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Strong Converse Result for Weighted Approximation by Baskakov-Kantorovich Operator

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We study weighted approximation of functions in L_p-norm by a Kantorovich modification of the classical Baskakov operator, where the weight has the form $(1+x)^\alpha$, $\alpha<0$. Defining an appropriate K-functional we prove a strong converse inequality of type B for the rate of approximation.

Keywords and Phrases: Baskakov operator, K-functional, weighted approximation, Baskakov-Kantorovich operator.

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1. Introduction

The classical Baskakov operator [1] is defined for every bounded function $f(x)$ on the interval $[0,\infty)$ by the formula

$$V_n(f,x)=\sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) v_{n,k}(x), \tag{1}$$

where

$$v_{n,k}(x)=\binom{n+k-1}{k} x^k(1+x)^{-n-k}. \tag{2}$$

However, V_n is not suitable for approximation of functions in the L_p-norm because it is not a bounded operator in L_p. A kind of modification is needed to overcome this difficulty.

Analogously to the Bernstein-Kantorovich operator, replacing the function values by mean values, Ditzian and Totik [4] defined two Kantorovich type

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modifications of the classical Baskakov operator V_n. For $0 \le x < \infty$ they introduced

$$\tilde{V}_n^*(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x)\, n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)\, du$$

and

$$\tilde{V}_n(fx) = \sum_{k=0}^{\infty} v_{n,k}(x)\,(n-1) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(u)\, du. \tag{3}$$

The reason for introducing the latter operator is that the first one is not a contraction and therefore it is not very suitable for approximating of functions in L_p-norm for $p < \infty$. For the operator $\tilde{V}_n^*$ a direct and converse result of weak type (in accordance with the terminology of Ditzian and Ivanov [3]) were proved by Ditzian and Totik:

Theorem 1 ([4, Theorem 10.1.3]). *Let $w(x) = x^{\gamma(0)}(1+x)^{\gamma(\infty)}$ where $\gamma(\infty)$ is arbitrary and $-1/p < \gamma(0) < 1 - 1/p$ for $1 \le p \le \infty$, $wf \in L_p[0,\infty)$ and either $1 \le p \le \infty$ and $\alpha < 1$, or $1 < p < \infty$ and $\alpha \le 1$. Then for $\tilde{V}_n^*$ the following equivalency holds true*

$$\left\| w(\tilde{V}_n^* f - f) \right\|_p = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \left\| w \Delta^2_{h\sqrt{\psi}} f \right\|_{L_p[2h^2, 1-2h^2]} = O(h^{2\alpha}).$$

Here $\| \circ \|_p$ and $\| \circ \|_{p(J)}$ stand for the usual L_p-norm, respectively on $[0,\infty)$ and the interval J,

$$\psi(x) = x(1+x)$$

and

$$\Delta^2_{h\sqrt{\psi(x)}}(f,x) = f\big(x - h\sqrt{\psi(x)}\,\big) - 2f(x) + f\big(x + h\sqrt{\psi(x)}\,\big).$$

The same equivalency can be proved for $\tilde{V}_n$ with minor changes in the proof. In some sense, for the operator $\tilde{V}_n^*$ the inequalities of this type are the best. But for the operator $\tilde{V}_n$ the things are different. By defining an appropriate K-functional strong direct and converse inequalities can be proved. In [11] the first author proved strong (in the sense that there are no other terms except the K-functional) direct inequality. In this paper we prove strong converse inequalities of type B using the same K-functional. Hence, by combining the direct and converse inequalities we characterize exactly the weighted error of approximation of functions by the Baskakov-Kantorovich operator $\tilde{V}_n$.

Before stating these results, let us introduce some notation. The weights under consideration in this paper are

$$w(x) = (1+x)^{\alpha}, \qquad \alpha < 0. \tag{4}$$

The first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^k g(x) = g^{(k)}(x)$ for $k \in \mathbb{N}$.

We define a differential operator $\tilde{D}$ by the formula

$$\tilde{D} = \frac{d}{dx}\Big(\psi(x)\frac{d}{dx}\Big) = D\psi D, \qquad \text{where} \quad \psi(x) = x(1+x).$$

The space $AC_{loc}(0,\infty)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,\infty)$. For $1 \le p \le \infty$ we set

$$L_p(w) = \{f : f, Df \in AC_{loc}(0,\infty),\ wf \in L_p[0,\infty)\},$$
$$W_p(w) = \big\{f : f, Df \in AC_{loc}(0,\infty),\ w\tilde{D}f \in L_p[0,\infty),\ \lim_{x\to 0^+} \psi(x)Df(x) = 0\big\},$$
$$L_p(w) + W_p(w) = \big\{f : f = f_1 + f_2,\ f_1 \in L_p(w),\ f_2 \in W_p(w)\big\}.$$

Also, we define a K-functional $K_w(f,t)_p$ for $t > 0$ by

$$K_w(f,t)_p = \inf\big\{\|w(f-g)\|_p + t\|w\tilde{D}g\|_p : f-g \in L_p(w),\ g \in W_p(w)\big\}. \quad (5)$$

In [11] the following direct inequality was proved.

Theorem 2 ([11]). *For $1 \le p \le \infty$, w defined by (4), $\tilde{V}_n$ defined by (3), and the K-functional given by (5) there exists a positive constant c such that for every $n \in \mathbb{N}$ and for all functions $f \in L_p(w) + W_p(w)$ there holds*

$$\big\|w\big(\tilde{V}_n f - f\big)\big\|_p \le cK_w\Big(f, \frac{1}{n}\Big)_p. \quad (6)$$

Our main result in this paper is the following strong converse inequality of type B (in the classification of Ditzian and Ivanov).

Theorem 3. *For $1 < p \le \infty$, $n \in \mathbb{N}$, w defined by (4), $\tilde{V}_n$ defined by (3), the K-functional given by (5), there exist absolute positive constants c and R, such that for every function $f \in L_p(w) + W_p(w)$ the inequality*

$$K_w\Big(f, \frac{1}{n}\Big)_p \le c\,\frac{\ell}{n}\big(\big\|w\big(\tilde{V}_n f - f\big)\big\|_p + \big\|w\big(\tilde{V}_\ell f - f\big)\big\|_p\big), \quad (7)$$

holds true for all $\ell \ge Rn$.

