)z}\,.$$

Multiplying both sides of this inequality by $2-(\alpha-1)z>0$ and arranging the powers of z, we arrive at the equivalent inequality

$$\sum_{\nu=2}^{m+1}\frac{(2-\nu)(\alpha+1)\alpha(\alpha-1)\dots(\alpha-\nu+2)}{\nu!}\,z^{\nu}=:\sum_{\nu=2}^{m+1}a_{\nu}\,z^{\nu}\le 0\,,$$

which is obviously true since $z>0$ and $a_{\nu}\le 0,\ 2\le\nu\le m+1$. □

Lemma 3.1 is a particular case of the following more general statement, which is of independent interest.

Proposition 3.1. *Let* $i,\,k\in\mathbb{N},\ i<k$.

(i) *If* $-1<\alpha\le 0$ *or* $\alpha\ge 1$, *then*

$$\Big(\frac{i}{k}\Big)^{\alpha}\le\frac{\frac{\Gamma(i+\alpha)}{\Gamma(i)}}{\frac{\Gamma(k+\alpha)}{\Gamma(k)}}\le\left(\frac{i+\frac{\alpha-1}{2}}{k+\frac{\alpha-1}{2}}\right)^{\alpha}.$$

(ii) *If* $0\le\alpha\le 1$, *then*

$$\Big(\frac{i}{k}\Big)^{\alpha}\ge\frac{\frac{\Gamma(i+\alpha)}{\Gamma(i)}}{\frac{\Gamma(k+\alpha)}{\Gamma(k)}}\ge\left(\frac{i+\frac{\alpha-1}{2}}{k+\frac{\alpha-1}{2}}\right)^{\alpha}.$$

The proof of Proposition 3.1 is omitted as we only need its part given in Lemma 3.1.

4. Proof of Theorem 1.1

As was mentioned in Sect. 2, $c_n^2(\alpha)=\mu_{\max}(\mathbf{A}_n)$, where $\mu_{\max}(\mathbf{A}_n)$ is the largest eigenvalue of the matrix $\mathbf{A}_n$ given by (2.6). It is well-known that

$$\mu_{\max}(\mathbf{A}_n)\le\|\mathbf{A}_n\|_{*}\,,$$

where $\|\cdot\|_*$ is any matrix norm. Here, we shall exploit $\|\cdot\|_\infty$,

$$\|\mathbf{A}_n\|_\infty = \max_{1\le k\le n}\sum_{i=1}^{n}|a_{k,i}| = \max_{1\le k\le n}\sum_{i=1}^{n}a_{k,i}$$

(notice that $a_{k,i}>0,\ 1\le i,k\le n$). Theorem 1.1 is an immediate consequence of the following statement.

Proposition 4.1. *The following inequality holds true:*

$$\|\mathbf{A}_n\|_\infty \le \frac{4n\big(n+2+\frac{3(\alpha+1)}{4}\big)}{\alpha^2+10\alpha+8}\,, \qquad \alpha\ge 2\,.$$

We shall need the following lemma, which is proved in [4].

Lemma 4.1. *Let $\alpha_i>0$, $\gamma_{\min}\le\gamma_i\le\gamma_{\max}$, $1\le i\le r$, and let*

$$f(x) := (x+\gamma_1)^{\alpha_1}(x+\gamma_2)^{\alpha_2}\cdots(x+\gamma_r)^{\alpha_r}, \qquad s := \sum_{i=1}^{r}\alpha_i\,.$$

Then, for any $x>x_0$, where $x_0+\gamma_{\min}\ge 0$, we have

$$\frac{1}{s+1}\big[(t+\gamma_{\min})f(t)\big]_{x_0}^{x} < \int_{x_0}^{x} f(t)\,dt < \frac{1}{s+1}\,(x+\gamma_{\max})f(x)\,.$$

Proof of Proposition 4.1. Let us assume first that $\alpha>2$. For a fixed k, $1\le i,k\le n$, we consider the sum of the elements in the k-th row of $\mathbf{A}_n$,

$$\sum_{i=1}^{n}a_{k,i} = \sum_{i=1}^{k-1}\frac{\beta_{i+1}}{\beta_{k+1}}\,a_{i,i} + a_{k,k} + \sum_{i=k+1}^{n}\frac{\beta_{k+1}}{\beta_{i+1}}\,a_{k,k}\,.$$

