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A Note on Orthogonality and Mixed Recurrence Equations

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The weight function $w(x)$ and the interval of orthogonality (a, b) determine the polynomials p_n in an orthogonal sequence up to a normalizing factor. We prove that, if $m \in \{1, 2, \ldots\}$, then $\int_a^b w(x)q(x)p_n(x)\,dx = 0$ with $\deg(q) = n+m$, if and only if q is a linear combination of $p_{n+1}, p_{n+2}, \ldots, p_{n+m}$. Furthermore, we determine the conditions on $k \in \mathbb{N}_0$, necessary and sufficient to obtain full interlacing between the n zeros of p_n and the $n-1$ zeros of $G_{m-1}g_{n-m,k}(x)$, where the polynomial G_{m-1} is the coefficient of p_{n-1} in a recurrence equation involving polynomials p_n, p_{n-1} and $g_{n-m,k}$, $m \in \{2, 3, \ldots, n-1\}$. The polynomial $g_{n-m,k}$ is orthogonal with respect to the weight $c_k(x)w(x)$ (or $c_{2k}(x)w(x)$) on (a, b), where $c_k(x) > 0$ is a polynomial of degree k. In this way, we extend (and correct the proof of) a result in [3].

1. Introduction

A sequence of real monic polynomials $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to a weight function $w(x) > 0$ on the (finite or infinite) interval (a, b), if

$$\langle p_m, p_n\rangle = \int_a^b w(x)p_m(x)p_n(x)\,dx = \delta_{mn}\langle p_n, p_n\rangle,$$

where $\langle p_n, p_n\rangle > 0$ and δ_{mn} is the Kronecker delta.

Further, the sequence $\{p_n\}_{n=0}^{\infty}$ satisfies a three term recurrence equation:

$$p_n(x) = (x - C_n)p_{n-1}(x) - \lambda_n p_{n-2}(x), \tag{1}$$

with

$$C_n = \frac{\langle xp_{n-1}, p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle}, \qquad \lambda_n = \frac{\langle xp_{n-1}, p_{n-2}\rangle}{\langle p_{n-2}, p_{n-2}\rangle}, \qquad \frac{\langle xp_{n-1}, p_n\rangle}{\langle p_n, p_n\rangle} = 1,$$

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since p_n is monic (cf. [2, p. 18, Theorem 4.1]). Let the zeros of the polynomial p_n be $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$. It is well known that the zeros of p_n and p_{n-1} interlace, i.e.,

$$x_{n,1} < x_{n-1,1} < x_{n,2} < \cdots < x_{n,n-1} < x_{n-1,n-1} < x_{n,n}.$$

Rainville [10, Theorem 54] proved an equivalent condition for orthogonality. He showed that a necessary and sufficient condition for the orthogonality of p_n is

$$\int_a^b w(x)x^j p_n(x)\,dx = 0, \qquad j \in \{0, 1, \ldots, n-1\},$$

which is equivalent to (cf. [2, p. 8, Theorem 2.1])

$$\int_a^b w(x)q(x)p_n(x)\,dx = 0, \qquad \deg(q) \in \{0, 1, \ldots, n-1\},$$

where q is an arbitrary polynomial.

In the proof in [3, Theorem 2.1], the assumption was made that, given p_n is an orthogonal sequence, it follows from $\int_a^b w(x)q(x)p_n(x)\,dx = 0$, with q arbitrary, that $\deg(q) \leq n-1$. This, however, is not true in general and in Section 2 we discuss the conditions when $\int_a^b w(x)q(x)p_n(x)\,dx = 0$, should $\deg(q) \geq n+1$. In Section 3 we extend (and provide an accurate proof for) the result in [3, Theorem 2.1]. We show how Christoffel's formula is applied to obtain mixed three term recurrence equations involving polynomials p_n and p_{n-1}, orthogonal with respect to a weight $w(x)$, and polynomials $g_{n-m,k}$, orthogonal with respect to a weight $c_k(x)w(x)$ (or $c_{2k}(x)w(x)$), where c_k is a polynomial of degree k. Furthermore, we provide necessary and sufficient conditions on k, such that the coefficient of the polynomial p_{n-1} in these equations is of degree exactly $m-1$, in which case we can apply the result in [4, Theorem 2.1] to obtain inner bounds for the extreme zeros of the polynomial p_n. Throughout this note we will assume that the polynomials under consideration are co-prime, i.e., they do not have any common zeros. The scenario of common zeros is discussed in [1, 4].

