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Voronovskaja Type Theorems for Positive Linear Operators Related to Squared Fundamental Functions

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For a sequence of positive linear approximation operators defined by means of the squared Bernstein basis polynomials, Favard-Szász-Mirakjan fundamental functions and Baskakov fundamental functions, we derive a complete asymptotic expansion. The initial coefficients are explicitly calculated. As a special case, we obtain a Voronovskaja type formula. Finally, we introduce two Durrmeyer-type variants and calculate the initial coefficients of their asymptotic expansions. In each case the trivial class will be determined. Finally, we study the asymptotic properties of operators defined by means of squared Meyer-König and Zeller fundamental functions.

Keywords and Phrases: Approximation by positive operators, Bernstein polynomials, rate of convergence, degree of approximation, asymptotic expansions.

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1. Introduction

For a probability distribution $P = (p_\nu)_{\nu=0}^{\infty}$ the sum

$$IC(P) = \sum_{\nu=0}^{\infty} p_\nu^2$$

is called the index of coincidence [14]. The Rényi entropy of order 2 and the Tsallis entropy of order 2 are given by $R(P) = -\log IC(P)$ and $T(P) = 1 - IC(P)$, respectively (see [23], [25]). The associated Shannon entropy is given by $H(P) = -\sum_{\nu=0}^{\infty} p_\nu \log p_\nu$. For instance, the index of coincidence for the binomial distribution with parameters n and x

$$F_n(x) = \sum_{\nu=0}^{\infty} \left(\binom{n}{\nu} x^\nu (1-x)^{n-\nu} \right)^2$$

is related to the Rényi entropy $R_n(x) = -\log F_n(x)$ and the Tsallis entropy $R_n(x) = 1 - F_n(x)$.

During the last years, there has been an increasing interest in monotonicity and convexity properties of sums of squared fundamental functions arising in approximation theory (see, e.g. [13, 20, 22, 12, 4]). For a certain interval $I \subseteq \mathbb{R}$, consider a positive linear approximation process

$$(L_n f)(x) = \sum_{\nu=0}^{\infty} \ell_{n,\nu}(x) f(x_{n,\nu}) \qquad (x \in I)$$

such that $\sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x) > 0$, for all $x \in I$ with nodes $x_{n,\nu} \in I$. In many concrete applications we have $\ell_{n,\nu}(x) = 0$, for $\nu > n$, i.e., the sum is finite. Otherwise, we apply L_n to functions for which the sum is convergent, for all $x \in I$. We associate to L_n the operators $L_n^{\hat{}2}$ defined by

$$\left(L_n^{\hat{}2} f\right)(x) = \frac{1}{\sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x)} \sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x) f(x_{n,\nu}).$$

It is clear that the operators $L_n^{\hat{}2}$ are linear and positive. In this paper we are interested in the asymptotic properties of the sequence $\left(L_n^{\hat{}2} f\right)$ as n tends to infinity. We study the pointwise asymptotic rate of the operators $L_n^{\hat{}2}$ by a Voronovskaja type result for various examples. The main result is a complete asymptotic expansion

$$\left(L_n^{\hat{}2} f\right)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{b_k(f,x)}{n^k} \qquad (n \to \infty),$$

for sufficiently smooth functions f. The latter formula means that for each positive integer q,

$$\left(L_n^{\hat{}2} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{b_k(f,x)}{n^k} + o(n^{-q}) \qquad (n \to \infty).$$

In order to approximate integrable functions we consider the following two Durrmeyer-type variants of the operators $L_n^{\hat{}2}$ defined by

$$\left(DL_n^{\hat{}2} f\right)(x) = \frac{1}{\sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x)} \sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x) \frac{(\ell_{n,\nu}^2, f)}{(\ell_{n,\nu}^2, e_0)}$$

and

$$\left(\tilde{D}L_n^{\hat{}2} f\right)(x) = \frac{1}{\sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x)} \sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x) \frac{(\ell_{n,\nu}, f)}{(\ell_{n,\nu}, e_0)},$$

where

$$(g,h) = \int_I g(t)h(t)\,dt$$

is the canonical inner product.

In the special instances we are looking for functions f with $b_1(f,x) = 0$, for all $x \in I$. In this context, the family of these functions f having the property $\big(L_n^{\hat{2}} f\big)(x) = f(x) + o(n^{-1})$ as $n \to \infty$ is named the trivial class of the operators $L_n^{\hat{2}}$.

Throughout the paper e_r denote the monomials $e_r(x) = x^r$ $(r = 0, 1, 2, \ldots)$. Furthermore, we define $\psi_x(t) = t - x$, for $t, x \in \mathbb{R}$.

2. A General Result

The following result gives an asymptotic estimate of the central moments which is essential for obtaining asymptotic expansions for the operators $L_n^{\hat{2}}$ as n tends to infinity. To the best of our knowledge the result appears here for the first time. It generalizes the result [7, Lemma 4] for the squared Bernstein polynomials.

Lemma 1. *Fix $x \in I$. Let $0 \le \ell_{n,\nu}(x) \le 1$, for sufficiently large n, and*

$$\sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x) \sim \frac{c}{n^\alpha} \qquad (n \to \infty),$$

for some real constants c and $\alpha > 0$. Suppose that, for sufficiently large integers s,

$$\big(L_n \psi_x^{2s}\big)(x) = O(n^{-s}) \qquad (n \to \infty).$$

Then, for any (arbitrary small number) $\varepsilon > 0$, the central moments of the operators $L_n^{\hat{2}}$ satisfy the estimate

$$\big(L_n^{\hat{2}} \psi_x^s\big)(x) = O(n^{-s/2+\varepsilon}) \qquad (n \to \infty).$$

Remark 1. We conjecture that in the case of the most approximation operators the more general and sharper inequality

$$\big(L_n^{\hat{2}} \psi_x^s\big)(x) = O\big(n^{-\lfloor (s+1)/2 \rfloor}\big) \qquad (n \to \infty) \tag{1}$$

is valid, for all non-negative integers s.

Remark 2. Note that, for the classical Bernstein polynomials, the relation $\sum_{\nu=0}^{n} p_{n,\nu}(x)\big(\frac{\nu}{n} - x\big)^{2s} = O(n^{-s})$ as $n \to \infty$ is well known.

3. The Bernstein Operators

The Bernstein operators B_n defined by

$$(B_n f)(x) = \sum_{\nu=0}^{\infty} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} f\Big(\frac{\nu}{n}\Big) \qquad (x \in [0,1])$$

are the most prominent approximation operators. They are the instance $I = [0,1]$, $x_{n,\nu} = \nu/n$ and

$$\ell_{n,\nu}(x) \equiv p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}, \quad n = 1,2,\ldots, \quad x \in [0,1]$$

are the Bernstein basis polynomials.