Combining Theorem 2 and Theorem 3 we immediately conclude:

Corollary 1. *Under the assumptions of Theorem 3 there exists $k \in \mathbb{N}$ such that*

$$K_w\Big(f, \frac{1}{n}\Big)_p \sim \big\|w(\tilde{V}_n f - f)\big\|_p + \big\|w(\tilde{V}_{kn} f - f)\big\|_p, \qquad p > 1.$$

Remark 1. In a series of papers [5, 6, 7], Draganov and Ivanov introduced new moduli of smoothness by which they characterize the K-functional

$$\tilde{K}_w(f,t)_p = \inf\big\{\|w(f-g)\|_p + t\|w\psi D^2 g\|_p : f-g \in L_p(w),\ \psi D^2 g \in L_p(w)\big\}.$$

In [10], the first author proved that for $1 < p < \infty$ the same moduli can be used in order to compute $K_w(f,t)_p$. But they cannot be used for $p = 1$ and $p = \infty$ in which cases new moduli of smoothness are needed.

Throughout this paper, c is assumed to be an absolute positive constant, which means it does not depend on f and n. Also, it may be different on each occurrence. The relation $\theta_1(f,t) \sim \theta_2(f,t)$ means that there exists a constant $C > 1$, independent of f and t, such that

$$C^{-1}\theta_1(f,t) \le \theta_2(f,t) \le C\,\theta_1(f,t).$$

2. Auxiliary Results

We start this section with some properties of the operators V_n, $\tilde{V}_n$ and the basis functions $v_{n,k}$, defined in (1), (3) and (2), respectively (see [1, 4, 8]):

Proposition 1. *(i)* V_n *is a positive linear operator with* $\|V_n f\|_\infty \le \|f\|_\infty$*;*
(ii) For all $x \in [0,\infty)$,

$$V_n(1,x) = 1, \qquad V_n(t-x,x) = 0, \qquad V_n\big((t-x)^2,x\big) = \frac{\psi(x)}{n}. \tag{8}$$

Proposition 2. *(i)* $\tilde{V}_n$ *is a positive linear operator with* $\|\tilde{V}_n\|_1 = 1$ *and*

$$\|\tilde{V}_n\|_p \le 1 \quad \textit{for} \quad p > 1; \tag{9}$$

(ii) For all $x \in [0,\infty)$,

$$\tilde{V}_n(1,x) = 1, \quad \tilde{V}_n(t-x,x) = \frac{2x+1}{2(n-1)} = \frac{1}{2(n-1)}D\psi(x), \tag{10}$$

$$\tilde{V}_n\big((t-x)^2,x\big) = \frac{3(n+1)\psi(x)+1}{3(n-1)^2}. \tag{11}$$

Proposition 3. *(i) We have*

$$Dv_{n,k}(x) = n\big(v_{n+1,k-1}(x) - v_{n+1,k}(x)\big) = \frac{n}{\psi(x)}\Big(\frac{k}{n} - x\Big)v_{n,k}(x); \tag{12}$$

(ii) For each integer m*, there exist constants* c*, depending only on* m*, such that*

$$\sum_{k=1}^{\infty}\Big(\frac{n}{k}\Big)^m v_{n,k}(x) \le cx^{-m}, \qquad x \in [0,\infty),$$

$$\sum_{k=0}^{\infty}\Big(1+\frac{k}{n}\Big)^m v_{n,k}(x) \le c(1+x)^m, \qquad x \in [0,\infty), \tag{13}$$

$$V_n\big((t-x)^{2m},x\big) \le c\Big(\frac{\psi(x)}{n}\Big)^m, \qquad x \ge \frac{1}{n}, \tag{14}$$

It was shown in [4] that the following representation holds true for every $m \in \mathbb{N}$:

$$D^m \tilde{V}_n(f,x) = \frac{(n+m-1)!}{(n-1)!} \sum_{k=0}^{\infty} \Delta^m a_{k,n-1}(f)\, v_{n+m,k}(x), \tag{15}$$

where

$$a_{k,n}(f) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)\,du, \quad \Delta a_{k,n} = a_{k+1,n} - a_{k,n}, \quad \Delta^m a_{k,n} = \Delta\big(\Delta^{m-1} a_{k,n}\big).$$

Also, by simple computations one can verify the identities:

$$\begin{aligned} \int_0^\infty v_{n,k}(x)\,dx &= \frac{1}{n-1}, \\ \int_0^\infty x v_{n,k}(x)\,dx &= \frac{k+1}{(n-1)(n-2)}, \\ \int_0^\infty x^2 v_{n,k}(x)\,dx &= \frac{(k+1)(k+2)}{(n-1)(n-2)(n-3)}. \end{aligned} \tag{16}$$

As a consequence we have

$$\int_0^\infty \Big(x - \frac{k}{n}\Big)^2 v_{n,k}(x)\,dx = \frac{1}{n^2}\, \psi\Big(\frac{k}{n}\Big) + O\Big(\frac{1}{n^3}\Big). \tag{17}$$

We will make use of the inequality (see [4, Section 10.3, p. 165]):

$$\|w\psi D^2 \tilde{V}_n f\|_p \le cn\|wf\|_p \qquad \text{for} \quad f \in L_p[0,\infty). \tag{18}$$

Below we prove a series of preliminary results.

Lemma 1. *For $\alpha \in \mathbb{R}$ there exists a constant c such that for every natural number $n \ge |\alpha|$ and $v_{n,k}$ defined by (2),*

$$\sum_{k=0}^{\infty} \Big(1 + \frac{k}{n}\Big)^\alpha v_{n,k}(x) \le c(1+x)^\alpha, \qquad x \in [0,\infty). \tag{19}$$

Proof. Let $m \in \mathbb{N}$ be the smallest integer such that $m > |\alpha|$. By Hölder's inequality we have

$$\sum_{k=0}^{\infty} v_{n,k}(x)\Big(1+\frac{k}{n}\Big)^\alpha \le \left\{\sum_{k=0}^{\infty} v_{n,k}(x)\Big(1+\frac{k}{n}\Big)^{m \operatorname{sign}\alpha}\right\}^{|\alpha|/m} \left\{\sum_{k=0}^{\infty} v_{n,k}(x)\right\}^{1-|\alpha|/m}$$

and the lemma follows from (13) and (8). $\square$

Lemma 2. *For arbitrary $\alpha \in \mathbb{R}$ the following inequality holds true:*

$$\int_0^\infty (1+x)^\alpha v_{n,k}(x)\,dx \le \frac{c}{n}\Big(1+\frac{k}{n}\Big)^\alpha. \tag{20}$$

Proof. If $s \in \mathbb{N}$, by using (16) we obtain

$$\begin{aligned}\int_0^\infty (1+x)^s v_{n,k}(x)\,dx &= \frac{(n+k-1)\cdots(n+k-s)}{(n-1)\cdots(n-s)}\int_0^\infty v_{n-s,k}(x)dx \\ &= \frac{1}{n-s-1}\prod_{j=1}^{s}\Big(1+\frac{k}{n-j}\Big) \\ &\le \frac{c}{n}\Big(1+\frac{k}{n}\Big)^s.\end{aligned}$$

