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Linking Baskakov Type Operators

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We consider the linking Baskakov type operators and investigate their behavior with respect to the class of Lipschitz functions. Concerning the relation between their basis functions and the B-splines, a conjecture is presented and some results supporting it.

Keywords and Phrases: Linking operators, Baskakov-Durrmeyer type operators, Kantorovich modifications of operators, Lipschitz functions, B-splines.

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1. Introduction

In 1957 the so-called Baskakov type operators depending on a real parameter c were introduced by Baskakov in [3]. For $c = -1$, $c = 0$ and $c = 1$, respectively, they are the classical Bernstein, Szász-Mirakjan and Baskakov operators, respectively. They can be applied to functions which are continuous on the underlying interval and, if $c \geq 0$, satisfy certain growth conditions.

For the approximation of integrable functions the Baskakov-Durrmeyer type operators play an important role. They have a lot of nice properties; they commute, they are self-adjoint, they commute with corresponding differential operators but they do not interpolate at finite endpoints of the underlying interval and only preserve constants.

For integrable functions having finite limits at finite endpoints of the interval one can apply the so-called genuine Baskakov-Durrmeyer type operators, which, like their classical counterparts, interpolate the function at finite endpoints of the interval and preserve linear functions.

In the last years a non-trivial link between classical Baskakov type operators and their genuine Durrmeyer type modification came into the focus of research.

It started with a paper by Păltănea [18], where he defined a non-trivial link between genuine Bernstein-Durrmeyer operators and classical Bernstein operators, depending on a positive real parameter ρ.

Linking operators for the Szász-Mirakjan case were defined by Păltănea in [19] and for Baskakov type operators by Heilmann and Raşa in [11].

In what follows for $c \in \mathbb{R}$ we use the notations

$$a^{c,\overline{j}} := \prod_{\ell=0}^{j-1}(a+c\ell), \quad a^{c,\underline{j}} := \prod_{\ell=0}^{j-1}(a-c\ell), \quad j \in \mathbb{N}, \quad a^{c,\overline{0}} = a^{c,\underline{0}} := 1$$

which enables us to state the results for the different operators in a unified form.

Let $c \in \mathbb{R}$, $n \in \mathbb{R}$, $n > c$ for $c \geq 0$ and $-n/c \in \mathbb{N}$ for $c < 0$. Furthermore let $\rho \in \mathbb{R}^+$, $j \in \mathbb{N}_0$, $x \in I_c$ with $I_c = [0, \infty)$ for $c \geq 0$ and $I_c = [0, -1/c]$ for $c < 0$. Then the basis functions are given by

$$p_{n,j}(x) = \begin{cases} \dfrac{n^j}{j!}\, x^j e^{-nx}, & c = 0, \\ \dfrac{n^{c,\overline{j}}}{j!}\, x^j (1+cx)^{-\left(\frac{n}{c}+j\right)}, & c \neq 0. \end{cases} \tag{1}$$

$$= \begin{cases} \dfrac{(-c)^{j+1}}{(n-c)B(j+1, -\frac{n}{c}-j+1)}\, x^j (1+cx)^{-\left(\frac{n}{c}+j\right)}, & c < 0, \\ \dfrac{n^j}{\Gamma(j+1)}\, x^j e^{-nx}, & c = 0, \\ \dfrac{c^{j+1}}{(n-c)B(j+1, \frac{n}{c}-1)}\, x^j (1+cx)^{-\left(\frac{n}{c}+j\right)}, & c > 0. \end{cases} \tag{2}$$

We remark that (2) is well defined also for $j \in \mathbb{R}$, $j \geq 0$, which will be considered below.

In the following definitions of the operators we omit the parameter c in the notations in order to reduce the necessary sub and superscripts.

We assume that $f : I_c \longrightarrow \mathbb{R}$ is given in such a way that the corresponding integrals and series are convergent.

Definition 1. The operators of Baskakov-type are defined by

$$(B_{n,\infty} f)(x) = \sum_{j=0}^{\infty} p_{n,j}(x) f\left(\frac{j}{n}\right),$$

and the genuine Baskakov-Durrmeyer type operators are denoted by

$$\begin{aligned} (B_{n,1} f)(x) &= f(0) p_{n,0}(x) + f\left(-\frac{1}{c}\right) p_{n,-\frac{n}{c}}(x) \\ &\quad + \sum_{j=1}^{-\frac{n}{c}-1} p_{n,j}(x)(n+c) \int_0^{-\frac{1}{c}} p_{n+2c,j-1}(t) f(t)\, dt \end{aligned}$$

for $c < 0$ and by

$$(B_{n,1} f)(x) = f(0) p_{n,0}(x) + \sum_{j=1}^{\infty} p_{n,j}(x)(n+c) \int_0^{\infty} p_{n+2c,j-1}(t) f(t)\, dt$$

for $c \geq 0$.