The rational Bernstein type operators $B_n^{\hat{}2} : C[0,1] \to C[0,1]$ defined by

$$\big(B_n^{\hat{}2} f\big)(x) = \frac{1}{\sum_{\nu=0}^n p_{n,\nu}^2(x)} \sum_{\nu=0}^n p_{n,\nu}^2(x) f\Big(\frac{\nu}{n}\Big) \tag{2}$$

were studied by Herzog [15] in 2009 who investigated them accompanied by a plenty of numerical experiments. It is clear that the operators $B_n^{\hat{}2}$ are linear, positive, preserve constants and interpolate f at the endpoints.

The second central moment of positive linear operators is crucial for their approximation properties. In 2017, Gavrea and Ivan [12, Eq. (12)] estimated the second central moments $\big(B_n^{\hat{}2}\psi_x^2\big)(x)$ of the operators $B_n^{\hat{}2}$ by the inequalities

$$\frac{1}{2}\,\frac{x(1-x)}{n} \le \big(B_n^{\hat{}2}\psi_x^2\big)(x) \le r M_r \frac{x(1-x)}{n},$$

where $M_r = \sup\limits_{0\le u\le 1} G_r(u)$ and

$$G_r(u) = \frac{\int_0^1 ut(1-ut)^{r-1}(t(1-t))^{-1/2}dt}{\int_0^1 (1-ut)^r (t(1-t))^{-1/2}dt}, \quad u \in [0,1], \quad r = 1,2,\ldots$$

([12, Eq. (1)]). Note that $x(1-x)/n = \big(B_n\psi_x^2\big)(x)$ is the second central moment of the classical Bernstein polynomials B_n. As a corollary, Gavrea and Ivan [12, Cor. 5] obtained an estimate of the rate of convergence

$$\big|\big(B_n^{\hat{}2} f\big)(x) - f(x)\big| \le \Big(1 + \sqrt{\sqrt{3}-1}\Big)\omega\Big(f; \sqrt{\frac{x(1-x)}{n}}\Big), \quad x \in [0,1], \quad n \ge 2.$$

Hence (see [12, Th. 6]), for any $f \in C[0,1]$, the sequence of operators $\big(B_n^{\hat{}2} f\big)$ converges to f uniformly on $[0,1]$.

Recently, Holhoş [16, Theorem 1] derived the following Voronovskaja-type result for the sequence $\big(B_n^{\hat{}2}\big)$.

Theorem 1 (Holhoş, 2018). *Let $f \in C[0,1]$ be such that the second derivative of f exists and is continuous in a neighborhood of $x \in (0,1)$. Then*

$$\lim_{n\to\infty} n\big(\big(B_n^{\hat{}2} f\big)(x) - f(x)\big) = \frac{2x-1}{4} f'(x) + \frac{x(1-x)}{4} f''(x).$$

The continuity of f'' in a neighborhood of x is not necessary. Following Corollary 2 it is sufficient that $f''(x)$ exists (see [7, Corollary 1], which appeared simultaneously and independently of [16]). In the following we gather the recent results of Abel and Kushnirevych in [7]:

Theorem 2. *Let $x \in (0,1)$, $q \in \mathbb{N}$ and $f \in C[0,1]$. If $f^{(2q+2)}(x)$ exists, the sequence of operators $\left(B_n^{\hat{}2}\right)$ satisfies the asymptotic relation*

$$\left(B_n^{\hat{}2} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{b_k(f,x)}{n^k} + o(n^{-q}) \qquad (n \to \infty),$$

where $b_k(f,x)$ are certain coefficients independent of n. The initial values are given by

$$\begin{aligned}
b_1(f,x) &= \frac{1}{4}(2x-1)f'(x) + \frac{1}{4}x(1-x)f''(x),\\
b_2(f,x) &= \frac{2x-1}{32x(1-x)}f'(x) + \frac{1}{32}f''(x) + \frac{1}{48}(2x-1)x(1-x)f^{(3)}(x)\\
&\quad + \frac{1}{32}x^2(1-x)^2 f^{(4)}(x),\\
b_3(f,x) &= \frac{(2x-1)(1-3x(1-x))}{64(x(1-x))^2}f'(x) + \frac{1-3x(1-x)}{32\cdot 2!\,x(1-x)}f''(x)\\
&\quad + \frac{2x-1}{16\cdot 3!}f^{(3)}(x) + \frac{x(1-x)(1+4x(1-x))}{16\cdot 4!}f^{(4)}(x)\\
&\quad + \frac{-5(x(1-x))^2(2x-1)}{16\cdot 5!}f^{(5)}(x) + \frac{15(x(1-x))^3}{8\cdot 6!}f^{(6)}(x).
\end{aligned}$$

Remark 3. Under the conditions of Theorem 2, if f is sufficiently smooth at the point x, the sequence of operators $\left(B_n^{\hat{}2}\right)$ possesses a complete asymptotic expansion

$$\left(B_n^{\hat{}2} f\right)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{b_k(f,x)}{n^k} \qquad (n \to \infty).$$

Remark 4. For $q = 1,2,3,4$, it is sufficient that $f^{(2q)}(x)$ exists instead of $f^{(2q+2)}(x)$. This would be sufficient, for all positive integers q, if the central moments of the operators $B_n^{\hat{}2}$ satisfy the estimate

$$\left(B_n^{\hat{}2} \psi_x^s\right)(x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty), \tag{3}$$

for all non-negative integers s. However, we can show only that, for any (arbitrary small number) $\varepsilon > 0$,

$$\left(B_n^{\hat{}2} \psi_x^s\right)(x) = O(n^{-s/2+\varepsilon}) \qquad (n \to \infty)$$

(see Lemma 1). We conjecture that the sharper estimate (3) is true.

Corollary 1. *Let $x \in (0,1)$ and $f \in C[0,1]$. If $f''(x)$ exists, the operators $B_n^{\hat{}2}$ satisfy the asymptotic relation*

$$\lim_{n\to\infty} n\left(\left(B_n^{\hat{}2} f\right)(x) - f(x)\right) = \frac{1}{4}(2x-1)f'(x) + \frac{1}{4}x(1-x)f''(x).$$

Remark 5. The trivial class of the operators $B_n^{\hat{}2}$ consists of the functions

$$f_0(x) = c_1 x^2 (3 - 2x) + c_2 \qquad (c_1, c_2 \in \mathbb{R}).$$

For these functions we have $\lim_{n\to\infty} n\big(\big(B_n^{\hat{}2} f_0\big)(x) - f_0(x)\big) = 0$ or, more precisely, $\big(B_n^{\hat{}2} f_0\big)(x) = f_0(x) + O(n^{-2})$ as $n \to \infty$. Theorem 2 implies that

$$\lim_{n\to\infty} n^2\big(\big(B_n^{\hat{}2} f_0\big)(x) - f_0(x)\big) = \frac{c_1}{4}\, x(1-x)(1-2x).$$

3.1. The First Durrmeyer Variant of the Squared Bernstein Polynomials

In order to approximate integrable functions we introduce a Durrmeyer variant $DB_n^{\hat{}2}$ of the squared Bernstein polynomials:

$$DB_n^{\hat{}2} f = \frac{1}{\sum_{\nu=0}^n p_{n,\nu}^2} \sum_{\nu=0}^n p_{n,\nu}^2 \frac{\big(p_{n,\nu}^2, f\big)}{\big(p_{n,\nu}^2, e_0\big)},$$

where

$$(g,h) = \int_0^1 g(t) h(t)\, dt.$$

Because of

$$\big(p_{n,\nu}^2, e_0\big) = \int_0^1 p_{n,\nu}^2(t)\, dt = \frac{\binom{n}{\nu}^2}{(2n+1)\binom{2n}{2\nu}}$$

we can rewrite the definition of $DB_n^{\hat{}2}$ in the more explicit form

$$\big(DB_n^{\hat{}2} f\big)(x) = \frac{2n+1}{\sum\limits_{\nu=0}^n p_{n,\nu}^2(x)} \sum_{\nu=0}^n p_{n,\nu}^2(x) \frac{\binom{2n}{2\nu}}{\binom{n}{\nu}^2} \int_0^1 f(t) p_{n,\nu}^2(t)\, dt.$$

We report the following results from [7, Theorem 2 and Corollary 2].