For a negative integer s, (16) yields

$$\begin{aligned}\int_0^\infty (1+x)^s v_{n,k}(x)\,dx &= \frac{(n-s-1)\cdots n}{(n-s+k-1)\cdots(n+k)}\int_0^\infty v_{n-s,k}(x)dx \\ &= \frac{1}{n-s-1}\prod_{j=1}^{-s}\frac{1}{1+\frac{k}{n-j+1}} \\ &\le \frac{c}{n}\Big(1+\frac{k}{n}\Big)^s.\end{aligned}$$

Therefore, for any integer s,

$$\int_0^\infty (1+x)^s v_{n,k}(x)\,dx \le \frac{c}{n}\Big(1+\frac{k}{n}\Big)^s. \tag{21}$$

Let m be the smallest natural number such that $m > |\alpha|$. By Hölder's inequality we have

$$\begin{aligned}&\int_0^\infty (1+x)^\alpha v_{n,k}(x)\,dx \\ &\quad\le \Big\{\int_0^\infty (1+x)^{m\,\mathrm{sign}\,\alpha} v_{n,k}(x)\,dx\Big\}^{|\alpha|/m}\Big\{\int_0^\infty v_{n,k}(x)\,dx\Big\}^{1-|\alpha|/m}.\end{aligned}$$

Then, (21) with $s = m\,\mathrm{sign}\,\alpha$ yields

$$\int_0^\infty (1+x)^\alpha v_{n,k}(x)\,dx \le \Big\{\frac{c}{n}\Big(1+\frac{k}{n}\Big)^{m\,\mathrm{sign}\,\alpha}\Big\}^{|\alpha|/m}\Big(\frac{1}{n-1}\Big)^{1-|\alpha|/m} \le \frac{c}{n}\Big(1+\frac{k}{n}\Big)^\alpha.$$

The constant c in inequality (20) depends only on α. □

The next lemma is proved in [4].

Lemma 3 ([4, p. 161]). *For $1 \le p \le \infty$ and for all $f \in L_p(w)$ we have*

$$\big\|w\tilde{V}_n f\big\|_p \le c\|wf\|_p. \tag{22}$$

In order to prove the converse inequality of Theorem 3 we apply a method, suggested by Ditzian and Ivanov in [3, Theorems 3.1 and 3.2]. Their approach is based on using the second iteration of the operator $\tilde{V}_n f$ in the weighted K-functional. Two further lemmas follow below.

Lemma 4. *For $p > 1$ and every $g \in W_p(w)$ we have*

$$\|wDg\|_p \le c\big\|w\tilde{D}g\big\|_p, \tag{23}$$

$$\|wD\psi \cdot Dg\|_p \le c\big\|w\tilde{D}g\big\|_p, \tag{24}$$

$$\|w\psi D^2 g\|_p \le c\big\|w\tilde{D}g\big\|_p. \tag{25}$$

Proof. Firstly, we prove (24). We have

$$\begin{aligned}|w(x)D\psi(x) \cdot Dg(x)| &\le 2w(x)(1+x)|Dg(x)| = 2w(x)\frac{1}{x}\,|\psi(x)Dg(x)| \\ &= 2w(x)\frac{1}{x}\int_0^x |\tilde{D}g(t)|\,dt \le \frac{2}{x}\int_0^x |w(t)\tilde{D}g(t)|\,dt.\end{aligned}$$

Then by using the classical Hardy's inequality we obtain

$$\begin{aligned}\int_0^\infty |w(x)D\psi(x) \cdot Dg(x)|^p\,dx &\le c\int_0^\infty \Big(\frac{1}{x}\int_0^x |w(t)\tilde{D}g(t)|\,dt\Big)^p dx \\ &\le c\int_0^\infty |w(x)\tilde{D}g(x)|^p\,dx,\end{aligned}$$

i.e.

$$\|wD\psi \cdot Dg\|_p \le c\|w\tilde{D}g\|_p.$$

Here, the constant c depends only on p.

Obviously, $|w(x)Dg(x)| \le |w(x)D\psi \cdot Dg(x)|$ for $x \in [0,\infty)$, hence the inequality (24) yields (23).

Finally, the inequality (25) follows from (24) and

$$\begin{aligned}\|w\psi D^2 g\|_p &\le \|w(\psi D^2 g + D\psi \cdot Dg)\|_p + \|wD\psi \cdot Dg\|_p \\ &= \|w\tilde{D}g\|_p + \|wD\psi \cdot Dg\|_p.\end{aligned}$$

□

Lemma 5. *For $1 \le p \le \infty$ we have*

$$\big\|wD\tilde{V}_n f\big\|_p \le cn\|wf\|_p \qquad \text{for } f \in L_p(w), \tag{26}$$

$$\big\|wD\tilde{V}_n f\big\|_p \le c\|wDf\|_p \qquad \text{for } f, Df \in L_p(w),$$

$$\big\|wD^2\tilde{V}_n f\big\|_p \le cn\|wDf\|_p \qquad \text{for } f, Df \in L_p(w), \tag{27}$$

$$\big\|wD^3\tilde{V}_n f\big\|_p \le cn^2\|wDf\|_p \qquad \text{for } f, Df \in L_p(w). \tag{28}$$

Proof. The proofs are analogous so we prove only (27). First we will show that the inequality (27) holds for $p = 1$ and $p = \infty$. Then by applying the Riesz-Thorin theorem we extend the inequality for every $p \geq 1$.