Depending on a parameter $\rho \in \mathbb{R}^+$ the linking operators are given by

$$(B_{n,\rho}f)(x) = \sum_{j=0}^{\infty} F_{n,j}^{\rho}(f) p_{n,j}(x)$$

where

$$F_{n,j}^{\rho}(f) = \begin{cases} f(0), & j = 0, \quad c \in \mathbb{R}, \\ f\big(-\frac{1}{c}\big), & j = -\frac{n}{c}, \quad c < 0, \\ \int\limits_{I_c} \mu_{n,j,\rho}(t) f(t)\, dt, & \text{otherwise}, \end{cases}$$

with

$$\mu_{n,j,\rho}(t) = \begin{cases} \dfrac{(-c)^{j\rho}}{B\big(j\rho, -(\frac{n}{c}+j)\rho\big)}\, t^{j\rho-1}(1+ct)^{-(\frac{n}{c}+j)\rho-1}, & c < 0, \\ \dfrac{(n\rho)^{j\rho}}{\Gamma(j\rho)}\, t^{j\rho-1} e^{-n\rho t}, & c = 0, \\ \dfrac{c^{j\rho}}{B\big(j\rho, \frac{n}{c}\rho+1\big)}\, t^{j\rho-1}(1+ct)^{-(\frac{n}{c}+j)\rho-1}, & c > 0. \end{cases}$$

Note that by (2) it is not necessary to use a different notation for the functions related to the linking parameter ρ, i. e.,

$$\mu_{n,j,\rho}(t) = (n\rho + c) p_{n\rho+2c, j\rho-1}(t).$$

For the special case $\rho \in \mathbb{N}$, this is also true with the usual representation (1) for the basis functions.

For $c < 0$, $B_{n,\rho}f$ is well defined if $f \in L_1[0,1]$ with finite limits at the endpoints of the interval $\big[0, -\frac{1}{c}\big]$, i.e.,

$$f(0) = \lim_{x\to 0^+} f(x) \qquad \text{and} \qquad f\Big(-\frac{1}{c}\Big) = \lim_{x\to(-1/c)^-} f(x).$$

For $c \geq 0$ the operators $B_{n,\rho}$ are well defined for functions $f \in W_n^{\rho}$ having a finite limit $f(0) = \lim_{x\to 0^+} f(x)$ where W_n^{ρ} denotes the space of functions $f \in L_{1,loc}[0,\infty)$ satisfying certain growth conditions, i. e., there exist constants $M > 0$, $0 \leq q < n\rho + c$, such that a. e. on $[0,\infty)$

$$|f(t)| \leq M e^{qt} \qquad \text{for } c = 0,$$
$$|f(t)| \leq M t^{q/c} \qquad \text{for } c > 0.$$

In [4, 11, 20] the k-th order Kantorovich modifications of the operators $B_{n,\rho}$ were considered, namely,

$$B_{n,\rho}^{(k)} := D^k \circ B_{n,\rho} \circ I_k, \qquad k \in \mathbb{N}_0,$$

where D^k denotes the k-th order ordinary differential operator and

$$I_0 f = f, \qquad (I_k f)(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)\, dt, \qquad k \in \mathbb{N}.$$

For $k = 0$ we omit the superscript (k) as indicated by the definition above.

These operators play an important role in the investigation of simultaneous approximation and this general definition contains many known operators as special cases. For $c = 0$ we get the k-th order Kantorovich modification of linking operators considered in [20]. For $\rho = 1$, $k \in \mathbb{N}$, we get the Baskakov-Durrmeyer type operators $B_{n,1}^{(1)}$ (see [17] for $c = 0$ and [8, (1.3)] for $c \geq 0$, named M_{n+c} there) and the auxiliary operators $B_{n,1}^{(k)}$ considered in [9, (3.5)], (named $M_{n+c,k-1}$ there).

2. Convergence to Classical Operators

The name linking operators is motivated by certain convergence properties to classical operators. Therefore, in this section, we summarize known results concerning the limit of the operators $B_{n,\rho}^{(k)}$ for $\rho \to \infty$.

Theorem 1 ([5, Theorem 2.3]). *Let $c = -1$. For every $f \in C[0,1]$,*

$$\lim_{\rho\to\infty} B_{n,\rho} f = B_{n,\infty} f$$

uniformly on $[0,1]$.

Theorem 2 ([20, Theorem 4]). *Let $c = 0$. Assume that $f : [0,\infty) \longrightarrow \mathbb{R}$ is integrable and there exist constants $M > 0$, $q \geq 0$ such that $|f(t)| \leq Me^{qt}$ for $t \in [0,\infty)$. Then for any $b > 0$ there is $\rho_0 > 0$, such that $B_{n,\rho} f$ exists for all $\rho \geq \rho_0$ and we have*

$$\lim_{\rho\to\infty} (B_{n,\rho} f)(x) = (B_{n,\infty} f)(x),$$

uniformly for $x \in [0,b]$.

The explicit representations for the images of polynomials for all operators $B_{n,\rho}^{(k)}$ led to the following result.

Theorem 3 ([11, Corollary 1]). *For each polynomial p we have*

$$\lim_{\rho\to\infty} \big(B_{n,\rho}^{(k)} p\big)(x) = \big(B_{n,\infty}^{(k)} p\big)(x),$$

uniformly on $[0,1]$ in case $c = -1$ and uniformly on every compact subinterval of $[0,\infty)$ in case $c \geq 0$.

A different function space was considered in [4] for the case $c \geq 0$.

Theorem 4 ([4, Lemma 5, Corollary 3]). *Let $f \in C^2[0,\infty)$ with the condition $\|f''\|_\infty < \infty$. Then we have*

$$\lim_{\rho\to\infty} (B_{n,\rho} f)(x) = (B_{n,\infty} f)(x),$$

uniformly on every compact subinterval of $[0,\infty)$.

Theorem 5 ([1, Theorem 2]). *Let $c,\gamma>0$. Assume that $f\in C[0,\infty)$ satisfies the growth condition $f(t)=O(t^\gamma)$ as $t\to\infty$. Then, for any $b>0$ there is a constant $\rho_0>0$ such that $B_{n,\rho}f$ exists for all $\rho\ge\rho_0$ and*

$$\lim_{\rho\to\infty}(B_{n,\rho}f)(x)=(B_{n,\infty}f)(x)$$

uniformly for $x\in[0,b]$.