Theorem 3. *Let $x \in (0,1)$ and suppose that f is bounded and integrable on $[0,1]$. If $f^{(4)}(x)$ exists, the sequence of operators $\big(DB_n^{\hat{}2}\big)$ satisfies the asymptotic relation*

$$\big(DB_n^{\hat{}2} f\big)(x) \sim f(x) + \frac{c_1(f,x)}{n} + \frac{c_2(f,x)}{n^2} \qquad (n \to \infty),$$

where

$$\begin{aligned} c_1(f,x) &= \frac{1}{4}\,(1-2x) f'(x) + \frac{1}{2}\, x(1-x) f''(x), \\ c_2(f,x) &= -\frac{1 + 6x - 24x^2 + 16x^3}{32x(1-x)} f'(x) + \frac{1}{32}\,(3 - 40x(1-x) f''(x) \\ &\quad + \frac{3}{8}\, x(1-x)(1-2x) f^{(3)}(x) + \frac{1}{8}\,(x(1-x))^2 f^{(4)}(x). \end{aligned}$$

Corollary 2. *Let $x \in (0,1)$ and suppose that f is bounded and integrable on $[0,1]$. If $f''(x)$ exists, the operators $D\hat{B_n^2}$ satisfy the asymptotic relation*

$$\lim_{n\to\infty} n\big(\big(D\hat{B_n^2} f\big)(x) - f(x)\big) = \frac{1}{4}\,(1-2x)f'(x) + \frac{1}{2}\,x(1-xt)f''(x).$$

Remark 6. The trivial class of the operators $D\hat{B_n^2}$ consists of the functions

$$f_0(x) = c_1 \arccos\sqrt{x} + c_2 \qquad (c_1, c_2 \in \mathbb{R}).$$

Theorem 3 implies that

$$\lim_{n\to\infty} n^2\big(\big(D\hat{B_n^2} f_0\big)(x) - f_0(x)\big) = c_1 \frac{1-2x}{18496(x(1-x))^{3/2}}.$$

4. An Alternative Durrmeyer Variant Related to Squared Bernstein Polynomials

In order to approximate integrable functions we introduce a second Durrmeyer variant $\tilde{D}\hat{B_n^2}$ of the operators $\hat{B_n^2}$. Replacing the point evaluation $f\big(\frac{\nu}{n}\big)$ in (2) with the functional $(n+1)\int_0^1 f(t)p_{n,\nu}(t)\,dt$ leads to the definition

$$\big(\tilde{D}\hat{B_n^2} f\big)(x) = \frac{n+1}{\sum_{\nu=0}^n p_{n,\nu}^2(x)} \sum_{\nu=0}^n p_{n,\nu}^2(x) \int_0^1 f(t)p_{n,\nu}(t)\,dt.$$

The following results can be found in [7, Theorem 3 and Corollary 3].

Theorem 4. *Let $x \in (0,1)$ and suppose that f is bounded and integrable on $[0,1]$. If $f^{(4)}(x)$ exists, the sequence of operators $\big(\tilde{D}\hat{B_n^2}\big)$ satisfies the asymptotic relation*

$$\big(\tilde{D}\hat{B_n^2} f\big)(x) \sim f(x) + \frac{\tilde{c}_1(f,x)}{n} + \frac{\tilde{c}_2(f,x)}{n^2} \qquad (n\to\infty),$$

where

$$\begin{aligned}
\tilde{c}_1(f,x) &= \frac{3}{4}\,(1-2x)f'(x) + \frac{3}{4}\,x(1-x)f''(x),\\
\tilde{c}_2(f,x) &= \frac{(2x-1)(1+48x(1-x))}{32x(1-x)}\,f'(x) + \frac{21}{32}\,(1-8x(1-x))f''(x)\\
&\quad + \frac{19(1-2x)x(1-x)}{16}\,f^{(3)}(x) + \frac{9}{32}\,(x(1-x))^2 f^{(4)}(x).
\end{aligned}$$

Corollary 3. *Let $x \in (0,1)$ and suppose that f is bounded and integrable on $[0,1]$. If $f''(x)$ exists, the operators $\tilde{D}B_n^{\hat{}2}$ satisfy the Voronovskaja-type formula*

$$\lim_{n\to\infty} n\big(\big(\tilde{D}B_n^{\hat{}2} f\big)(x) - f(x)\big) = \frac{3}{4}\big(x(1-x)f'(x)\big)'.$$

Remark 7. Note that the classical Bernstein-Durrmeyer operators DB_n possess the complete asymptotic expansion

$$\big(\tilde{D}B_n\big)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{\big(x^k(1-x)^k f^{(k)}(x)\big)^{(k)}}{k!\,(n+2)^{\overline{k}}} \qquad (n\to\infty).$$

Here $z^{\overline{k}} := z(z+1)\cdots(z+k-1)$, for $k\in\mathbb{N}$, denotes the rising factorial. In particular, we have the Voronovskaja-type formula

$$\lim_{n\to\infty} n\big(\big(\tilde{D}B_n f\big)(x) - f(x)\big) = \big(x(1-x)f'(x)\big)'.$$

(see [3] and [5]).

Remark 8. Note that the Voronovskaja-type formula in Corollary 3 coincides with that of the classical Bernstein–Durrmeyer operators $\tilde{D}B_n$ aside from the factor $3/4$. Hence the trivial class is the same like that of the classical Bernstein-Durrmeyer operators:

$$f_0(x) = c_1(\log x - \log(1-x)) + c_2 \qquad (c_1, c_2 \in \mathbb{R}).$$

5. The Favard-Szász-Mirakjan Operators

In this section we announce several results without presenting any proofs.