Let $p = 1$. From (15) we have

$$D^2 \tilde{V}_n(f, x) = n(n+1) \sum_{k=0}^{\infty} \Delta^2 a_{k,n-1}(f)\, v_{n+2,k}(x).$$

Consequently, by (20) we obtain

$$\begin{aligned} \left\| w D^2 \tilde{V}_n f \right\|_1 &\leq 2n^2 \left\| w(x) \sum_{k=0}^{\infty} \Delta^2 a_{k,n-1}(f)\, v_{n+2,k}(x) \right\|_1 \\ &\leq cn^2 \sum_{k=0}^{\infty} |\Delta^2 a_{k,n-1}(f)| \int_0^{\infty} w(x) v_{n+2,k}(x)\, dx \\ &\leq cn \sum_{k=0}^{\infty} |\Delta a_{k,n-1}(f)|\, w\Big(\frac{k}{n}\Big). \end{aligned}$$

Also,

$$\begin{aligned} |\Delta a_{k,n-1}(f)| &= (n-1) \Big| \int_{\frac{k+1}{n-1}}^{\frac{k+2}{n-1}} f(t)\, dt - \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)\, dt \Big| \\ &= (n-1) \Big| \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} \Big(\int_0^{\frac{1}{n-1}} Df(u+v)\, dv \Big) du \Big| \\ &\leq \int_{\frac{k}{n-1}}^{\frac{k+2}{n-1}} |Df(u)|\, du \\ &= \int_{\frac{k}{n-1}}^{\frac{k+2}{n-1}} \frac{|w(u) Df(u)|}{w(u)}\, du \\ &\leq \frac{c}{w\big(\frac{k}{n}\big)} \int_{\frac{k}{n-1}}^{\frac{k+2}{n-1}} |w(u) Df(u)|\, du. \end{aligned}$$

Therefore,

$$\left\| w D^2 \tilde{V}_n f \right\|_1 \leq cn \sum_{k=0}^{\infty} \int_{\frac{k}{n-1}}^{\frac{k+2}{n-1}} |w(u) Df(u)|\, du \leq cn \|w Df\|_1.$$

Now, we consider the case $p = \infty$. As above we obtain

$$|\Delta a_{k,n-1}(f)| \leq \frac{c}{w\big(\frac{k}{n}\big)} \int_{\frac{k}{n-1}}^{\frac{k+2}{n-1}} |w(u) Df(u)|\, du \leq \frac{2c}{n-1} \Big(1 + \frac{k}{n}\Big)^{-\alpha} \|w Df\|_{\infty}.$$

Similarly to the case $p = 1$, for every $x \in [0, \infty)$ we have

$$\begin{aligned} |w(x)D^2\tilde{V}_n(f,x)| &\le cn^2 w(x) \sum_{k=0}^{\infty} |\Delta^2 a_{k,n-1}(f)| v_{n+2,k}(x) \\ &\le cn^2 w(x) \sum_{k=0}^{\infty} |\Delta a_{k,n-1}(f)| v_{n+2,k}(x) \\ &\le cn\|wDf\|_\infty\, w(x) \sum_{k=0}^{\infty} \Big(1+\frac{k}{n}\Big)^{-\alpha} v_{n+2,k}(x). \end{aligned}$$

Then, applying Lemma 1 it follows that

$$|w(x)D^2\tilde{V}_n(f,x)| \le cn\|wDf\|_\infty\,(1+x)^{\alpha}(1+x)^{-\alpha}.$$

Hence $\big\|wD^2\tilde{V}_n f\big\|_\infty \le cn\|wDf\|_\infty$ and the lemma is proved. □

We complete this section with the proof of two more inequalities which are important of their own. The first one is a Voronovskaya type of inequality.

Lemma 6. *If $p > 1$ and $n \in \mathbb{N}$, then the inequality*

$$\begin{aligned} &\Big\|w\Big(\tilde{V}_n f - f - \frac{1}{2(n-1)}\tilde{D}f\Big)\Big\|_p \\ &\qquad \le c\bigg(\frac{\big\|w\psi^{3/2}D^3 f\big\|_p}{n^{3/2}} + \frac{\big\|wD^2 f\big\|_p}{n^2} + \frac{\big\|w\psi D^2 f\big\|_p}{n^2} + \frac{\big\|wD^3 f\big\|_p}{n^3}\bigg) \end{aligned} \tag{29}$$

holds true for every $f \in C^3[0,\infty)$ for which the right-hand side is finite.

Proof. By Taylor's formula, for $x \in [0, \infty)$ we have

$$f(u) = f(x) + (u-x)Df(x) + \frac{1}{2}\,(u-x)^2 D^2 f(x) + R(f,x;u), \tag{30}$$

where

$$R(f,x;u) = \frac{1}{2}\int_x^u (u-v)^2 D^3 f(v)\,dv.$$

Having in mind properties (10) and (11) we apply the operator $\tilde{V}_n$ to the functions in both sides of identity (30) and obtain

$$\tilde{V}_n(f,x) = f(x) + \frac{D\psi(x)}{2(n-1)}\,Df(x) + \frac{3(n+1)\psi(x)+1}{6(n-1)^2}\,D^2 f(x) + \tilde{V}_n\big(R(f,x;\cdot),x\big).$$

Hence,

$$\begin{aligned} &w(x)\Big(\tilde{V}_n(f,x) - f(x) - \frac{\tilde{D}f(x)}{2(n-1)}\Big) \\ &\qquad = \frac{\psi(x)+1/6}{(n-1)^2}\,w(x)D^2 f(x) + w(x)\tilde{V}_n\big(R(f,x;\cdot),x\big). \end{aligned} \tag{31}$$

Now we estimate the last term in the RHS of (31).

Case 1: $0 \le x < 1/n$. In this case $w(x) \sim 1$ and we proceed as in [9, p. 392] only replacing $D^3 f$ by $wD^3 f$ to obtain

$$\big|w(x)\tilde{V}_n\big(R(f,x;\cdot),\,x\big)\big| \le \frac{c}{n^3}\,M(wD^3 f,x), \tag{32}$$

where

$$M(h,x) = \sup_{u>x} \frac{1}{u-x}\int_x^u |h(t)|\,dt$$

is the Hardy maximal function.

Case 2: $x \ge 1/n$. By using Cauchy's inequality, (19) and (14) we obtain for $m = 0,1,2,3$:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{w(x)}{w\big(\frac{k}{n}\big)}\Big|x-\frac{k}{n}\Big|^m v_{n,k}(x)\\
&\quad\le \bigg(\sum_{k=0}^{\infty}\frac{w^2(x)}{w^2\big(\frac{k}{n}\big)}\,v_{n,k}(x)\bigg)^{1/2}\bigg(\sum_{k=0}^{\infty}\Big(x-\frac{k}{n}\Big)^{2m} v_{n,k}(x)\bigg)^{1/2}\\
&\quad\le c\Big(\frac{\psi(x)}{n}\Big)^{m/2}.
\end{aligned}$$

Taking into account that $\frac{1}{\psi(x)} \le n$ for $x \ge 1/n$ it easily follows that

$$\psi^{-3/2}(x)\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n}\big)}\Big(\Big|x-\frac{k}{n}\Big|+\frac{1}{n}\Big)^3 v_{n,k}(x) \le \frac{c}{n^{3/2}}. \tag{33}$$

We consider two subcases.