By Lemma 2.1 and Lemma 3.1, we have

$$a_{\nu,\nu} = \frac{\nu}{1+\alpha}\,, \qquad \frac{\beta_{\mu+1}}{\beta_{\nu+1}} \le \left(\frac{\mu+\frac{\alpha+1}{2}}{\nu+\frac{\alpha+1}{2}}\right)^{\alpha/2}, \qquad \mu<\nu\,,\ \ \alpha\ge 1\,,$$

hence

$$\begin{aligned}\sum_{i=1}^{n}a_{k,i} \le \frac{1}{1+\alpha}\Big[\Big(k+\frac{\alpha+1}{2}\Big)^{-\alpha/2}\sum_{i=1}^{k-1}i\Big(i+\frac{\alpha+1}{2}\Big)^{\alpha/2}+k \\ +k\Big(k+\frac{\alpha+1}{2}\Big)^{\alpha/2}\sum_{i=k+1}^{n}\Big(i+\frac{\alpha+1}{2}\Big)^{-\alpha/2}\Big] \\ =: \frac{1}{1+\alpha}\Big[\Big(k+\frac{\alpha+1}{2}\Big)^{-\alpha/2}S_1+k+k\Big(k+\frac{\alpha+1}{2}\Big)^{\alpha/2}S_2\Big]\,.\end{aligned}$$

To obtain an upper bound for S_1, we observe that $f_1(x) = x\big(x+\frac{\alpha+1}{2}\big)^{\alpha/2}$ is an increasing function in $(0,\infty)$, hence we can estimate the sum by an integral and then estimate the integral with the help of Lemma 4.1. This yields

$$S_1 \le \int_0^k f_1(x)\,dx < \frac{1}{\frac{\alpha}{2}+2}\,k\Big(k+\frac{\alpha+1}{2}\Big)^{\alpha/2+1} = \frac{2}{\alpha+4}\,k\Big(k+\frac{\alpha+1}{2}\Big)^{\alpha/2+1}.$$

To estimate S_2 from above, we observe that $f_2(x) = \big(x+\frac{\alpha+1}{2}\big)^{-\alpha/2}$ is a decreasing function in $(0,\infty)$, hence

$$S_2 \le \int_k^n f_2(x)\,dx = \frac{2}{\alpha-2}\Big(k+\frac{\alpha+1}{2}\Big)^{1-\alpha/2}\left[1-\Big(\frac{k+\frac{\alpha+1}{2}}{n+\frac{\alpha+1}{2}}\Big)^{\alpha/2-1}\right].$$

Substituting the above upper bounds for S_1 and S_2, we obtain

$$\begin{aligned}\sum_{i=1}^n a_{k,i} &\le \frac{2k\big(k+\frac{\alpha+1}{2}\big)}{(\alpha+1)(\alpha-2)}\left[\frac{2(\alpha+1)}{\alpha+4}-\Big(\frac{k+\frac{\alpha+1}{2}}{n+\frac{\alpha+1}{2}}\Big)^{\alpha/2-1}\right]+\frac{k}{\alpha+1}\\ &=: \frac{k}{\alpha+1}+\frac{2\Big(n+\frac{\alpha+1}{2}\Big)^2}{(\alpha+1)(\alpha-2)}\,\psi_\alpha(k)\varphi_\alpha(y)\,, \qquad (4.1)\end{aligned}$$

where

$$\varphi_\alpha(y) := \frac{2(\alpha+1)}{\alpha+4}\,y^2 - y^{\alpha/2+1}\,, \qquad y := \frac{k+\frac{\alpha+1}{2}}{n+\frac{\alpha+1}{2}} \in (0,1]\,,$$

$$\psi_\alpha(k) := \frac{k}{k+\frac{\alpha+1}{2}}\,,$$

For a fixed $\alpha > 2$, the function φ_α has a unique local extremum in $[0,1]$, a maximum, which is attained at

$$y_\alpha = \Big(\frac{8(\alpha+1)}{(\alpha+2)(\alpha+4)}\Big)^{2/(\alpha-2)} = \Big(1-\frac{\alpha(\alpha-2)}{(\alpha+2)(\alpha+4)}\Big)^{2/(\alpha-2)} \in (0,1) \qquad (4.2)$$

and

$$\max_{y\in[0,1]}\varphi_\alpha(y) = \varphi_\alpha(y_\alpha) = \frac{2(\alpha+1)(\alpha-2)}{(\alpha+2)(\alpha+4)}\,y_\alpha^2 > 0\,. \qquad (4.3)$$