5.1. The Classical Favard-Szász-Mirakjan Operators

The Favard-Szász-Mirakjan operators S_n defined by

$$(S_n f)(x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!} f\Big(\frac{\nu}{n}\Big) \qquad (x\in[0,\infty)\,)$$

are the instance $I = [0,\infty)$, $x_{n,\nu} = \nu/n$ and

$$\ell_{n,\nu}(x) \equiv s_{n,\nu}(x) = e^{-nx}\frac{(nx)^\nu}{\nu!}, \quad n = 1,2,\ldots, \quad x\in[0,\infty).$$

In this subsection we consider the operators

$$\left(\hat{S_n^2} f\right)(x) = \frac{1}{\sum_{\nu=0}^{\infty} \frac{(nx)^{2\nu}}{(\nu!)^2}} \sum_{\nu=0}^{\infty} \frac{(nx)^{2\nu}}{(\nu!)^2} f\left(\frac{\nu}{n}\right) \qquad (x \in [0,\infty)\,).$$

While the operators are defined for functions of exponential growth, i.e., f satisfies $f(t) = O(e^{\alpha t})$ as $t \to \infty$ with a positive constant α, we restrict ourselves to functions of polynomial growth.

Theorem 5. *Let $x > 0$, $q \in \mathbb{N}$ and $f \in C[0,\infty)$ be a function of polynomial growth as the variable tends to infinity. If $f^{(2q+2)}(x)$ exists, the sequence of operators $\left(\hat{S_n^2}\right)$ satisfies the asymptotic relation*

$$\left(\hat{S_n^2} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{b_k(f,x)}{n^k} + o(n^{-q}) \qquad (n \to \infty),$$

where $b_k(f,x)$ are certain coefficients independent of n. The initial values are given by

$$\begin{aligned}
b_1(f,x) &= \frac{-1}{4} f'(x) + \frac{1}{4} x f''(x), \\
b_2(f,x) &= \frac{-1}{32x} f'(x) + \frac{1}{32} f''(x) - \frac{1}{48} x f^{(3)}(x) + \frac{1}{32} x^2 f^{(4)}(x), \\
b_3(f,x) &= \frac{-1}{64x^2} f'(x) + \frac{1}{64x} f''(x) - \frac{1}{96} f^{(3)}(x) + \frac{x}{384} f^{(4)}(x) \\
&\quad + \frac{x^2}{384} f^{(5)}(x) + \frac{15x^3}{384} f^{(6)}(x).
\end{aligned}$$

Remark 9. The trivial class of the operators $\hat{S_n^2}$ consists of the functions

$$f_0(x) = c_1 x^2 + c_2 \qquad (c_1, c_2 \in \mathbb{R}).$$

It is remarkable that the operator $\hat{S_n^2}$ preserves the quadratic function e_2. We have

$$\hat{S_n^2} e_r = e_r \qquad (r \in \{0, 2\}).$$

For the study of the operators $\hat{S_n^2}$ one can use the fact that the moments of $\hat{S_n^2}$ can be expressed in terms of the modified Bessel functions I_j. The modified Bessel function I_j of order j is a solution of the differential equation

$$z^2 \left(\frac{d}{dz}\right)^2 I_j(z) + z \frac{d}{dz} I_j(z) - (z^2 + j^2) I_j(z) = 0$$

(see [8, Eq. (9.6.1)]) and it is given by the power-series expansion

$$I_j(z) = \left(\frac{z}{2}\right)^j \sum_{\nu=0}^{\infty} \frac{1}{\nu!\,\Gamma(\nu+j+1)} \left(\frac{z^2}{4}\right)^\nu$$

(see [8, Eq. (9.6.10)]). One can show that

$$\sum_{\nu=0}^{\infty} s_{n,\nu}^2(x)\left(\frac{\nu}{n}\right)^r = n^{-r}\sum_{j=0}^{r} j!\,\sigma(r,j)\sum_{\nu=0}^{\infty} s_{n,\nu}^2(x)\binom{\nu}{j},$$

where $\sigma(r,j)$ denote the Stirling numbers of the second kind. For instance, we have

$$\left(S_n^{\hat{}2} e_1\right)(x) = \frac{I_1(2nx)}{I_0(2nx)}\,x \qquad (x>0).$$

Using elementary methods one can show the inequality

$$1-\frac{5}{z} \le \frac{I_1(z)}{I_0(z)} \le 1 \qquad (z>0),$$

which implies the estimates

$$\left|\left(S_n^{\hat{}2}\psi_x\right)(x)\right| = \left| -x\left(1-\frac{I_1(2nx)}{I_0(2nx)}\right)\right| \le \frac{5}{2n}, \qquad (x>0)$$

and

$$\left(S_n^{\hat{}2}\psi_x^2\right)(x) = 2x^2\left(1-\frac{I_1(2nx)}{I_0(2nx)}\right) \le \frac{5x}{n}, \qquad (x>0).$$

Standard arguments imply the following estimate of the rate of convergence in terms of the first modulus of continuity [9, Theorem 5.1.2].

Theorem 6. *Let $x>0$ and $n\in\mathbb{N}$. For every bounded function $f\in C[0,\infty)$ and $\delta>0$, we have*

$$\left|\left(S_n^{\hat{}2} f\right)(x) - f(x)\right| \le \left(1+\frac{1}{\delta}\sqrt{\frac{5x}{n}}\right)\omega(f;\delta).$$

Moreover, if f is differentiable on $[0,\infty)$ with f' bounded on $[0,\infty)$, we also have

$$\left|\left(S_n^{\hat{}2} f\right)(x) - f(x)\right| \le \frac{5}{2n}\left|f'(x)\right| + \sqrt{\frac{5x}{n}}\left(1+\frac{1}{\delta}\sqrt{\frac{5x}{n}}\right)\omega(f';\delta).$$

Theorem 6 applied to $\delta=\sqrt{5x/n}$ implies

Corollary 4. *Let $x>0$ and $n\in\mathbb{N}$. For every bounded function $f\in C[0,\infty)$, we have*

$$\left|\left(S_n^{\hat{}2} f\right)(x) - f(x)\right| \le 2\omega\left(f;\sqrt{\frac{5x}{n}}\right) \qquad (x>0, \quad n\ge 1).$$

Moreover, if f is differentiable on $[0,\infty)$ with f' bounded on $[0,\infty)$, we also have

$$\left|\left(S_n^{\hat{}2} f\right)(x) - f(x)\right| \le \frac{5}{2n}\left|f'(x)\right| + 2\sqrt{\frac{5x}{n}}\,\omega\left(f';\sqrt{\frac{5x}{n}}\right).$$