3. Representation for *k*-th Order Kantorovich Modifications

For $\rho=1$ and $\rho=\infty$ nice explicit representations are well-known, i.e.,

$$\big(B_{n,1}^{(k)}f\big)(x)=\frac{n^{c,\overline{k}}}{n^{c,\underline{k-1}}}\sum_{j=0}^{\infty}p_{n+ck,j}(x)\int_0^\infty p_{n-c(k-2),j+k-1}(t)f(t)\,dt, \tag{3}$$

$$B_{n,\infty}^{(k)}(f;x)=n^{c,\overline{k}}\sum_{j=0}^{\infty}p_{n+ck,j}(x)\Delta_{1/n}^{k}I_k\Big(f;\frac{j}{n}\Big), \tag{4}$$

where the forward difference of order k with the step h for a function g is given by $\Delta_h^k g(x)=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}g(x+ih)$. For (3) see [9, (3.5)] with $B_{n,1}^{(k)}$ named $M_{n+c,k-1}$ there. To verify (4) one can use the same arguments as in [16] for the Bernstein operators, i.e., by substituting

$$p_{n,j}^{(k)}(x)=n^{c,\overline{k}}\sum_{\ell=0}^{\min(k,j)}(-1)^{k-\ell}\binom{\ell}{k}p_{n+ck,j-\ell}(x)$$

into $\dfrac{d^k}{dx^k}B_{n,\infty}(I_kf;x)$ and rearranging the terms to get

$$\frac{d^k}{dx^k}B_{n,\infty}(I_kf;x)=n^{c,\overline{k}}\sum_{j=0}^{\infty}p_{n+ck,j}(x)\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{k-\ell}I_k\Big(f;\frac{j+\ell}{n}\Big)$$

with

$$\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{k-\ell}I_k\Big(f;\frac{j+\ell}{n}\Big)=\Delta_{1/n}^{k}I_k\Big(f;\frac{j}{n}\Big).$$

By using Peano's representation theorem for divided differences (see, e.g., [21, p. 137]), (4) can also be written as

$$B_{n,\infty}^{(k)}(f;x)=\frac{n^{c,\overline{k}}}{n^{k-1}}\sum_{j=0}^{\infty}p_{n+kc,j}(x)\int_0^\infty N_{n,k,j}(t)f(t)\,dt,$$

where $N_{n,k,j}$ denotes the B-spline of order k with equidistant knots $\frac{j}{n}, \dots, \frac{j+k}{n}$, defined by

$$N_{n,1,j}(t) = \begin{cases} 1, & \frac{j}{n} < t < \frac{j+1}{n}, \\ \frac{1}{2}, & t \in \{\frac{j}{n}, \frac{j+1}{n}\}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

$$N_{n,k,j}(t) = \frac{n}{k-1}\Big\{\Big(t - \frac{j}{n}\Big)N_{n,k-1,j}(t) + \Big(\frac{j+k}{n} - t\Big)N_{n,k-1,j+1}(t)\Big\}. \tag{6}$$

In different papers [12, 14, 15] two of the authors proved the following representations in case $\rho \in \mathbb{N}$.

Theorem 6. *Let $n, k \in \mathbb{N}$, $n - k \geq 1$, $\rho \in \mathbb{N}$ and $f \in W_n^\rho$. Then we have the representation*

$$B_{n,\rho}^{(k)}(f;x) = \frac{n^{c,\overline{k}}}{(n\rho)^{c,\underline{k-1}}} \sum_{j=0}^{\infty} p_{n+kc,j}(x) \times \int_0^\infty \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} p_{n\rho-c(k-2),j\rho+i_1+\cdots+i_k+k-1}(t) f(t)\, dt.$$

Remark 1. Observe that for $k = 1$ the kernel $\sum_{i=0}^{\rho-1} p_{n\rho+c,i+j\rho}(t)$ can also be written in terms of classical Baskakov type operators applied to the characteristic functions $\chi_{\left[\frac{j\rho}{n\rho+c}, \frac{(j+1)\rho}{n\rho+c}\right)}$, i. e.,

$$\begin{aligned} B_{n\rho+c,\infty}\big(\chi_{\left[\frac{j\rho}{n\rho+c}, \frac{(j+1)\rho}{n\rho+c}\right)};t\big) &= \sum_{i=0}^{\infty} p_{n\rho+c,i}(t)\chi_{\left[\frac{j\rho}{n\rho+c}, \frac{(j+1)\rho}{n\rho+c}\right)}\Big(\frac{i}{n\rho+c}\Big) \\ &= \sum_{i=j\rho}^{(j+1)\rho-1} p_{n\rho+c,i}(t) = \sum_{i=0}^{\rho-1} p_{n\rho+c,i+j\rho}(t). \end{aligned}$$

4. Results for Functions with Certain Lipschitz Properties

In [5, Theorem 4.1] Gonska and Păltănea and [20, Theorem 6] Păltănea, respectively, considered convexity properties of the operators $B_{n,\rho}$ in case $c = -1$ and $c = 0$, respectively. In long and tricky proofs it was shown that for each $\rho \in \mathbb{R}$, $\rho > 0$, $(B_{n,\rho}f)^{(r)} \geq 0$ for each function $f \in C^r[0,1]$ and $f \in W_n^\rho \cap C^r[0,\infty)$, respectively, $f^{(r)} \geq 0$.