We proceed with a further estimation of y_α^2. From (4.2) and $\ln(1+x) \le x$, $x > -1$, we have

$$\ln y_\alpha^2 = \frac{4}{\alpha-2}\ln\Big(1-\frac{\alpha(\alpha-2)}{(\alpha+2)(\alpha+4)}\Big) < -\frac{4\alpha}{(\alpha+2)(\alpha+4)}\,,$$

hence

$$y_\alpha^2 \le e^{-\frac{4\alpha}{(\alpha+2)(\alpha+4)}} \le \frac{1}{1+\frac{4\alpha}{(\alpha+2)(\alpha+4)}} = \frac{(\alpha+2)(\alpha+4)}{\alpha^2+10\alpha+8}\,,$$

where for the last inequality we have used that $e^{-x} \le \frac{1}{1+x}$, $x \ge 0$. Replacing this bound in (4.3), we obtain

$$\max_{y\in[0,1]} \varphi_\alpha(y) \le \frac{2(\alpha+1)(\alpha-2)}{\alpha^2+10\alpha+8}\,.$$

This estimate and

$$\max_{1\le k\le n} \psi_\alpha(k) = \psi_\alpha(n) = \frac{n}{n+\frac{\alpha+1}{2}}$$

yield

$$\frac{2\big(n+\frac{\alpha+1}{2}\big)^2}{(\alpha+1)(\alpha-2)} \max_{y\in[0,1]} \varphi_\alpha(y) \max_{1\le k\le n} \psi_\alpha(k) \le \frac{4n\big(n+\frac{\alpha+1}{2}\big)}{\alpha^2+10\alpha+8}\,.$$

Now we obtain from (4.1)

$$\begin{aligned}
\sum_{i=1}^{n} a_{k,i} &\le \frac{n}{\alpha+1} + \frac{2\big(n+\frac{\alpha+1}{2}\big)^2}{(\alpha+1)(\alpha-2)} \max_{y\in[0,1]} \varphi_\alpha(y) \max_{1\le k\le n} \psi_\alpha(k) \\
&\le \frac{4}{\alpha^2+10\alpha+8}\, n\Big(n+\frac{\alpha+1}{2}+\frac{\alpha^2+10\alpha+8}{4(\alpha+1)}\Big) \\
&< \frac{4}{\alpha^2+10\alpha+8}\, n\Big(n+\frac{\alpha+1}{2}+\frac{\alpha^2+10\alpha+9}{4(\alpha+1)}\Big) \\
&= \frac{4}{\alpha^2+10\alpha+8}\, n\Big(n+2+\frac{3(\alpha+1)}{4}\Big)\,.
\end{aligned}$$

The latter bound is also an upper bound for $\|\mathbf{A}_n\|_\infty$, therefore Proposition 4.1 is proved in the case $\alpha > 2$.

The proof of the case $\alpha = 2$ is similar (and somewhat simpler), and therefore is omitted. □

Remark 4.1. Actually, the above proof works also in the case $1 \le \alpha < 2$ (with a minor modification, e.g., φ_α has a minimum instead of maximum in $(0,1)$ but appears in (4.1) with a negative factor, etc.), yielding a similar upper bound for $\|\mathbf{A}_n\|_\infty$, and hence for $c_n^2(\alpha)$. However, for small α the upper bound for $c_n^2(\alpha)$ implied by the estimation of $\|\mathbf{A}_n\|_\infty$ is worse than the upper bound given in Theorem A, and also than the upper bound obtained through the Frobenius norm of $\mathbf{A}_n$.