5.2. The Durrmeyer Variants of the Favard-Szász-Mirakjan Operators

In order to approximate integrable functions, we introduce the following Durrmeyer variants of the operators $S_n^{\hat{}2}$ defined by

$$DS_n^{\hat{}2} f = \frac{1}{\sum_{\nu=0}^{\infty} s_{n,\nu}^2} \sum_{\nu=0}^{\infty} s_{n,\nu}^2 \frac{\left(s_{n,\nu}^2, f\right)}{\left(s_{n,\nu}^2, e_0\right)}$$

and

$$\tilde{D}S_n^{\hat{}2} f = \frac{1}{\sum_{\nu=0}^{\infty} s_{n,\nu}^2} \sum_{\nu=0}^{\infty} s_{n,\nu}^2 \frac{(s_{n,\nu}, f)}{(s_{n,\nu}, e_0)},$$

where

$$(g, h) = \int_0^{\infty} g(t)h(t)\, dt.$$

Because of

$$\left(s_{n,\nu}^2, e_0\right) = \int_0^{\infty} s_{n,\nu}^2(t)\, dt = \frac{\binom{2\nu}{\nu}}{n\, 2^{2\nu+1}},$$

$$\left(s_{n,\nu}, e_0\right) = \int_0^{\infty} s_{n,\nu}(t)\, dt = \frac{1}{n}$$

we can rewrite the both definitions in the more explicit form

$$DS_n^{\hat{}2} f = \frac{n}{\sum_{\nu=0}^{\infty} s_{n,\nu}^2} \sum_{\nu=0}^{\infty} s_{n,\nu}^2 \frac{2^{2\nu+1}}{\binom{2\nu}{\nu}} \int_0^{\infty} s_{n,\nu}^2(t) f(t)\, dt$$

and

$$\tilde{D}S_n^{\hat{}2} f = \frac{n}{\sum_{\nu=0}^{\infty} s_{n,\nu}^2} \sum_{\nu=0}^{\infty} s_{n,\nu}^2 \int_0^{\infty} s_{n,\nu}(t) f(t)\, dt.$$

Theorem 7. *Let $x > 0$ and suppose that f is of polynomial growth, locally bounded and integrable on $[0, \infty)$. If $f^{(4)}(x)$ exists, the sequences of operators $\left(DS_n^{\hat{}2}\right)$ and $\left(\tilde{D}S_n^{\hat{}2}\right)$ satisfy the asymptotic relations*

$$\left(DS_n^{\hat{}2} f\right)(x) \sim f(x) + \frac{c_1(f,x)}{n} + \frac{c_2(f,x)}{n^2} \qquad (n \to \infty),$$

$$\left(\tilde{D}S_n^{\hat{}2} f\right)(x) \sim f(x) + \frac{\tilde{c}_1(f,x)}{n} + \frac{\tilde{c}_2(f,x)}{n^2} \qquad (n \to \infty),$$

where

$$c_1(f,x) = \frac{1}{4} f'(x) + \frac{x}{2} f''(x),$$

$$c_2(f,x) = -\frac{1}{32x} f'(x) + \frac{3}{32} f''(x) + \frac{3x}{8} f^{(3)}(x) + \frac{x^2}{8} f^{(4)}(x),$$

$$\tilde{c}_1(f,x) = \frac{3}{4} f'(x) + \frac{3x}{4} f''(x),$$

$$\tilde{c}_2(f,x) = -\frac{1}{32x} f'(x) + \frac{21}{32} f''(x) + \frac{19x}{16} f^{(3)}(x) + \frac{27x^2}{96} f^{(4)}(x).$$

Corollary 5. *Let $x > 0$ and suppose that f is of polynomial growth, locally bounded and integrable on $[0, \infty)$. If $f''(x)$ exists, the operators $\big(DS_n^{\hat{}2}\big)$ and $\big(\tilde{D}S_n^{\hat{}2}\big)$ satisfy the asymptotic relation*

$$\lim_{n\to\infty} n\big(\big(DS_n^{\hat{}2} f\big)(x) - f(x)\big) = \frac{1}{4} f'(x) + \frac{x}{2} f''(x),$$
$$\lim_{n\to\infty} n\big(\big(\tilde{D}S_n^{\hat{}2} f\big)(x) - f(x)\big) = \frac{3}{4} f'(x) + \frac{3x}{4} f''(x).$$

Remark 10. The trivial class of the operators $DS_n^{\hat{}2}$ consists of the functions

$$f_0(x) = c_1\sqrt{x} + c_2 \qquad (c_1, c_2 \in \mathbb{R}).$$

Theorem 7 implies that

$$\lim_{n\to\infty} n^2\big(\big(DS_n^{\hat{}2} f_0)(x) - f_0(x)\big) = \frac{-c_1}{64x^{3/2}}.$$

Remark 11. The trivial class of the operators $\tilde{D}S_n^{\hat{}2}$ consists of the functions

$$f_0(x) = c_1 \log x + c_2 \qquad (c_1, c_2 \in \mathbb{R}).$$

Theorem 7 implies that

$$\lim_{n\to\infty} n^2\big(\big(\tilde{D}S_n^{\hat{}2} f_0\big)(x) - f_0(x)\big) = 0.$$

That this limit is equal to zero is quite unusual. One could ask whether the operators $\tilde{D}S_n^{\hat{}2}$ preserve the logarithm function. We propose the

Problem 1 (Open Problem). *Is the equation $\tilde{D}S_n^{\hat{}2} \log = \log$ valid for all $n \in \mathbb{N}$?*

6. The Baskakov Operators

In this section we announce several results. We present full proofs for the classical definition and omit the details in the case of the both Durrmeyer variants.

6.1. The Classical Baskakov Operators

The Baskakov operators V_n were introduced by Baskakov [10] in 1957. They are the instance $I = [0, \infty)$, $x_{n,\nu} = \nu/n$ and

$$\ell_{n,\nu}(x) \equiv b_{n,\nu}(x) = \binom{\nu + n - 1}{\nu} x^{\nu} (1 + x)^{-n-\nu}, \quad n = 1, 2, \ldots, \quad x \in [0, \infty)$$

are the Baskakov fundamental functions. In this section we study the operators

$$\left(V_n^{\hat{}2} f\right)(x) = \frac{1}{\sum_{\nu=0}^{\infty} b_{n,\nu}^2(x)} \sum_{\nu=0}^{\infty} b_{n,\nu}^2(x) f\left(\frac{\nu}{n}\right).$$

Theorem 8. *Let $x > 0$, $q \in \mathbb{N}$ and $f \in C[0,\infty)$ be a function of polynomial growth as the variable tends to infinity. If $f^{(2q+2)}(x)$ exists, the sequence of operators $\left(V_n^{\hat{}2}\right)$ satisfies the asymptotic relation*

$$\left(V_n^{\hat{}2} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{b_k(f,x)}{n^k} + o(n^{-q}) \qquad (n \to \infty), \tag{4}$$

where $b_k(f,x)$ are certain coefficients independent of n. The initial values are given by

$$\begin{aligned}
b_1(f,x) &= -\frac{1+2x}{4} f'(x) + \frac{x(1+x)}{4} f''(x),\\
b_2(f,x) &= \frac{-(1+2x) + 24x(1+x)^2}{32x(1+x)} f'(x) + \frac{1+24x(1+x)}{32} f''(x)\\
&\quad - \frac{1}{48} x(1+x)(1+2x) f^{(3)}(x) + \frac{1}{32} x^2(1+x)^2 f^{(4)}(x),\\
b_3(f,x) &= \frac{1 - 13x + 99x^2 + 318x^3 + 276x^4 + 72x^5}{64(x(1+x))^2} f'(x)\\
&\quad + \frac{-1 + 51x + 327x^2 + 516x^3 + 240x^4}{64x(1+x)} f''(x)\\
&\quad - \frac{1 + 38x + 54x^2 + 18x^3}{48} f^{(3)}(x)\\
&\quad + \frac{x(1+x)(1+44x(1+x))}{16 \cdot 4!} f^{(4)}(x)\\
&\quad + \frac{5(x(1+x))^2(1+2x)}{16 \cdot 5!} f^{(5)}(x) + \frac{15(x(1+x))^3}{8 \cdot 6!} f^{(6)}(x).
\end{aligned}$$

For $q = 1,2,3,4$, it is sufficient that $f^{(2q)}(x)$ exists instead of $f^{(2q+2)}(x)$.