This was generalized to $B_{n,\rho}^{(k)}$, $c = 0$, $k \in \mathbb{N}$ (see [4, Theorem 3]).

Restricting the consideration to $\rho \in \mathbb{N}$ we know from the representation in Theorem 6 that all the operators $B_{n,\rho}^{(k)}$ are positive and the convexity properties for $B_{n,\rho}^{(k)}$, $\rho \in \mathbb{N}$ now follow as a corollary (see [12, Corollary 3], [14, Corollary 1].

Let I be an interval and $\beta > 0$. Define

$$Lip_\beta(I) := \left\{ f \in C(I) \,:\, |f(x) - f(y)| \le \beta |x-y|,\ x, y \in I \right\}.$$

Proposition 1. *Let $f \in C(I)$. Then $f \in Lip_1(I)$ iff the functions $e_1 + f$ and $e_1 - f$ are increasing.*

Proof.

$$\begin{aligned} f \in Lip_1(I) &\Longleftrightarrow |f(x) - f(y)| \le x - y,\ \forall x, y \in I,\ x > y \\ &\Longleftrightarrow y - x \le f(x) - f(y) \le x - y,\ \forall x, y \in I,\ x > y \\ &\Longleftrightarrow e_1 + f \text{ and } e_1 - f \text{ are increasing functions.} \end{aligned}$$

□

Proposition 2. *Let $L : C(I) \to C(I)$ be a positive linear operator such that $Le_0 = \alpha e_0$ and $Le_1 = \beta e_1 + \gamma e_0$ for some constants $\alpha > 0$, $\beta > 0$, $\gamma \in \mathbb{R}$. Suppose that L maps increasing functions into increasing functions. Then $L(Lip_1(I)) \subset Lip_\beta(I)$.*

Proof. Let $f \in Lip_1(I)$. Then $e_1 + f$ and $e_1 - f$ are increasing, and consequently $e_1 - \frac{\gamma}{\alpha} e_0 + f$ and $e_1 - \frac{\gamma}{\alpha} e_0 - f$ are increasing. It follows that $Le_1 - \frac{\gamma}{\alpha} Le_0 + Lf$ and $Le_1 - \frac{\gamma}{\alpha} Le_0 - Lf$ are also increasing, which means that $\beta e_1 + Lf$ and $\beta e_1 - Lf$ are increasing. Finally, $e_1 \pm \frac{1}{\beta} Lf$ are increasing, and Proposition 1 shows that $Lf \in Lip_\beta(I)$. □

Applying Proposition 2 to the operators $B_{n,\rho}^{(k)}$ leads to the following result.

Corollary 1. *$B_{n,\rho}^{(k)}(Lip_1(I_c)) \subset Lip_\beta(I_c)$, with $\beta = \dfrac{\rho^{k+1} n^{c,\overline{k+1}}}{(n\rho)^{c,\underline{k+1}}}$.*

Proof. For the operator $L := B_{n,\rho}^{(k)}$ we have (see [10, Corollary 1] and [11, Corollary 2]):

$$\alpha = \frac{\rho^k}{(n\rho)^{c,\underline{k}}} n^{c,\overline{k}}, \qquad \beta = \frac{\rho^{k+1} n^{c,\overline{k+1}}}{(n\rho)^{c,\underline{k+1}}}, \qquad \gamma = \frac{k(\rho+1)\rho^{k+1} n^{c,\overline{k}}}{2\rho (n\rho)^{c,\underline{k+1}}}.$$

Moreover, according to the results summarized at the beginning of this section, for $c \le 0$ and $\rho \in \mathbb{R}$, $\rho > 0$, respectively for $c > 0$ and $\rho \in \mathbb{N}$, $B_{n,\rho}^{(k)}$ transforms increasing functions into increasing functions. □

Remark 2. Proposition 2 when $\frac{\beta}{\alpha} < 1$ is useful in studying the iterates of the operator $\frac{1}{\alpha} L$ (see [2, 6, 13]). For $L = B_{n,\rho}^{(k)}$ the condition $\frac{\beta}{\alpha} < 1$ is equivalent to $c < 0$.

5. Remarks on Convergence to B-splines

In [12] the authors considered the topic for the linking Bernstein operators with a different approach. In [5, Theorem 2.3] Gonska and Păltănea proved the uniform convergence on $[0,1]$ of the linking Bernstein operators to the classical Bernstein operators for every function $f \in C[0,1]$. With explicit representations for the images of monomials and density arguments this can be extended to the k-th order Kantorovich modifications (see [12, Introduction]). From the representation of the k-th order Kantorovich modifications of the linking Bernstein operator and the classical Bernstein operator (see [12, Theorem 2, (2)]) one can derive immediately that

$$\lim_{\rho\to\infty} \frac{1}{\rho^{k-1}} \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} \tilde{p}_{n\rho+k-2,j\rho+i_1+\cdots+i_k+k-1}(t) = N_{n,k,j}(t)$$

for $t \in [0,1]$ with the Bernstein basis functions $\tilde{p}_{n,j}(t) = \binom{n}{j} t^j (1-t)^{n-j}$.

As the situation concerning the convergence of the operators in the Baskakov-type case is more complicated (see Theorems 2, 3 and 5) we can only conjecture that

$$\lim_{\rho\to\infty} \frac{1}{\rho^{k-1}} \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} p_{n\rho-c(k-2),j\rho+i_1+\cdots+i_k+k-1}(t) = N_{n,k,j}(t)$$

for $t \in [0,\infty)$.