5. The Frobenius Norm of $\mathbf{A}_n$

Let us recall that the Frobenius norm $\|\cdot\|_F$ of a matrix $\mathbf{B} = (b_{i,j})_{n\times n}$ with real elements is defined by

$$\|\mathbf{B}\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}^2 = \operatorname{tr}(\mathbf{B}^\top \mathbf{B})\,.$$

Since $\mathbf{A}_n$ is a symmetric and positive definite matrix, we have

$$\|\mathbf{A}_n\|_F^2 = \operatorname{tr}(\mathbf{A}_n^2) = \mu_1^2 + \mu_2^2 + \cdots + \mu_n^2\,, \tag{5.1}$$

where $0 < \mu_1 < \mu_2 < \cdots < \mu_n = \mu_{\max}(\mathbf{A}_n)$ are the eigenvalues of $\mathbf{A}_n$, i.e., the zeros of the characteristic polynomial $P_n(\mu) = \det(\mu\mathbf{E}_n - \mathbf{A}_n)$,

$$P_n(\mu) = \mu^n - b_1\,\mu^{n-1} + b_2\,\mu^{n-2} - b_3\,\mu^{n-3} + \cdots + (-1)^n b_n\,.$$

In [3] we evaluated coefficients b_i, $1 \le i \le 3$, as a part of the proof of Theorem A. These coefficients are given below:

$$\begin{aligned} b_1 &= \operatorname{tr}(\mathbf{A}_n) = \frac{n(n+1)}{2(\alpha+1)}\,,\\ b_2 &= \frac{(n-1)n(n+1)}{24(\alpha+1)(\alpha+2)(\alpha+3)}\big[3(\alpha+2)n + 2(\alpha+6)\big]\,,\\ b_3 &= (n-2)(n-1)n(n+1)\\ &\quad\times\frac{\big[5(\alpha+2)(\alpha+4)n(n+1) + 8(7\alpha+20)n + 12(\alpha+20)\big]}{240(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\,. \end{aligned}$$

Estimates for $c_n^2(\alpha) = \mu_{\max}(\mathbf{A}_n)$ are also possible in terms of solely the first two coefficients, b_1 and b_2. Indeed, since $\operatorname{tr}(\mathbf{A}_n) = b_1$ and, by (5.1), $\|\mathbf{A}_n\|_F^2 = b_1^2 - 2b_2$, we have

$$b_1 - 2\,\frac{b_2}{b_1} = \frac{\|\mathbf{A}_n\|_F^2}{\operatorname{tr}(\mathbf{A}_n)} \le \mu_{\max}(\mathbf{A}_n) \le \|\mathbf{A}_n\|_F = \big(b_1^2 - 2b_2\big)^{1/2}\,.$$

Replacing b_1 and b_2 in the first and the last expression, we obtain the estimates

$$c_n^4(\alpha) \le \frac{n(n+1)}{2(\alpha+1)^2(\alpha+3)}\Big(n^2 + \frac{2\alpha^2+5\alpha+6}{3(\alpha+2)}\,n + \frac{(\alpha+1)(\alpha+6)}{3(\alpha+2)}\Big)\,, \tag{5.2}$$

$$c_n^2(\alpha) \ge \frac{n^2}{(\alpha+1)(\alpha+3)} + \frac{(2\alpha^2+5\alpha+6)\,n}{3(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{\alpha+6}{3(\alpha+2)(\alpha+3)}\,, \tag{5.3}$$

the second being nothing but the lower estimate (1.2) of Dörfler.

Slightly weaker but simpler estimates can be obtained on the basis of (5.2) and (5.3).

Proposition 5.1. *For all $n \ge 3$, the best Markov constant $c_n(\alpha)$ satisfies the inequalities*

$$c_n^2(\alpha) \le \frac{(n+1)\sqrt{n\big(n+\frac{2(\alpha+1)}{3}\big)}}{(\alpha+1)\sqrt{2(\alpha+3)}}\,, \qquad \alpha > -1\,, \tag{5.4}$$

$$c_n^2(\alpha) \geq \begin{cases} \dfrac{n\big(n+\frac{7}{8}\big)}{(\alpha+1)(\alpha+3)}\,, & \alpha \in (-1,0)\,, \\ \dfrac{n(n+1)}{(\alpha+1)(\alpha+3)}\,, & \alpha \in [0,1]\,, \\ \dfrac{n\big(n+\frac{2\alpha+1}{3}\big)}{(\alpha+1)(\alpha+3)}\,, & \alpha \geq 1\,. \end{cases} \tag{5.5}$$

Proof. Inequality (5.4) follows from (5.2) and the inequality

$$n^2 + \frac{2\alpha^2+5\alpha+6}{3(\alpha+2)}\,n + \frac{(\alpha+1)(\alpha+6)}{3(\alpha+2)} \leq (n+1)\Big(n+\frac{2\alpha+1}{3}\Big)\,.$$

The latter simplifies to the inequality

$$\frac{(\alpha+1)(4n+\alpha-2)}{3(\alpha+2)} \geq 0\,,$$

which is obviously true.