2. Orthogonality

Theorem 1. *Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials, orthogonal on the (finite or infinite) interval (a, b) with respect to the weight $w(x) > 0$. Then, for each $n \in \mathbb{N}$,*

$$\int_a^b w(x)x^n p_{n-1}(x)\,dx = 0 \tag{2}$$

if and only if $C_n = 0$, where C_n is given in (1).

Proof. Let $n \in \mathbb{N}$. Consider

$$p_n(x) = x^n + a_{n,n-1}x^{n-1} + \ldots + a_{n,1}x + a_{n,0}.$$

Since $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials, we know that, for each n,

$$\begin{aligned}
0 &= \int_a^b w(x)p_n(x)p_{n-1}(x)\,dx \\
&= \int_a^b w(x)\big(x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x + a_{n,0}\big)p_{n-1}(x)\,dx \\
&= \int_a^b w(x)x^n p_{n-1}(x)\,dx + a_{n,n-1}\int_a^b w(x)x^{n-1}p_{n-1}(x)\,dx.
\end{aligned}$$

Due to orthogonality, $\int_a^b w(x)x^{n-1}p_{n-1}(x)\,dx = \langle p_{n-1}, p_{n-1}\rangle > 0$. It therefore follows that, for each n, $\int_a^b w(x)x^n p_{n-1}(x)\,dx = 0$ if and only if $a_{n,n-1} = 0$. In comparing the coefficients of x^{n-1} in (1), we obtain

$$C_n = a_{n-1,n-2} - a_{n,n-1}$$

and the result follows. □

Remark 1. $C_n = 0$ for the following polynomial sequences, orthogonal with respect to an even weight function on $(-b, b)$, $b \in \mathbb{R}$ (cf. [2, p. 21, Theorem 4.3]):

(i) Hermite polynomials [8, Section 9.5], orthogonal with respect to e^{-x^2} on $\mathbb{R}$;

(ii) Jacobi polynomials [8, Section 9.8] with $\alpha = \beta > -1$, orthogonal with respect to $(1 - x^2)^{\alpha}$ on $(-1, 1)$;

(iii) Meixner-Pollaczek polynomials [8, Section 9.7], with $\phi = 0$, orthogonal with respect to $|\Gamma(\lambda + ix)|^2$ for $\lambda > 0$ on $(-\infty, \infty)$;

(iv) Pseudo-Jacobi polynomials [8, Section 9.9], with $\nu = 0$, orthogonal with respect to $(1 + x^2)^{-N-1}$ for $N \in \mathbb{N}_0$ on $(-\infty, \infty)$.

The weight function $w(x)$ and the interval of orthogonality (a, b) determine the polynomials p_n in an orthogonal sequence up to a normalizing factor (cf. [9, Chapter 2]). The scenario where $\int_a^b w(x)q(x)p_n(x)\,dx = 0$ for each $n \geq 0$ and the degree of q is $n + m$, for a fixed $m \in \{1, 2, \ldots\}$, is discussed in the following result:

Theorem 2. *Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the interval (a, b) with respect to the weight function $w(x) > 0$. Then, for $m \in \mathbb{N}$,*

$$\int_a^b w(x)p_j(x)Q_{n+m}(x)\,dx = 0, \qquad j \in \{0, 1, \ldots, n\}, \tag{3}$$

if and only if there exist coefficients b_k, $k \in \{n + 1, n + 2, \ldots, n + m\}$, such that $Q_{n+m}(x) = b_{n+1}p_{n+1}(x) + b_{n+2}p_{n+2}(x) + \cdots + b_{n+m}p_{n+m}(x)$.