We fortify our conjecture by the following illustrations (see Figure 1) where we choose $k=1$, $c=1$, $n=5$ and $j=1$.

Moreover, in the case $c=0$ we provide more evidence supporting our conjecture. So, consider $c=0$ and $k \in \{1,2,3\}$.

Case $k=1$: In this case,

$$p_{n,j}(x) = \frac{n^j}{j!} x^j e^{-nx}, \qquad x \in [0,\infty), \quad j \in \mathbb{N}_0,$$

and

$$S_{n\rho} f(x) = \sum_{i=0}^{\infty} p_{n\rho,i}(x) f\Big(\frac{i}{n\rho}\Big), \qquad \rho \in \mathbb{N}.$$

Let us remark that

$$\frac{j}{n} \le \frac{i}{n\rho} < \frac{j+1}{n} \iff j\rho \le i < (j+1)\rho.$$

Using (5) we get

$$S_{n\rho} N_{n,1,j} = \sum_{i=j\rho}^{(j+1)\rho-1} p_{n\rho,i} = \sum_{i_1=0}^{\rho-1} p_{n\rho,j\rho+i_1}.$$

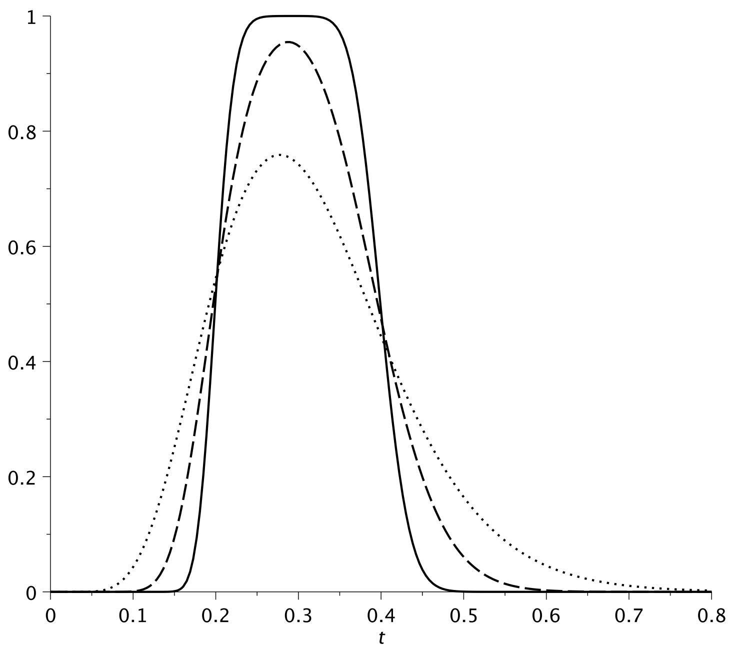

Figure 1. The dotted line belongs to $\rho = 10$, the dashed line to $\rho = 30$ and the solid line to $\rho = 150$.

It remains to prove that $S_{n\rho}N_{n,1,j}(x) \to N_{n,1,j}(x)$, for $x \geq 0$ and $\rho \to \infty$. To do this, we shall use the notation of [7], the inequality after (3.3) with $f = N_{n,1,j}$:

$$\left|S_{n\rho}(f;x) - \bar{f}(x)\right| \leq \frac{2x+1}{n\rho}\sum_{k=0}^{n\rho} V_{I_k}(g_x) + \frac{2(8x^2+6x+1)}{\sqrt{n\rho x}}\tilde{f}(x). \qquad (7)$$

Clearly, we may restrict to the case $x \in \left\{\frac{j}{n}, \frac{j+1}{n}\right\}$. For $x = \frac{j}{n}$, we get

$$\bar{f}\Big(\frac{j}{n}\Big) = \frac{1}{2}, \qquad \tilde{f}\Big(\frac{j}{n}\Big) = 1, \qquad g_{\frac{j}{n}}(t) = \begin{cases} 0, & t < \frac{j+1}{n}, \\ -1, & t \geq \frac{j+1}{n}. \end{cases}$$

But $\frac{j+1}{n} \leq \dfrac{j}{n} + \frac{1}{\sqrt{k}} \iff V_{I_k}(g_x) = 1 \iff k \leq n^2$. Therefore,

$$\left|S_{n\rho}(f;x) - \bar{f}(x)\right| \leq \frac{2x+1}{n\rho}(n^2+1) + \frac{2(8x^2+6x+1)}{\sqrt{n\rho x}},$$

and

$$\lim_{\rho\to\infty} S_{n\rho}\Big(f;\frac{j}{n}\Big) = \bar{f}\Big(\frac{j}{n}\Big) = \frac{1}{2}.$$

For $x = \frac{j+1}{n}$, it follows

$$g_{\frac{j+1}{n}}(t) = \begin{cases} 0, & t \geq \frac{j}{n}, \\ -1, & t < \frac{j}{n}, \end{cases}$$

and

$$\frac{j+1}{n} - \frac{1}{\sqrt{k}} < \frac{j}{n} \iff V_{I_k}(g_x) = 1 \iff k < n^2.$$

From (7) we obtain

$$\lim_{\rho\to\infty} S_{n\rho}\Big(f; \frac{j+1}{n}\Big) = \bar{f}\Big(\frac{j+1}{n}\Big) = \frac{1}{2},$$

and consequently

$$\sum_{i_1=0}^{\rho-1} p_{n\rho, j\rho+i_1} = S_{n\rho} N_{n,1,j} \longrightarrow N_{n,1,j} \qquad (\rho \to \infty).$$