From (5.3) we have

$$c_n^2(\alpha) \geq \frac{n\big(n+\frac{2\alpha^2+5\alpha+6}{3(\alpha+2)}\big)}{(\alpha+1)(\alpha+3)} = \frac{n\big(n+\frac{2\alpha+1}{3}+\frac{4}{3(\alpha+2)}\big)}{(\alpha+1)(\alpha+3)}\,,$$

whence the case $\alpha \geq 1$ in (5.5) readily follows. For the proof of the cases $\alpha \in (-1,0)$ and $\alpha \in [0,1]$, we observe that $g(\alpha) = \frac{2\alpha^2+5\alpha+6}{3(\alpha+2)}$ has a unique local extremum in $(-1,1]$, a minimum, which is attained at $\alpha_* = \sqrt{2}-2 \in (-1,0)$. Hence, $g(\alpha) \geq g(\alpha_*) = \frac{4\sqrt{2}}{3} - 1 > \frac{7}{8}$ for $\alpha \in (-1,0)$, and $g(\alpha) \geq g(0) = 1$ for $\alpha \in [0,1]$. □

Remark 5.1. Estimates (5.2) and (5.3) and their consequences (5.4) and (5.5) are inferior to the estimates in Theorem A in the sense that they imply weaker estimates for the asymptotic Markov constant $c(\alpha)$. It can be also shown that the upper estimate in Theorem A is superior to (5.4) for every $\alpha > -1$ and $n \geq 3$. On the other hand, for small n Dörfler's lower estimate (5.3) and the lower estimates in Proposition 5.1 are superior to the lower estimate in Theorem A.

6. Proof of Corollaries 1.1–1.3

Proof of Corollary 1.1. The right-hand side inequality follows from Theorem 1.1: for $\alpha \geq 2$ we have

$$c_n^2(\alpha) \leq \frac{4n\big(n+2+\frac{3(\alpha+1)}{4}\big)}{\alpha^2+10\alpha+8} < \frac{4n(n+\alpha+3)}{\alpha^2+9\alpha+8} = \frac{4n(n+\alpha+3)}{(\alpha+1)(\alpha+8)}\,.$$

For the left-hand side inequality we make use of estimate (5.5), the case $\alpha \geq 1$. For $n \geq 3$ we have

$$\begin{aligned} c_n^2(\alpha) &\geq \frac{n\big(n+\frac{2\alpha+1}{3}\big)}{(\alpha+1)(\alpha+3)} > \frac{2n\big(n+\alpha+\frac{1}{2}\big)}{3(\alpha+1)(\alpha+3)} \\ &\geq \frac{2n\big(n+\alpha+\frac{1}{2}+5\big)}{3(\alpha+1)(\alpha+8)} > \frac{2n(n+\alpha+3)}{3(\alpha+1)(\alpha+8)}\,, \end{aligned}$$

where for the first inequality in the second line we have used that $f(x) = \frac{x+a}{x+b}$ is a decreasing function in $(0,\infty)$ when $a > b > 0$. This proves the left-hand side inequality in Corollary 1.1. □

Proof of Corollary 1.2. (i) From (5.3) we deduce that

$$\lim_{\alpha\to -1}(\alpha+1)c_n^2(\alpha) \geq \frac{n(n+1)}{2}\,,$$

while from the upper estimate in Theorem A we obtain

$$\lim_{\alpha\to -1}(\alpha+1)c_n^2(\alpha) \leq \frac{n(n+1)}{2}$$

(notice that the same conclusion follows from (5.4)).

(ii) The right-hand side inequality follows from Theorem 1.1, and the left-hand side inequality follows from (5.5). □

Proof of Corollary 1.3. The lower estimate is a consequence from Theorem A, while the upper estimates follow from Theorem A and Theorem 1.1, respectively. □

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