Proof. (a) If $Q_{n+m}(x) = b_{n+1}p_{n+1}(x) + b_{n+2}p_{n+2}(x) + \cdots + b_{n+m}p_{n+m}(x)$, then (3) follows from orthogonality.

(b) Let $n \in \mathbb{N}$ and suppose (3) is true for $m \in \{1, 2, \ldots\}$ fixed. Since any arbitrary polynomial can be represented as a linear combination of the polynomials in an orthogonal sequence, $Q_{n+m}(x)$ can be represented as a linear combination of the polynomials $p_0(x), p_1(x), \ldots, p_{n+m}(x)$, say

$$Q_{n+m}(x) = b_0p_0(x) + b_1p_1(x) + \cdots + b_{n+m}p_{n+m}(x), \qquad b_{n+m} \neq 0.$$

Then, for $k \in \{0, 1, \ldots, n+m\}$, we have

$$\langle Q_{n+m}, p_k\rangle = \langle b_0p_0 + b_1p_1 + \ldots + b_{n+m}p_{n+m}, p_k\rangle = \langle b_kp_k, p_k\rangle$$

and

$$b_k = \frac{\langle Q_{n+m}, p_k\rangle}{\langle p_k, p_k\rangle} = 0$$

if $k \leq n$ (using (3)) and the result follows. □

3. On Mixed Three Term Recurrence Equations

We consider the following result that provides a three term recurrence equation involving p_n, p_{n-1} and p_{n-m}, $m \in \{2, 3, \ldots, n\}$.

Lemma 1 ([1, Theorem 4]). *Suppose* $\{p_n\}_{n=0}^{\infty}$ *is a sequence of polynomials, satisfying* (1). *Then, given* n, *there exists a sequence of real orthogonal polynomials* $S_m^{(n)}(x)$, $m \in \{0, 1, \ldots, n\}$, *of exact degree* m, *satisfying the three term recurrence equation*

$$S_m^{(n)}(x) = (x - C_{n-(m-1)})S_{m-1}^{(n)}(x) - \lambda_{n-(m-2)}S_{m-2}^{(n)}(x), \quad m \in \{1, 2, \ldots, n\}, \quad (4)$$

with $S_0^{(n)}(x) := 1$ *and* $S_{-1}^{(n)}(x) := 0$ *such that, for* $m \in \{2, 3, \ldots, n\}$,

$$\lambda_n\lambda_{n-1}\cdots\lambda_{n-m+2}p_{n-m}(x) = S_{m-1}^{(n)}(x)p_{n-1}(x) - S_{m-2}^{(n-1)}(x)p_n(x). \quad (5)$$

The polynomials $\{S_m^{(n)}\}_{m=0}^{n}$, $n \in \{0, 1, \ldots\}$, will be referred to as the *associated* polynomials of the sequence $\{p_n\}_{n=0}^{\infty}$ and they are part of an orthogonal sequence [2, p. 21, Theorem 4.4].

Remark 2. (i) Equation (5) differs from [1, Eq. (10)]. When we replace m with 2 in [1, Eq. (10)], taking into account that the polynomial S_m is of degree $m-1$, we do not obtain [1, Eq. (2c)] and we deduce that [1, Eq. (10)] is not completely correct. Furthermore, in order to obtain (5), we replaced n by $n-m$ in [1, Eq. (10)].