Case $k = 2$*:* In this case

$$\frac{1}{\rho} \sum_{i_1=0}^{\rho-1} \sum_{i_2=0}^{\rho-1} p_{n\rho, j\rho+i_1+i_2+1} = \frac{1}{\rho} \left\{ \sum_{\ell=1}^{\rho} \ell\, p_{n\rho, j\rho+\ell} + \sum_{\ell=1}^{\rho-1} (\rho-\ell) p_{n\rho,(j+1)\rho+\ell} \right\},$$

and

$$N_{n,2,j}(t) = \begin{cases} nt - j, & \frac{j}{n} \leq t < \frac{j+1}{n}, \\ j + 2 - nt, & \frac{j+1}{n} \leq t < \frac{j+2}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

From

$$\frac{j}{n} \leq \frac{i}{n\rho} < \frac{j+1}{n} \iff j\rho \leq i < (j+1)\rho,$$

and

$$\frac{j+1}{n} \leq \frac{i}{n\rho} < \frac{j+2}{n} \iff (j+1)\rho \leq i < (j+2)\rho,$$

we get

$$\begin{aligned}
S_{n\rho}N_{n,2,j} &= \sum_{i=j\rho}^{(j+2)\rho-1} p_{n\rho,i}N_{n,2,j}\Big(\frac{i}{n\rho}\Big)\\
&= \sum_{i=j\rho+1}^{(j+1)\rho} p_{n\rho,i}N_{n,2,j}\Big(\frac{i}{n\rho}\Big) + \sum_{i=(j+1)\rho+1}^{(j+2)\rho-1} p_{n\rho,i}N_{n,2,j}\Big(\frac{i}{n\rho}\Big)\\
&= \sum_{\ell=1}^{\rho} p_{n\rho,j\rho+\ell}N_{n,2,j}\Big(\frac{j\rho+\ell}{n\rho}\Big) + \sum_{\ell=1}^{\rho-1} p_{n\rho,(j+1)\rho+\ell}N_{n,2,j}\Big(\frac{(j+1)\rho+\ell}{n\rho}\Big)\\
&= \sum_{\ell=1}^{\rho} p_{n\rho,j\rho+\ell}\frac{\ell}{\rho} + \sum_{\ell=1}^{\rho-1} p_{n\rho,(j+1)\rho+\ell}\Big(1-\frac{\ell}{\rho}\Big)\\
&= \frac{1}{\rho}\left\{\sum_{\ell=1}^{\rho}\ell\, p_{n\rho,j\rho+\ell} + \sum_{\ell=1}^{\rho-1}(\rho-\ell)p_{n\rho,(j+1)\rho+\ell}\right\}.
\end{aligned}$$

Therefore,

$$\frac{1}{\rho}\sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1} p_{n\rho,j\rho+i_1+i_2+1} = S_{n\rho}N_{n,2,j} \longrightarrow N_{n,2,j} \qquad (\rho\to\infty).$$

Case $k=3$*:* To simplify the notation, we shall replace $p_{n\rho,j\rho+\ell}$ by P_ℓ. Then, we get

$$\begin{aligned}
&\sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1}\sum_{i_3=0}^{\rho-1} p_{n\rho,j\rho+i_1+i_2+i_3+2}\\
&\qquad= \sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1}\sum_{i_3=0}^{\rho-1} P_{i_1+i_2+1+i_3+1}\\
&\qquad= \sum_{\ell=1}^{\rho}\sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1} P_{i_1+i_2+1+\ell}\\
&\qquad= \sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1} P_{i_1+i_2+1+1} + \sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1} P_{i_1+i_2+1+2} + \cdots + \sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1} P_{i_1+i_2+1+\rho}.
\end{aligned}$$

The following matrix of coefficients shows the number of occurrences of P_ℓ:

P_2	P_3	P_4	$\cdots$	$P_{\rho+1}$	$P_{\rho+2}$	$\cdots$	$P_{2\rho-1}$	$P_{2\rho}$	$P_{2\rho+1}$	$\cdots$	$P_{3\rho-2}$	$P_{3\rho-1}$
1	2	3	$\cdots$	ρ	$\rho-1$	$\cdots$	2	1	0	$\cdots$	0	0
0	1	2	$\cdots$	$\rho-1$	ρ	$\cdots$	3	2	1	$\cdots$	0	0
0	0	1	$\cdots$	$\rho-2$	$\rho-1$	$\cdots$	4	3	2	$\cdots$	0	0
0	0	0	$\cdots$	$\rho-3$	$\rho-2$	$\cdots$	5	4	3	$\cdots$	0	0
0	0	0	$\cdots$	$\rho-4$	$\rho-3$	$\cdots$	6	5	4	$\cdots$	0	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	0	0	$\cdots$	2	3	$\cdots$	ρ	$\rho-1$	$\rho-2$	$\cdots$	1	0
0	0	0	$\cdots$	1	2	$\cdots$	$\rho-1$	ρ	$\rho-1$	$\cdots$	2	1