Remark 12. The special case $q = 1$ is the Voronovskaja-type result

$$\lim_{n\to\infty} n\left(\left(V_n^{\hat{}2} f\right)(x) - f(x)\right) = -\frac{1+2x}{4} f'(x) + \frac{x(1+x)}{4} f''(x),$$

which was recently derived by Holhoş [17, Corollary 11] under certain conditions on the function f.

Remark 13. The trivial class of the operators $V_n^{\hat{}2}$ consists of the functions

$$f_0(x) = c_1 x^2 (3+2x) + c_2 \qquad (c_1, c_2 \in \mathbb{R}).$$

For these functions we have $\lim_{n\to\infty} n\big(\big(V_n^{\hat{}2} f_0\big)(x) - f_0(x)\big) = 0$ or, more precisely, $\big(V_n^{\hat{}2} f_0\big)(x) = f_0(x) + O(n^{-2})$ as $n \to \infty$. Theorem 8 implies that

$$\lim_{n\to\infty} n^2\big(\big(V_n^{\hat{}2} f_0\big)(x) - f_0(x)\big) = \frac{c_1}{4}\, x(1+x)(35+52x).$$

As a first step we study the moments $V_n^{\hat{}2} e_r$ of the operators $V_n^{\hat{}2}$. In the case $f = e_r$ we have to consider

$$\sum_{\nu=0}^{\infty} b_{n,\nu}^2(x) \left(\frac{\nu}{n}\right)^r = n^{-r} \sum_{k=0}^{r} k!\,\sigma(r,k) \sum_{\nu=0}^{\infty} b_{n,\nu}^2(x) \binom{\nu}{k}. \tag{5}$$

In the latter equality, we made use of the identities

$$\nu^r = \sum_{k=0}^{r} k!\,\sigma(r,k)\binom{\nu}{k},$$

where $\sigma(r,k)$ denote the Stirling numbers of the second kind. The next lemma provides an integral representation of the inner sum in Eq. (5). For $x > 0$ and $n, j \ge 0$, we define

$$I_{n,j}(x) := \frac{1}{\pi}\int_0^{\pi} \big(1 + 2x(1+x)(1-\cos\theta)\big)^{-n} \cos(j\theta)\, d\theta. \tag{6}$$

Lemma 2. *Let $x > 0$. Then, for $n = 1, 2, 3, \ldots,$ and $k = 0, 1, 2, \ldots,$*

$$\sum_{\nu=0}^{\infty} b_{n,\nu}^2(x)\binom{\nu}{k} = \binom{k+n-1}{k} \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} x^{2k-j}(1+x)^j I_{n+k,j}(x).$$

Proof. Our starting point is the generating function of the fundamental functions

$$\sum_{\nu=0}^{\infty} b_{n,\nu}(x)(te^{i\theta})^{\nu} = (1 + x - xte^{i\theta})^{-n}.$$

Differentiating both sides k times with respect to t, and taking $t = 1$, we obtain

$$\sum_{\nu=0}^{\infty} b_{n,\nu}(x)\binom{\nu}{k} e^{i\nu\theta} = \binom{k+n-1}{k} x^k (1 + x - xe^{i\theta})^{-n-k} e^{ik\theta}. \tag{7}$$

Taking advantage of the orthogonality relation

$$\int_0^{2\pi} e^{i\nu\theta}\, \overline{e^{i\mu\theta}}\, d\theta = 2\pi\delta_{\nu,\mu} \qquad (\nu, \mu \in \mathbb{Z})$$

we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{\nu=0}^{\infty} b_{n,\nu}(x)\binom{\nu}{k} e^{i\nu\theta}\right)\left(\sum_{\mu=0}^{\infty} b_{n,\mu}(x)e^{-i\mu\theta}\right) d\theta = \sum_{\nu=0}^{\infty} b_{n,\nu}^2(x)\binom{\nu}{k}.$$

Using Eq. (7) leads to

$$
\begin{aligned}
2\pi & \sum_{\nu=0}^{\infty} b_{n,\nu}^2(x)\binom{\nu}{k} \\
&= \binom{k+n-1}{k} x^k \int_0^{2\pi} (1+x-xe^{i\theta})^{-n-k} e^{ik\theta} (1+x-xe^{-i\theta})^{-n} d\theta \\
&= \binom{k+n-1}{k} x^k \int_0^{2\pi} \big(1+2x(1+x)(1-\cos\theta)\big)^{-n-k} \big((1+x)e^{i\theta}-x\big)^k \, d\theta.
\end{aligned}
$$

Applying the binomial formula

$$
\big((1+x)e^{i\theta}-x\big)^k = \sum_{j=0}^{k} \binom{k}{j} (-x)^{k-j} (1+x)^j e^{ij\theta},
$$

after some simplification we arrive at the desired formula. □

Now we derive a complete asymptotic expansion for the integrals $I_{n,j}(x)$ as n tends to infinity.

Lemma 3. *Let $x>0$. Then, for $j=0,1,2,\ldots$,*

$$
I_{n,j}(x) \sim \frac{1}{\pi} \sum_{\nu=0}^{\infty} a_{j,\nu}(x) \frac{\Gamma\big(\nu+\frac{1}{2}\big)}{n^{\nu+\frac{1}{2}}} \qquad (n\to\infty),
$$

where $a_{j,\nu}(x)$ are the coefficients in the power series expansion

$$
T_j\Big(1-\frac{e^t-1}{2x(1+x)}\Big) \frac{e^t\sqrt{t}}{\sqrt{e^t-1}\sqrt{4x(1+x)-e^t+1}} = \sum_{\nu=0}^{\infty} a_{j,\nu}(x) t^\nu
$$

and T_j denotes the Chebyshev polynomial of degree j.