(ii) Let $S_1^{(k)}(x) = x - C_k$, k is an integer. By replacing n by $n+1$ in (1), we obtain

$$\begin{aligned} p_{n+1}(x) &= (x - C_{n+1})p_n(x) - \lambda_{n+1}p_{n-1}(x), \\ &= S_1^{(n+1)}(x)p_n(x) - \lambda_{n+1}p_{n-1}(x). \end{aligned} \tag{6}$$

By replacing n by $n+2$ in (1) and using (6), we obtain

$$\begin{aligned} p_{n+2}(x) &= (x - C_{n+2})p_{n+1}(x) - \lambda_{n+2}p_n(x) \\ &= (x - C_{n+2})\big((x - C_{n+1})p_n(x) - \lambda_{n+1}p_{n-1}(x)\big) - \lambda_{n+2}p_n(x), \\ &= \big((x - C_{n+2})(x - C_{n+1}) - \lambda_{n+2}\big)p_n(x) - \lambda_{n+1}(x - C_{n+2})p_{n-1}(x) \\ &= S_2^{(n+2)}(x)p_n(x) - \lambda_{n+1}S_1^{(n+2)}(x)p_{n-1}(x), \end{aligned}$$

where $S_m^{(n+m)}(x)$, $m \in \{1, 2, \ldots\}$, satisfy (4) with n replaced by $n+m$. From this iterating process we obtain the recurrence equation:

$$p_{n+m}(x) = S_m^{(n+m)}(x)p_n(x) - \lambda_{n+1}S_{m-1}^{(n+m)}(x)p_{n-1}(x), \tag{7}$$

In the specific case where $m = 1$, the polynomials $S_n^{(n+1)}$ satisfy the same three term recurrence equation as the numerator polynomials $p_n^{(1)}$ in [2, p. 86, Definition 4.1] and the associated polynomials in [12].

Let $m \in \{2, 3, \ldots, n\}$ and $k \in \mathbb{N}_0$. The following theorem corrects and extends the result in [3, Theorem 2.1].

Theorem 3. *Let $k \in \mathbb{N}_0$ and $m \in \{2, 3, \ldots, n\}$ be fixed and let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the interval (a, b) with respect to the weight function $w(x) > 0$. The sequence of polynomials $\{g_{n,k}\}_{n=0}^{\infty}$, orthogonal with respect to $c_k(x)w(x) > 0$ on (a, b), where $c_k(x)$ is a polynomial of degree k, satisfies*

$$c_k(x)g_{n-m,k}(x) = R(x)p_n(x) - G(x)p_{n-1}(x), \qquad n \in \{2, 3, \ldots\},$$

where R and G are polynomials with $\deg(R) = \max\{m-2, k-m\}$ *and* $\deg(G) = \max\{m-1, k-m-1\}$. *Furthermore,* $\deg(G) = m-1$ *iff* $k \in \{0, 1, \ldots, 2m\}$.

Proof. Fix $k \in \mathbb{N}_0$ and $m \in \{2, 3, \ldots, n\}$. We apply Christoffel's formula [7, Theorem 2.7.1] to $g_{n-m,k}$, to obtain

$$\begin{aligned} U_{n-m,k}^{k\times k} c_k(x) g_{n-m,k}(x) &= \begin{vmatrix} p_{n-m}(x_1) & p_{n-m+1}(x_1) & \cdots & p_{n-m+k}(x_1) \\ p_{n-m}(x_2) & p_{n-m+1}(x_2) & \cdots & p_{n-m+k}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-m}(x_k) & p_{n-m+1}(x_k) & \cdots & p_{n-m+k}(x_k) \\ p_{n-m}(x) & p_{n-m+1}(x) & \cdots & p_{n-m+k}(x) \end{vmatrix} \\ &= \sum_{j=0}^{k} (-1)^{k+j} U_{n-m,j}^{k\times k} p_{n-m+j}(x), \end{aligned} \tag{8}$$

where x_i, $i \in \{1, 2, \ldots, k\}$, are the zeros of c_k and $U^{k\times k}_{n-m,j}$, $j \in \{0, 1, \ldots, k\}$ is the determinant of the $k \times k$ matrix that we obtain from the $(k+1) \times (k+1)$ matrix in (8) by deleting the $(k+1)^{th}$ row and the $(j+1)^{th}$ column.