Therefore,

$$
\begin{aligned}
&\sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1}\sum_{i_3=0}^{\rho-1} p_{n\rho,j\rho+i_1+i_2+i_3+2}\\
&= \frac{1\cdot 2}{2}\,P_2+\frac{2\cdot 3}{2}\,P_3+\frac{3\cdot 4}{2}\,P_4+\cdots+\frac{\rho(\rho+1)}{2}\,P_{\rho+1}\\
&\quad+\Big(\frac{(2+\rho)(\rho-1)}{2}+\rho-1\Big)P_{\rho+2}\\
&\quad+\Big(\frac{(3+\rho)(\rho-2)}{2}+\frac{[(\rho-1)+(\rho-2)]\cdot 2}{2}\Big)P_{\rho+3}\\
&\quad+\cdots+\Big(\frac{[(\rho-1)+\rho]\cdot 2}{2}+\frac{[2+(\rho-1)](\rho-2)}{2}\Big)P_{2\rho-1}\\
&\quad+\frac{\rho(\rho+1)}{2}\,P_{2\rho}+\frac{(\rho-1)\rho}{2}\,P_{2\rho+1}+\frac{(\rho-2)(\rho-1)}{2}\,P_{2\rho+2}\\
&\quad+\cdots+\frac{2\cdot 3}{2}\,P_{3\rho-2}+\frac{1\cdot 2}{2}\,P_{3\rho-1}\\
&= \frac{1}{2}\Bigg\{\sum_{\ell=1}^{\rho}\ell(\ell-1)p_{n\rho,j\rho+\ell}+\sum_{\ell=\rho+1}^{2\rho-1}\big[\ell(2\rho-\ell+1)+(3\rho-\ell)(\ell-\rho-1)\big]p_{n\rho,j\rho+\ell}\\
&\qquad+\sum_{\ell=2\rho}^{3\rho-1}(3\rho-\ell)(3\rho-\ell+1)p_{n\rho,j\rho+\ell}\Bigg\}.
\end{aligned}
$$

Denote

$$
\begin{aligned}
A_\rho := \frac{1}{2\rho^2}\Bigg\{&\sum_{\ell=1}^{\rho}\ell(\ell-1)p_{n\rho,j\rho+\ell}\\
&+\sum_{\ell=\rho+1}^{2\rho-1}\big[\ell(2\rho-\ell+1)+(3\rho-\ell)(\ell-\rho-1)\big]p_{n\rho,j\rho+\ell}\\
&+\sum_{\ell=2\rho}^{3\rho-1}(3\rho-\ell)(3\rho-\ell+1)p_{n\rho,j\rho+\ell}\Bigg\}.
\end{aligned}
$$

We have, using (6),

$$N_{n,3,j}(t) = \frac{n}{2}\Big\{\Big(t - \frac{j}{n}\Big)N_{n,2,j}(t) + \Big(\frac{j+3}{n} - t\Big)N_{n,2,j+1}(t)\Big\},$$

and so

$$N_{n,3,j}(t) = \frac{n^2}{2}\begin{cases}\big(t-\frac{j}{n}\big)^2, & \frac{j}{n} \le t < \frac{j+1}{n},\\ \big(t-\frac{j}{n}\big)\big(\frac{j+2}{n}-t\big) + \big(\frac{j+3}{n}-t\big)\big(t-\frac{j+1}{n}\big), & \frac{j+1}{n} \le t < \frac{j+2}{n},\\ \big(\frac{j+3}{n}-t\big)^2, & \frac{j+2}{n} \le t < \frac{j+3}{n},\\ 0, & \text{otherwise.}\end{cases}$$

From $\frac{j}{n} \le \frac{i}{n\rho} < \frac{j+3}{n} \iff j\rho \le i < (j+3)\rho$, we deduce that

$$\begin{aligned} S_{n\rho}N_{n,3,j} &= \sum_{i=0}^{\infty} p_{n\rho,i}N_{n,3,j}\Big(\frac{i}{n\rho}\Big) = \sum_{i=j\rho+1}^{(j+3)\rho-1} p_{n\rho,i}N_{n,3,j}\Big(\frac{i}{n\rho}\Big)\\ &= \sum_{\ell=1}^{3\rho-1} p_{n\rho,j\rho+\ell}N_{n,3,j}\Big(\frac{j\rho+\ell}{n\rho}\Big)\\ &= \frac{1}{2\rho^2}\Big\{\sum_{\ell=1}^{\rho}\ell^2 p_{n\rho,j\rho+\ell} + \sum_{\ell=\rho+1}^{2\rho-1}\big[\ell(2\rho-\ell)+(3\rho-\ell)(\ell-\rho)\big]p_{n\rho,j\rho+\ell}\\ &\qquad + \sum_{\ell=2\rho}^{3\rho-1}(3\rho-\ell)^2 p_{n\rho,j\rho+\ell}\Big\}, \end{aligned}$$

and thus

$$S_{n\rho}N_{n,3,j}-A_\rho=\frac{1}{2\rho^2}\Big\{\sum_{\ell=1}^{\rho}\ell\, p_{n\rho,j\rho+\ell}+\sum_{\ell=\rho+1}^{2\rho-1}(3\rho-2\ell)p_{n\rho,j\rho+\ell}+\sum_{\ell=2\rho}^{3\rho-1}(\ell-3\rho)p_{n\rho,j\rho+\ell}\Big\}.$$