(a) If $u \ge x$, then

$$\begin{aligned}
&\Big|\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u (u-v)^2|w(v)D^3 f(v)|\,dv\Big)du\Big|\\
&\quad= \sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u \frac{(u-v)^2}{\psi^{3/2}(v)}|w(v)\psi^{3/2}(v)D^3 f(v)|\,dv\Big)du\\
&\quad\le \sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u \frac{(u-v)^2}{\psi^{3/2}(x)}|w(v)\psi^{3/2}(v)D^3 f(v)|\,dv\Big)du\\
&\quad= \sum_{k=0}^{\infty}\psi^{-3/2}(x)\frac{w(x)}{w\big(\frac{k}{n}\big)}\,v_{n,k}(x)\\
&\qquad\qquad\times\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(u-x)^3\Big|\frac{1}{u-x}\int_x^u |w(v)\psi^{3/2}(v)D^3 f(v)|\,dv\Big|du
\end{aligned}$$

$$\le \Bigg(\sum_{k=0}^{\infty}\psi^{-3/2}(x)\frac{w(x)}{w\big(\frac{k}{n}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(u-x)^3du\Bigg)\,M\big(w\psi^{3/2}D^3f,x\big)$$

$$\le \frac{c}{n-1}\,M\big(w\psi^{3/2}D^3f,x\big)\psi^{-3/2}(x)\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n}\big)}\Big(\Big|x-\frac{k}{n}\Big|+\frac{1}{n}\Big)^3 v_{n,k}(x).$$

Inequality (33) yields that

$$\Big|\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u(u-v)^2|w(v)D^3f(v)|\,dv\Big)du\Big|$$
$$\le \frac{c}{(n-1)n^{3/2}}M\big(w\psi^{3/2}D^3f,x\big). \tag{34}$$

Therefore

$$\big|w(x)\tilde V_n\big(R(f,x;\cdot),\,x\big)\big|$$
$$\le \frac{n-1}{2}\Big|\sum_{k=0}^{\infty}w(x)v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u(u-v)^2\frac{|w(v)D^3f(v)|}{w(v)}\,dv\Big)du\Big|$$
$$\le \frac{n-1}{2}\Big|\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k+1}{n-1}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u(u-v)^2|w(v)D^3f(v)|\,dv\Big)du\Big|$$
$$\le \frac{n-1}{2}\Big|\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u(u-v)^2|w(v)D^3f(v)|\,dv\Big)du\Big|$$

and from (34) it follows that

$$\big|w(x)\tilde V_n\big(R(f,x;\cdot),\,x\big)\big|\le\frac{c}{n^{3/2}}M\big(w\psi^{3/2}D^3f,x\big). \tag{35}$$

(b) $u<x$. Observe that for $v\in[u,x]$ we have $\frac{(u-v)^2}{v^{3/2}}\le\frac{(x-u)^2}{x^{3/2}}$. Then

$$\Bigg|\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n-1}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_x^u(u-v)^2|w(v)D^3f(v)|\,dv\Big)du\Bigg|$$
$$\le\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n-1}\big)}\,v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\Big(\int_u^x\frac{(u-v)^2}{\psi^{3/2}(v)}|w(v)\psi^{3/2}(v)D^3f(v)|\,dv\Big)du$$
$$\le\sum_{k=0}^{\infty}\frac{w(x)}{w\big(\frac{k}{n-1}\big)}\,\frac{v_{n,k}(x)}{x^{3/2}}$$
$$\times\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}\frac{1}{(1+u)^{3/2}}\int_u^x(u-v)^2|w(v)\psi^{3/2}(v)D^3f(v)|\,dv\,du$$

$$\begin{aligned}
&\le \sum_{k=0}^{\infty} \frac{w(x)}{w\big(\frac{k}{n-1}\big)}\, \frac{v_{n,k}(x)}{x^{3/2}} \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} \frac{(x-u)^2}{(1+u)^{3/2}} \int_u^x |w(v)\psi^{3/2}(v)D^3 f(v)|\, dv\, du\\
&\le M\big(w\psi^{3/2}D^3 f, x\big)\frac{w(x)}{x^{3/2}} \sum_{k=0}^{\infty} \frac{v_{n,k}(x)}{w\big(\frac{k}{n-1}\big)} \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} \frac{(x-u)^3}{(1+u)^{3/2}}\, du\\
&\le M\big(w\psi^{3/2}D^3 f, x\big)\frac{w(x)}{x^{3/2}} \sum_{k=0}^{\infty} \frac{v_{n,k}(x)}{w\big(\frac{k}{n-1}\big)} \Big(1+\frac{k}{n-1}\Big)^{-3/2}\Big(\Big|x-\frac{k}{n}\Big|+\frac{1}{n}\Big)^3\\
&= M\big(w\psi^{3/2}D^3 f, x\big)\frac{w(x)}{x^{3/2}} \sum_{k=0}^{\infty} v_{n,k}(x)\Big(1+\frac{k}{n-1}\Big)^{-\alpha-3/2}\Big(\Big|x-\frac{k}{n}\Big|+\frac{1}{n}\Big)^3.
\end{aligned}$$

Now, by using Cauchy's inequality, Lemma 1 and (14) we obtain

$$\begin{aligned}
&\frac{w(x)}{x^{3/2}} \sum_{k=0}^{\infty} v_{n,k}(x)\Big(1+\frac{k}{n-1}\Big)^{-\alpha-3/2}\Big(\Big|x-\frac{k}{n}\Big|+\frac{1}{n}\Big)^3\\
&\quad\le \frac{w(x)}{x^{3/2}}\left\{\sum_{k=0}^{\infty} v_{n,k}(x)\Big(1+\frac{k}{n-1}\Big)^{-2\alpha-3}\right\}^{1/2}\\
&\qquad\times\left\{\sum_{k=0}^{\infty} v_{n,k}(x)\Big(\Big|x-\frac{k}{n}\Big|+\frac{1}{n}\Big)^6\right\}^{1/2}\\
&\quad\le \frac{w(x)}{x^{3/2}}\,(1+x)^{-\alpha-3/2}\Big(\frac{\psi(x)}{n}\Big)^{3/2}\\
&\quad\le \frac{c}{n^{3/2}}.
\end{aligned}$$

So, in subcase (b) we proved the same estimation as (35) in subcase (a):

$$\begin{aligned}
&\big|w(x)\tilde{V}_n\big(R(f,x;\cdot),\,x\big)\big|\\
&\quad=\left|\sum_{k=0}^{\infty} \frac{w(x)}{w\big(\frac{k}{n-1}\big)}\, v_{n,k}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} \Big(\int_x^u (u-v)^2 |w(v)D^3 f(v)|\, dv\Big) du\right|\\
&\quad\le \frac{c}{n^{3/2}} M\big(w\psi^{3/2}D^3 f, x\big).
\end{aligned}$$

From (32) and (35), by applying the Hardy inequality about the maximal function (for $p>1$) we obtain the estimation

$$\big\|w(x)\tilde{V}_n\big(R(f,x;\cdot),\,x\big)\big\|_p \le \frac{c}{n^{3/2}}\big\|w\psi^{3/2}D^3 f\big\|_p + \frac{c}{n^3}\big\|wD^3 f\big\|_p.$$

Hence, the statement of the lemma follows immediately from (31). □

Remark 2. The assertion of Lemma 6 is valid for a wider class of functions, but this form is quite enough for our purpose. We will apply it for the second iteration operator $\tilde{V}_n^2 f$ and $f \in L_p(w)+W_p(w)$. For the proof of Theorem 3 we will show that $\tilde{V}_n^2 f$ satisfies the condition of Lemma 6 for all such functions f.