In particular,

$$
\begin{aligned}
a_{j,0}(x) &= \frac{1}{2\sqrt{x(1+x)}}, \\
a_{j,1}(x) &= \frac{1-4j^2+6x(1+x)}{16(x(1+x))^{3/2}}, \\
a_{j,2}(x) &= \frac{9-40j^2+16j^4+(60-240j^2)x(1+x)+100(x(1+x))^2}{768(x(1+x))^{5/2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_{n,j}(x) = & \frac{1}{2\sqrt{\pi x(1+x)n}} + \frac{1-4j^2+6x(1+x)}{16\sqrt{\pi}(x(1+x)n)^{3/2}} \\
& + \frac{9-40j^2+16j^4+(60-240j^2)x(1+x)+100(x(1+x))^2}{384\sqrt{\pi}(x(1+x)n)^{5/2}} \\
& + O(n^{-7/2}) \qquad (n\to\infty).
\end{aligned}
$$

Proof of Lemma 3. We make the change of variable

$$t = \log\big(1 + 2x(1+x)(1-\cos\theta)\big),$$
$$e^t dt = 2x(1+x)\sin\theta\, d\theta$$

in the integral (6). Recall that the Chebyshev polynomials T_j $(j = 0, 1, 2, \ldots)$ satisfy $\cos(j\theta) = T_j(\cos\theta)$. Hence,

$$\cos(j\theta) = T_j\Big(1 - \frac{e^t - 1}{2x(1+x)}\Big).$$

Observing that

$$\sin\theta = \sqrt{1 - \cos^2\theta} = (2x(1+x))^{-1}\sqrt{e^t - 1}\sqrt{4x(1+x) - e^t + 1},$$

for $\theta \in [0, \pi]$, we obtain

$$\begin{aligned} &I_{n,j}(x) \\ &= \frac{1}{\pi}\int_0^{\log(1+4x(1+x))} e^{-nt} T_j\Big(1 - \frac{e^t - 1}{2x(1+x)}\Big) \frac{e^t}{\sqrt{e^t - 1}\sqrt{4x(1+x) - e^t + 1}}\, dt \\ &= \frac{1}{\pi}\int_0^{\log(1+4x(1+x))} e^{-nt} \sum_{\nu=0}^{\infty} a_{j,\nu} t^{\nu - 1/2}\, dt. \end{aligned}$$

Now the result follows by Watson's lemma (see, e.g. [21, Theorem 3.1, p. 71]). □

For abbreviation we define, for $s = 0, 1, 2, \ldots$,

$$T_{n,s}(x) := \sum_{\nu=0}^{\infty} b_{n,\nu}^2(x)\Big(\frac{\nu}{n} - x\Big)^s.$$

By Eq. (5) and Lemma 2, we have

$$\begin{aligned} T_{n,s}(x) = &\sum_{r=0}^{s}\binom{s}{r}(-x)^{s-r} n^{-r} \sum_{k=0}^{r} k!\,\sigma(r,k)\binom{k+n-1}{k} \\ &\times \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} x^{2k-j}(1+x)^j I_{n+k,j}(x). \end{aligned}$$

Now the result on the integrals $I_{n,j}(x)$ in Lemma 3 leads to complete asymptotic expansions for the central moments $T_{n,s}(x)$. In the particular case $s = 0$, we obtain the well-known relation

$$T_{n,0}(x) = \sum_{\nu=0}^{\infty} b_{n,\nu}^2(x) = I_{n,0}(x) \sim \frac{1}{2\sqrt{\pi x(1+x)n}} \qquad (n \to \infty).$$

This formula is the special case $r = 2$ of the recent result by Holhoş [17, Eq. (7)]

$$\sum_{\nu=0}^{\infty} b_{n,\nu}^{r}(x) \sim \frac{1}{\sqrt{r}}(2\pi x(1+x)n)^{(1-r)/2} \qquad (n \to \infty, \quad r = 1, 2, 3, \ldots).$$

The central moments of the operators $V_n^{\hat{}2}$ are given by

$$\left(V_n^{\hat{}2}\psi_x^s\right)(x) = \frac{T_{n,s}(x)}{T_{n,0}(x)}.$$

Proof of Theorem 8. It is well-known that, for $s = 0, 1, 2, \ldots$, the central moments of the classical Baskakov operators satisfy $\left(V_n\psi_x^s\right)(x) = O\left(n^{-\lfloor (s+1)/2\rfloor}\right)$ as $n \to \infty$. Hence, the existence of a complete asymptotic expansion follows, by Lemma 1 and a slight generalization of general approximation theorems due to Sikkema [24, Theorem 1]. By direct computation, it can be verified that Eq. (1) is valid, for $0 \le s \le 8$. Hence, for $q = 1, 2, 3, 4$, the expansion (4) is valid under the weaker condition that $f^{(2q)}(x)$ exists. □

6.2. Durrmeyer Variants Related to Squared Baskakov Operators

In order to approximate integrable functions, we introduce the following Durrmeyer variants of the operators $V_n^{\hat{}2}$ defined by

$$DV_n^{\hat{}2} f = \frac{1}{\sum_{\nu=0}^{\infty} b_{n,\nu}^2} \sum_{\nu=0}^{\infty} b_{n,\nu}^2 \frac{\left(b_{n,\nu}^2, f\right)}{\left(b_{n,\nu}^2, e_0\right)}$$

and

$$\tilde{D}V_n^{\hat{}2} f = \frac{1}{\sum_{\nu=0}^{\infty} b_{n,\nu}^2} \sum_{\nu=0}^{\infty} b_{n,\nu}^2 \frac{\left(b_{n,\nu}, f\right)}{\left(b_{n,\nu}, e_0\right)},$$

where

$$(g, h) = \int_0^{\infty} g(t)h(t)\,dt.$$

Because of

$$\begin{aligned}\left(b_{n,\nu}^2, e_0\right) &= \int_0^{\infty} b_{n,\nu}^2(t)\,dt = \frac{2}{n+\nu}\binom{2n+2\nu}{n+\nu}^{-1}\binom{2\nu}{\nu}\binom{2n-2}{n-1},\\ \left(b_{n,\nu}, e_0\right) &= \int_0^{\infty} b_{n,\nu}(t)\,dt = \frac{1}{n-1}\end{aligned}$$

we can rewrite the both definitions in the more explicit form

$$DV_n^{\hat{}2} f = \frac{1}{\binom{2n-2}{n-1}\sum_{\nu=0}^{\infty} b_{n,\nu}^2} \sum_{\nu=0}^{\infty} b_{n,\nu}^2 \frac{(n+\nu)\binom{2n+2\nu}{n+\nu}}{2\binom{2\nu}{\nu}} \int_0^{\infty} b_{n,\nu}^2(t) f(t)\,dt$$

and

$$\tilde{D}V_n^{\hat{}2} f = \frac{n-1}{\sum_{\nu=0}^{\infty} b_{n,\nu}^2} \sum_{\nu=0}^{\infty} b_{n,\nu}^2 \int_0^{\infty} b_{n,\nu}(t) f(t)\, dt.$$

The asymptotic behaviour of these operators can be obtained in the same manner as in Theorem 3 and Theorem 4. We omit the details. Asymptotic results on the ordinary Baskakov–Durrmeyer operators can be found in [6].