On the other hand,

$$S_{n\rho}N_{n,2,j} = \frac{1}{\rho}\Big\{\sum_{\ell=1}^{\rho}\ell\, p_{n\rho,j\rho+\ell} + \sum_{\ell=\rho+1}^{2\rho-1}(2\rho-\ell)p_{n\rho,j\rho+\ell}\Big\},$$

and

$$\begin{aligned} S_{n\rho}N_{n,2,j+1} &= \frac{1}{\rho}\Big\{\sum_{\ell=1}^{\rho}\ell\, p_{n\rho,(j+1)\rho+\ell} + \sum_{\ell=\rho+1}^{2\rho-1}(2\rho-\ell)p_{n\rho,(j+1)\rho+\ell}\Big\}\\ &= \frac{1}{\rho}\Big\{\sum_{\ell=\rho+1}^{2\rho-1}(\ell-\rho)p_{n\rho,j\rho+\ell} + \sum_{\ell=2\rho}^{3\rho-1}(3\rho-\ell)p_{n\rho,j\rho+\ell}\Big\}. \end{aligned}$$

Therefore,

$$S_{n\rho}N_{n,3,j} - A_\rho = \frac{1}{2\rho^2}\left\{ \sum_{\ell=1}^{\rho} \ell\, p_{n\rho,j\rho+\ell} + \sum_{\ell=\rho+1}^{2\rho-1} (2\rho-\ell) p_{n\rho,j\rho+\ell} \right.$$
$$\left. + \sum_{\ell=\rho+1}^{2\rho-1} (\rho-\ell) p_{n\rho,j\rho+\ell} + \sum_{\ell=2\rho}^{3\rho-1} (\ell-3\rho) p_{n\rho,j\rho+\ell} \right\}$$
$$= \frac{1}{2\rho}\big(S_{n\rho}N_{n,2,j} - S_{n\rho}N_{n,2,j+1}\big).$$

Thus,

$$\frac{1}{\rho^2}\sum_{i_1=0}^{\rho-1}\sum_{i_2=0}^{\rho-1}\sum_{i_3=0}^{\rho-1} p_{n\rho,j\rho+i_1+i_2+i_3+2} \longrightarrow N_{n,3,j} \qquad (\rho\to\infty).$$

References

[1] U. Abel, M. Heilmann and V. Kushnirevych, Convergence of linking Baskakov-type operators, submitted.

[2] F. Altomare and I. Raşa, Lipschitz contraction, unique ergodicity and asymptotics of Markov semigroups, *Boll. Unione Mat. Ital.* **5** (2012), 1–17.

[3] V. A. Baskakov, An instance of a sequence of positive linear operators in the space of continuous functions, *Dokl. Akad. Nauk SSSR* **113** (1957), no. 2, 249–251.

[4] K. Baumann, M. Heilmann and I. Raşa, Further results for kth order Kantorovich modification of linking Baskakov type operators, *Results Math.* **69** (2016), no. 3, 297–315.

[5] H. Gonska and R. Păltănea, Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions, *Czech. Math. J.* **60** (2010), 783–799.

[6] H. Gonska, I. Raşa and M. D. Rusu, Applications of an Ostrowski-type inequality, *J. Comput. Anal. Appl.* **14(1)** (2012), 19–31.

[7] S.-S. Guo and M. K. Khan, On the rate of convergence of some operators on functions of bounded variation, *J. Approx. Theory* **58** (1989), 90–101.

[8] M. Heilmann, Direct and converse results for operators of Baskakov-Durrmeyer operators, *Approx. Theory Appl.* **5** (1989), no. 1, 105–127.

[9] M. Heilmann, "Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren", Habilitationschrift Universität Dortmund, 1992.

[10] M. Heilmann and I. Raşa, k-th order Kantorovich type modification of the operators $U_{n,\rho}$, *J. Appl. Funct. Anal.* **9** (2014), no. 3–4, 320–334.

[11] M. Heilmann and I. Raşa, k-th order Kantorovich modification of linking Baskakov type operators, *in:* "Recent Trends in Mathematical Analysis and its Applications" (P. N. Agrawal et al., Eds.), pp. 229–242, *Springer Proc. Math. Stat.*, vol. 143, Springer, 2015.

[12] M. Heilmann and I. Raşa, A nice representation for a link between Bernstein-Durrmeyer and Kantorovich operators, *in:* "Proceedings of ICMC 2017" (D. Giri et. al., Eds.), pp. 312–320, *Commun. Comput. Inf. Sci.*, vol. 655, Springer, 2017.

[13] M. Heilmann and I. Rasa, Eigenstructure and iterates for uniquely ergodic Kantorovich modifications of operators, *Positivity* **21** (2017), 897–910.

[14] M. Heilmann and I. Raşa, A nice representation for a link between Baskakov and Szsz-Mirakjan-Durrmeyer operators and their Kantorovich variants, *Results Math.* **74** (2019), no. 9, DOI: 10.1007/s00025-018-0932-4.

[15] M. Heilmann and I. Raşa, Note on a proof for the representation of the k-th order Kantorovich modification of linking Baskakov type operators, 2019 (submitted).

[16] G. G. Lorentz, "Bernstein Polynomials", Chelsea Publishing Company, New York, 1986.

[17] S. M. Mazhar and V. Totik, Approximation by modified Szász operators, *Acta Sci. Math.* **49** (1985), 257–269.

[18] R. Păltănea, A class of Durrmeyer type operators preserving linear functions, *Ann. Tiberiu Popoviciu Sem. Funct. Eq. Approx. Conv. (Cluj-Napoca)* **5** (2007), 109–117.

[19] R. Păltănea, Modified Szász-Mirakjan operators of integral form, *Carpathian J. Math.* **24(3)** (2008), 378–385.

[20] R. Păltănea, Simultaneous approximation by a class of Szász-Mirakjan operators, *J. Appl. Funct. Anal.* **9** (2014), no. 3–4, 356–368.

[21] L. L. Schumaker, "Spline Functions: Basic Theory", Cambridge University Press, 2007.

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