The last preliminary result we need is the following Bernstein type inequality.

Lemma 7. *Let $1 \le p \le \infty$ and $n \in \mathbb{N}$. If $f \in W_p(w)$ there exists an absolute constant $c > 0$ such that*

$$\big\|w\psi^{3/2}D^3\tilde{V}_n f\big\|_p \le cn^{1/2}\big\|w\psi D^2 f\big\|_p. \tag{36}$$

Proof. From (15) and (12) we have

$$\begin{aligned} &w(x)\psi^{3/2}(x)D^3\tilde{V}_n f(x)\\ &\quad= n(n+1)w(x)\psi^{1/2}(x)\sum_{k=0}^{\infty}\Delta^2 a_{k,n-1}(f)Dv_{n+2,k}(x)\\ &\quad= n(n+1)(n+2)w(x)\psi^{1/2}(x)\sum_{k=0}^{\infty}\Delta^2 a_{k,n-1}(f)\Big(\frac{k}{n+2}-x\Big)v_{n+2,k}(x). \end{aligned} \tag{37}$$

First we prove (36) for $p = 1$ and $p = \infty$. Then by applying the Riesz-Thorin theorem we obtain the statement of the lemma for every $1 \le p \le \infty$.

Case $p = 1$: From the above representation we have

$$\begin{aligned} &\big\|w\psi^{3/2}D^3\tilde{V}_n f\big\|_1\\ &\quad\le cn^3\sum_{k=0}^{\infty}\big|\Delta^2 a_{k,n-1}(f)\big|\int_0^{\infty} w(x)\psi^{1/2}(x)\Big|\frac{k}{n+2}-x\Big|v_{n+2,k}(x)\,dx\\ &\quad= cn^3\sum_{k=0}^{\infty}\big|\Delta^2 a_{k,n-1}(f)\big|\int_0^{\infty}\sqrt{x}\,(1+x)^{\alpha+1/2}\Big|\frac{k}{n+2}-x\Big|v_{n+2,k}(x)\,dx. \end{aligned}$$

Applying Cauchy's inequality we obtain

$$\begin{aligned} &\int_0^{\infty}\sqrt{x}\,(1+x)^{\alpha+1/2}\Big|\frac{k}{n+2}-x\Big|v_{n+2,k}(x)\,dx\\ &\quad\le\Big(\int_0^{\infty}x(1+x)^{2\alpha+1}v_{n+2,k}(x)dx\Big)^{1/2}\Big(\int_0^{\infty}\Big(\frac{k}{n+2}-x\Big)^2 v_{n+2,k}(x)dx\Big)^{1/2}. \end{aligned}$$

We now estimate the two integrals in the RHS of this inequality. Using the identity

$$xv_{n+2,k}(x) = \frac{k+1}{n+1}v_{n+1,k+1}(x),$$

and Lemma 2, for the first integral we have

$$\int_0^\infty x(1+x)^{2\alpha+1} v_{n+2,k}(x)\,dx = \frac{k+1}{n+1}\int_0^\infty (1+x)^{2\alpha+1} v_{n+1,k+1}(x)\,dx \le \frac{c(k+1)}{(n+1)^2}\Big(1+\frac{k+1}{n+1}\Big)^{2\alpha+1}.$$

Inequality (17) yields the following estimate for the second integral

$$\int_0^\infty \Big(\frac{k}{n+2}-x\Big)^2 v_{n+2,k}(x)\,dx \le \frac{c}{(n+2)^2}\,\psi\Big(\frac{k}{n+2}\Big).$$

Consequently, we obtain

$$\big\|w\psi^{3/2}D^3\tilde V_n f\big\|_1 \le cn^{3/2}\sum_{k=0}^{\infty} w\Big(\frac{k+1}{n}\Big)\psi\Big(\frac{k+1}{n}\Big)\big|\Delta^2 a_{k,n-1}(f)\big|. \tag{38}$$

For $k=0$, we have $w\big(\frac{k+1}{n}\big)\sim 1$, hence as in [9, p. 395] we get the estimation

$$w\Big(\frac{k+1}{n}\Big)\psi\Big(\frac{k+1}{n}\Big)\big|\Delta^2 a_{k,n-1}(f)\big| \le \frac{c}{n}\,\big\|w\psi D^2 f\big\|_1. \tag{39}$$

In the case $k\ge 1$, $(w\psi)\big(\frac{k+1}{n}\big)\sim (w\psi)(x)$ for $x\in\big[\frac{k}{n-1},\frac{k+3}{n-1}\big]$. Following the considerations in [9, p. 395] we obtain

$$\sum_{k=1}^{\infty} w\Big(\frac{k+1}{n}\Big)\psi\Big(\frac{k+1}{n}\Big)\big|\Delta^2 a_{k,n-1}(f)\big| \le \frac{c}{n}\,\big\|w\psi D^2 f\big\|_1. \tag{40}$$

Now, inequality (36) follows from (38)–(40) and the lemma is proved for $p=1$.