7. The Meyer-König and Zeller Operators

The operators of Meyer-König and Zeller [19] in the slight modification of Cheney and Sharma [11], defined by

$$(M_n f)(x) = (1-x)^{n+1} \sum_{\nu=0}^{\infty} \binom{\nu+n}{\nu} f\Big(\frac{\nu}{\nu+n}\Big) x^{\nu} \qquad (0 \le x < 1),$$

(also called Bernstein power series) were the object of several investigations in approximation theory. They are the instance $I = [0,1)$, $x_{n,\nu} = \frac{\nu}{\nu+n}$ and

$$\ell_{n,\nu}(x) \equiv m_{n,\nu}(x) = (1-x)^{n+1} \binom{\nu+n}{\nu} x^{\nu}, \quad n = 1, 2, \dots, \quad x \in [0,1).$$

Usually, for $x = 1$, one defines $(M_n f)(1) = f(1)$. It is well known that, for every continuous function f on the interval $[0,1]$, we have

$$\lim_{n\to\infty} (M_n f)(x) = f(x) \qquad (x \in [0,1]).$$

In 1970, Lupas and Müller [18] obtained the Voronovskaja-type formula

$$\lim_{n\to\infty} n\big((M_n f)(x) - f(x)\big) = \frac{x(1-x)^2}{2} f''(x).$$

In the paper [2] (see also [1]) the complete asymptotic expansion for the Meyer-König and Zeller operators is presented in the form

$$(M_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} a_k(f,x) n^{-k} \qquad (n \to \infty),$$

provided that the function f possesses derivatives of sufficiently high order at $x \in [0,1]$.

In this section we study, for $0 \le x < 1$, the operators

$$\big(M_n^{\hat{}2} f\big)(x) = \frac{1}{\sum_{\nu=0}^{\infty} m_{n,\nu}^2(x)} \sum_{\nu=0}^{\infty} m_{n,\nu}^2(x) f\Big(\frac{\nu}{\nu+n}\Big).$$

The first three moments of the operators $M_n^{\hat{}2}$ can be represented in terms of hypergeometric functions. It is easy to see that

$$\begin{aligned} M_n^{\hat{}2} e_0 &= e_0, \\ \left(M_n^{\hat{}2} e_1\right)(x) &= (n+1)x^2 \, \frac{{}_2F_1(n+2, n+1; 2; x^2)}{{}_2F_1(n+1, n+1; 1; x^2)}, \\ M_n^{\hat{}2} e_2 &= e_2. \end{aligned}$$

Hence, $M_n^{\hat{}2}$ preserves constant functions and quadratic functions $c \cdot e_2$ $(c \in \mathbb{R})$.

Without a proof we announce the following result.

Theorem 9. *Let $x \in [0,1)$, $q \in \mathbb{N}$ and $f \in C[0,1)$ be a bounded function. If $f^{(2q+2)}(x)$ exists, the sequence of operators $\left(M_n^{\hat{}2}\right)$ satisfies the asymptotic relation*

$$\left(M_n^{\hat{}2} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{b_k(f,x)}{n^k} + o(n^{-q}) \qquad (n \to \infty), \tag{8}$$

where $b_k(f,x)$ are certain coefficients independent of n. The initial values are given by

$$\begin{aligned} b_1(f,x) &= \frac{1}{4}\,(1-x)^2 (x f''(x) - f'(x)), \\ b_2(f,x) &= \frac{(1-x^2)^2}{32x}\,(x f''(x) - f'(x)) - \frac{1}{48}\, x(1-x)^3 (1+7x) f^{(3)}(x) \\ &\quad + \frac{1}{32}\, x^2 (1-x)^4 f^{(4)}(x). \end{aligned}$$

For $q = 1, 2, 3$, the expansion (8) is valid under the weaker condition that $f^{(2q)}(x)$ exists.

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References

[1] U. Abel, The moments for the Meyer-König and Zeller operators, *J. Approx. Theory* **82** (1995), 352–361.

[2] U. Abel, The complete asymptotic expansion for the Meyer-König and Zeller operators, *J. Math. Anal. Appl.* **208** (1997), 109–119.

[3] U. Abel, Asymptotic approximation by Bernstein-Durrmeyer operators and their derivatives, *Approx. Theory Appl.* **16:2** (2000), 1–12.

[4] U. Abel, W. Gawronski and Th. Neuschel, Complete monotonicity and zeros of sums of squared Baskakov functions, *Appl. Math. Comput.* **258** (2015), 130–137.

[5] U. Abel and M. Heilmann, The complete asymptotic expansion for Bernstein-Durrmeyer operators with Jacobi weights, *Mediterr. J. Math.* **1** (2004), 487–499.

[6] U. Abel, M. Ivan and Zh.-K. Li, Local approximation by generalized Baskakov-Durrmeyer operators, *Numer. Funct. Anal. Optim.* **28** (2007), 245–264.

[7] U. Abel and V. Kushnirevych, Voronovskaja type theorems for positive linear operators related to squared Bernstein polynomials, *Positivity* **23** (2019), 697–710.

[8] M. Abramowitz and A. Stegun, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables", Applied Mathematics Series 55, National Bureau of Standards, Washington D. C., 1972.

[9] F. Altomare and M. Campiti, "Korovkin-type Approximation Theory and Its Applications", de Gruyter, Berlin, New York, 1994.

[10] V. Baskakov, An example of a sequence of linear positive operators in the space of continuous functions, *Dokl. Akad. Nauk SSSR* **113** (1957), 249–251.

[11] E. W. Cheney and A. Sharma, Bernstein power series, *Canad. J. Math.* **16** (1964), 241–253.

[12] I. Gavrea and M. Ivan, On a new sequence of positive linear operators related to squared Bernstein polynomials, *Positivity* **21** (2017), 911–917.

[13] H. Gonska, I. Raşa and M. Rusu, Chebyshev-Grüss-type inequalities via discrete oscillations, *Bull. Acad. Stiinte Repub. Mold. Mater.* **1** (2014), 63–89.

[14] P. Harremoës and F. Topsøe, Information diagrams: entropy, index of coincidence and probability of error, *IEEE Trans. Inform. Theory* **47** (2001), 2944–2960.

[15] F. Herzog, "Heuristische Untersuchungen der Konvergenzrate spezieller linearer Approximationsprozesse", Diploma thesis, Friedberg, 2009.

[16] A. Holhoş, Voronovskaya theorem for a sequence of positive linear operators related to squared Bernstein polynomials, *Positivity* **23** (2019), 571–580.

[17] A. Holhoş, A sequence of positive linear operators related to powered Baskakov basis, *Carpathian J. Math.* **35** (2019), No. 1, 51–58.

[18] A. Lupas and M. W. Müller, Approximation properties of the M_n-operators, *Aequationes Math.* **5** (1970), 19–37.

[19] W. Meyer-König and K. Zeller, Bernsteinsche Potenzreihen, *Studia Math.* **19** (1960), 89–94.

[20] G. Nikolov, Inequalities for ultraspherical polynomials. Proof of a conjecture of I. Raşa, *J. Math. Anal. Appl.* **418** (2014), 852–860.

[21] F. W. J. Olver, "Asymptotics and Special Functions", A K Peters, Ltd., 1997.

[22] I. Raşa, Special functions associated with positive linear operators, preprint arXiv:1409.1015v2.

[23] A. Rényi, On measures of entropy and information, *in* "Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability", Vol. 1, pp. 547–561, University of California Press, 1961.

[24] P. C. Sikkema, On some linear positive operators, *Indag. Math.* **32** (1970), 327–337.

[25] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, *J. Stat. Phys.* **52** (1988), 479–487.

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