Case $p=\infty$: We have

$$\begin{aligned}
\big|\Delta^2 a_{k,n-1}(f)\big| &= (n-1)\Big(\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)\,dt - 2\int_{\frac{k+1}{n-1}}^{\frac{k+2}{n-1}} f(t)\,dt + \int_{\frac{k+2}{n-1}}^{\frac{k+3}{n-1}} f(t)\,dt\Big)\\
&= (n-1)\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}} D^2 f\Big(\frac{k}{n-1}+t_1+t_2+t_3\Big)\,dt_1\,dt_2\,dt_3\\
&\le (n-1)\|w\psi D^2 f\|_\infty\\
&\quad\times \int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}} (w\psi)^{-1}\Big(\frac{k}{n-1}+t_1+t_2+t_3\Big)\,dt_1\,dt_2\,dt_3\\
&\le \frac{c(n-1)}{\big(1+\frac{k}{n-1}\big)^{\alpha+1}}\,\|w\psi D^2 f\|_\infty \int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}}\int_0^{\frac{1}{n-1}} \frac{dt_1\,dt_2\,dt_3}{\frac{k}{n-1}+t_1+t_2+t_3}\\
&\le \frac{c(n-1)}{\big(1+\frac{k}{n-1}\big)^{\alpha+1}}\,\|w\psi D^2 f\|_\infty \Big(\int_0^{\frac{1}{n-1}}\Big(\frac{k}{n-1}+t\Big)^{-1/3}dt\Big)^3,
\end{aligned}$$

hence

$$\left|\Delta^2 a_{k,n-1}(f)\right| \le \begin{cases} \dfrac{c}{n-1}\|w\psi D^2 f\|_\infty, & k=0, \\ \dfrac{c}{(n-1)^2 (w\psi)\left(\frac{k}{n-1}\right)}\|w\psi D^2 f\|_\infty, & k \ge 1. \end{cases} \tag{41}$$

Then from representation (37) and inequality (41), for $x \in [0,\infty)$,

$$\begin{aligned} &\left|w(x)\psi^{3/2}(x) D^3 \tilde{V}_n f(x)\right| \\ &\quad \le cn^3 w(x)\psi^{1/2}(x) \sum_{k=0}^{\infty} \left|\Delta^2 a_{k,n-1}(f)\right| \left|\frac{k}{n+2} - x\right| v_{n+2,k}(x) \\ &\quad \le \Bigg(cn^2 w(x)\psi^{1/2}(x) x v_{n+2,0}(x) \\ &\qquad + cnw(x)\psi^{1/2}(x) \sum_{k=1}^{\infty} \frac{1}{(w\psi)\left(\frac{k}{n-1}\right)} \left|\frac{k}{n+2} - x\right| v_{n+2,k}(x)\Bigg) \|w\psi D^2 f\|_\infty \end{aligned}$$

For the first term in the RHS we have

$$n^2 w(x)\psi^{1/2}(x)\, x v_{n+2,0}(x) = \frac{\sqrt{n}\,(nx)^{3/2}}{(1+x)^{n+3/2-\alpha}} \le c\sqrt{n}. \tag{42}$$

Indeed, for $0 \le x \le \frac{1}{n}$, the last inequality is obvious. For $x > \frac{1}{n}$, from $(1+x)^{n+3/2-\alpha} \ge c(nx)^2$ it follows that

$$\frac{\sqrt{n}\,(nx)^{3/2}}{(1+x)^{n+3/2-\alpha}} \le \frac{c\sqrt{n}\,(nx)^{3/2}}{(nx)^2} = \frac{c}{\sqrt{x}} \le c\sqrt{n}.$$

By the Cauchy inequality we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{(w\psi)\left(\frac{k}{n-1}\right)} \left|\frac{k}{n+2} - x\right| v_{n+2,k}(x) \\ &\qquad \le \left\{\sum_{k=1}^{\infty} \left(\frac{k}{n+2} - x\right)^2 v_{n+2,k}(x)\right\}^{1/2} \left\{\sum_{k=1}^{\infty} \frac{v_{n+2,k}(x)}{(w\psi)^2\left(\frac{k}{n-1}\right)}\right\}^{1/2}. \end{aligned}$$

From (8) we have

$$\sum_{k=1}^{\infty} \left(\frac{k}{n+2} - x\right)^2 v_{n+2,k}(x) \le \sum_{k=0}^{\infty} \left(\frac{k}{n+2} - x\right)^2 v_{n+2,k}(x) = \frac{\psi(x)}{n+2} \le \frac{\psi(x)}{n}.$$

Moreover,

$$\sum_{k=1}^{\infty} \frac{v_{n+2,k}(x)}{(w\psi)^2\left(\frac{k}{n-1}\right)} \le c(w\psi)^{-2}(x).$$

By applying Lemma 1 we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{v_{n+2,k}(x)}{(w\psi)^2\big(\frac{k}{n-1}\big)} &= \psi^{-2}(x) \sum_{k=1}^{\infty} \frac{v_{n+2,k}(x)}{w^2\big(\frac{k}{n-1}\big)\big(\frac{k}{n-1}\big)^2\big(1+\frac{k}{n-1}\big)^2} \\
&\le c\psi^{-2}(x) \sum_{k=1}^{\infty} \frac{v_{n-2,k+2}(x)}{w^2\big(\frac{k}{n-1}\big)} \\
&\le c\psi^{-2}(x) \sum_{k=1}^{\infty} \Big(1+\frac{k}{n-1}\Big)^{-2\alpha} v_{n-2,k+2}(x) \\
&\le c\psi^{-2}(x) \sum_{k=3}^{\infty} \Big(1+\frac{k}{n-2}\Big)^{-2\alpha} v_{n-2,k}(x) \\
&\le c\psi^{-2}(x) \sum_{k=0}^{\infty} \Big(1+\frac{k}{n-2}\Big)^{-2\alpha} v_{n-2,k}(x) \\
&\le c\psi^{-2}(x)(1+x)^{-2\alpha} \\
&\le \frac{c}{(w\psi)^2(x)}.
\end{aligned}$$

The lemma is proved. □

3. Proof of Theorem 3

In order to estimate the K-functional $K_w(f,t)_p$ defined in (5) we consider the function $g(x) := \tilde{V}_n^2(f,x) := \tilde{V}_n\big(\tilde{V}_n(f,x)\big)$. By Lemma 3, we have

$$\begin{aligned}
\big\|w(f-g)\big\|_p &= \big\|w\big(f-\tilde{V}_n f+\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p \\
&\le \big\|w\big(f-\tilde{V}_n f\big)\big\|_p + \big\|w\tilde{V}_n\big(f-\tilde{V}_n f\big)\big\|_p \\
&\le 2\big\|w\big(f-\tilde{V}_n f\big)\big\|_p.
\end{aligned}$$

It remains to estimate the weighted function $\tilde{D}g = \tilde{D}\tilde{V}_n^2 f$. Applying the Voronovskaya type inequality (29) in Lemma 6 to the function $g = \tilde{V}_n^2 f$ we obtain

$$\begin{aligned}
\frac{\big\|w\tilde{D}\tilde{V}_n^2 f\big\|_p}{2(\ell-1)} &\le \big\|w\big(\tilde{V}_\ell\tilde{V}_n^2 f-\tilde{V}_n^2 f\big)\big\|_p \\
&+ c\left(\frac{\big\|w\psi^{3/2}D^3\tilde{V}_n^2 f\big\|_p}{\ell^{3/2}} + \frac{\big\|wD^2\tilde{V}_n^2 f\big\|_p}{\ell^2} + \frac{\big\|w\psi D^2\tilde{V}_n^2 f\big\|_p}{\ell^2} + \frac{\big\|wD^3\tilde{V}_n^2 f\big\|_p}{\ell^3}\right).
\end{aligned}$$

Recall that everywhere c stands for an absolute positive constant, which may be different in the different occurrences.

From inequality (36) in Lemma 7, by replacing f with $\tilde{V}_n f$, we have

$$\begin{aligned}\big\|w\psi^{3/2}D^3\tilde{V}_n^2 f\big\|_p &\le c\sqrt{n}\big\|w\psi D^2\tilde{V}_n f\big\|_p\\ &\le c\sqrt{n}\big\|w\psi D^2\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p + c\sqrt{n}\big\|w\psi D^2\tilde{V}_n^2 f\big\|_p.\end{aligned}$$

From inequalities (27) and (23) it follows that

$$\begin{aligned}\big\|wD^2\tilde{V}_n^2 f\big\|_p &\le cn\big\|wD\tilde{V}_n f\big\|_p\\ &\le cn\big\|wD\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p + cn\big\|wD\tilde{V}_n^2 f\big\|_p\\ &\le cn\big\|wD\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p + cn\big\|w\tilde{D}\tilde{V}_n^2 f\big\|_p.\end{aligned}$$

Inequality (25) in Lemma 4 yields

$$\big\|w\psi D^2\tilde{V}_n^2 f\big\|_p \le c\big\|w\tilde{D}\tilde{V}_n^2 f\big\|_p.$$

From (28) and (23) we have

$$\begin{aligned}\big\|wD^3\tilde{V}_n^2 f\big\|_p &\le cn^2\big\|wD\tilde{V}_n f\big\|_p\\ &\le cn^2\big\|wD\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p + cn^2\big\|wD\tilde{V}_n^2 f\big\|_p\\ &\le cn^2\big\|wD\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p + cn^2\big\|w\tilde{D}\tilde{V}_n^2 f\big\|_p.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\|w\tilde{D}\tilde{V}_n^2 f\|_p}{2(\ell-1)} &\le \big\|w\big(\tilde{V}_\ell\tilde{V}_n^2 f-\tilde{V}_n^2 f\big)\big\|_p + c\,\frac{\sqrt{n}}{\ell^{3/2}}\,\big\|w\psi D^2\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p\\ &+ c\Big(\frac{n}{\ell^2}+\frac{n^2}{\ell^3}\Big)\big\|wD\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p\\ &+ c\Big(\frac{\sqrt{n}}{\ell^{3/2}}+\frac{n}{\ell^2}+\frac{n^2}{\ell^3}\Big)\big\|w\tilde{D}\tilde{V}_n^2 f\big\|_p.\end{aligned}$$

Let us choose $\ell \ge nR$, $R>0$ and $\ell\in\mathbb{N}$, such that

$$c\Big(\frac{\sqrt{n}}{\ell^{3/2}}+\frac{n}{\ell^2}+\frac{n^2}{\ell^3}\Big) \le \frac{1}{4(\ell-1)}.$$

Then we have

$$\begin{aligned}\frac{1}{4(\ell-1)}\big\|w\tilde{D}\tilde{V}_n^2 f\big\|_p &\le \big\|w\big(\tilde{V}_\ell\tilde{V}_n^2 f-\tilde{V}_n^2 f\big)\big\|_p + c\,\frac{\sqrt{n}}{\ell^{3/2}}\big\|w\psi D^2\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p\\ &+ c\Big(\frac{n}{\ell^2}+\frac{n^2}{\ell^3}\Big)\big\|wD\big(\tilde{V}_n f-\tilde{V}_n^2 f\big)\big\|_p.\end{aligned}$$

Again by Lemma 3 we obtain

$$\begin{aligned}\big\|w\big(\tilde V_\ell \tilde V_n^2 f - \tilde V_n^2 f\big)\big\|_p &= \big\|w\big(\tilde V_\ell \tilde V_n^2 f - \tilde V_\ell f + \tilde V_\ell f - f + f - \tilde V_n^2 f\big)\big\|_p \\ &\le \big\|w\big(\tilde V_\ell \tilde V_n^2 f - \tilde V_\ell f\big)\big\|_p + \big\|w\big(\tilde V_\ell f - f\big)\big\|_p + \big\|w\big(f - \tilde V_n^2 f\big)\big\|_p \\ &\le \big\|w\big(\tilde V_\ell f - f\big)\big\|_p + c\big\|w\big(f - \tilde V_n^2 f\big)\big\|_p \\ &\le \big\|w\big(\tilde V_\ell f - f\big)\big\|_p + c\big\|w\big(f - \tilde V_n f + \tilde V_n f - \tilde V_n^2 f\big)\big\|_p \\ &\le \big\|w\big(\tilde V_\ell f - f\big)\big\|_p + c\big\|w\big(f - \tilde V_n f\big)\big\|_p + c\big\|w\big(\tilde V_n f - \tilde V_n^2 f\big)\big\|_p \\ &\le c\big\|w\big(\tilde V_n f - f\big)\big\|_p + \big\|w\big(\tilde V_\ell f - f\big)\big\|_p .\end{aligned}$$

Also, from (18) it follows that

$$\big\|w\psi D^2\big(\tilde V_n f - \tilde V_n^2 f\big)\big\|_p \le cn\big\|w\big(\tilde V_n f - f\big)\big\|_p$$

and from (26),

$$\big\|wD\big(\tilde V_n f - \tilde V_n^2 f\big)\big\|_p \le cn\big\|w\big(\tilde V_n f - f\big)\big\|_p .$$

Then

$$\frac{\big\|w\tilde D \tilde V_n^2 f\big\|_p}{4(\ell-1)} \le \big\|w\big(\tilde V_\ell f - f\big)\big\|_p + c\Big(1 + \frac{n^{3/2}}{\ell^{3/2}} + \frac{n^2}{\ell^2} + \frac{n^3}{\ell^3}\Big)\big\|w\big(\tilde V_n f - f\big)\big\|_p .$$

By taking R big enough we obtain

$$\frac{1}{n}\,\big\|w\tilde D \tilde V_n^2 f\big\|_p \le c\,\frac{\ell}{n}\big(\big\|w\big(\tilde V_n f - f\big)\big\|_p + \big\|w\big(\tilde V_\ell f - f\big)\big\|_p\big)$$

and we accomplish the proof of Theorem 3. □

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