We need to write $U^{k\times k}_{n-m,k} c_k(x) g_{n-m,k}(x)$, and therefore $p_{n-m+j}(x)$, in terms of p_n and p_{n-1}. We use (5) to express the polynomials p_{n-m+j}, $j \in \{0, 1, \ldots, m-2\}$, in terms of p_n and p_{n-1}. When we replace m by $-m+j$ in (7), we obtain an expression for p_{n-m+j}, $j \in \{m+1, m+2, \ldots, k\}$. Finally, by collecting the coefficients of p_n and p_{n-1}, we obtain

$$U^{k\times k}_{n-m,k} c_k(x) g_{n-m,k}(x) = R^*(x) p_n(x) + G^*(x) p_{n-1}(x),$$

i.e.,

$$c_k(x) g_{n-m,k}(x) = R(x) p_n(x) + G(x) p_{n-1}(x),$$

with

$$R(x) = \sum_{j=0}^{m-2} \frac{(-1) d_j \, S^{(n-1)}_{m-j-2}(x)}{\lambda_n \lambda_{n-1} \cdots \lambda_{n-(m-j-2)}} + \sum_{j=m}^{k} d_j \, S^{(j-m+n)}_{j-m}(x) = \frac{R^*(x)}{U^{k\times k}_{n-m,k}},$$

$$G(x) = \sum_{j=0}^{m-2} \frac{d_j \, S^{(n)}_{m-j-1}(x)}{\lambda_n \lambda_{n-1} \cdots \lambda_{n-m+j+2}} + d_{m-1} - \lambda_{n+1} \sum_{j=m+1}^{k} d_j \, S^{(j-m+n)}_{j-m-1}(x) = \frac{G^*(x)}{U^{k\times k}_{n-m,k}}$$

and $d_j = \dfrac{(-1)^{k+j} U^{k\times k}_{n-m,j}}{U^{k\times k}_{n-m,k}}$, $j \in \{0, 1, \ldots, k\}$. It follows from the above representations of $R(x)$ and $G(x)$ and Lemma 1 that $\deg(R) = \max\{m-2, k-m\}$ and $\deg(G) = \max\{m-1, k-m-1\}$ (see Remark 3 (i) below).

The final statement that $\deg(G) = m-1$ if and only if $k \in \{0, 1, \ldots, 2m\}$, follows directly from the fact that $\deg(G) = \max\{m-1, k-m-1\}$. □

Remark 3. (i) It is clear that $d_k = 1$. In [7, Theorem 2.7.1] it is proved that $U^{k\times k}_{n-m,k} \neq 0$. In the same way we can prove that $U^{k\times k}_{n-m,0} \neq 0$, i.e., $d_0 \neq 0$, therefore the equalities $\deg(R) = \max\{m-2, k-m\}$ and $\deg(G) = \max\{m-1, k-m-1\}$ hold.

(ii) Should a zero x_i of c_k be of multiplicity $s > 1$, the corresponding rows of the determinant in (8) are replaced by the derivatives of order $0, 1, \ldots, s-1$ of the polynomials $p_{n-m}(x)$, $p_{n-m+1}(x), \ldots, p_{n-m+k}(x)$ at $x = x_i$ [11, Section 2.5].

(iii) In [4, Theorem 2.1] the location of the zeros of polynomials g_{n-m} and p_n was discussed in detail, provided these polynomials satisfy the equation

$$f(x) g_{n-m}(x) = H(x) p_n(x) + G_{m-1}(x) p_{n-1}(x),$$

where $f(x) \neq 0$ on (a, b), H and G_k are polynomials and $\deg(G_k) = k$. In Theorem 3 we provide necessary and sufficient conditions for the existence of recurrence equations of this type, i.e., for the coefficient of p_{n-1} to be of degree exactly $m-1$.

(iv) Theorem 3 complements the result in [5, Theorem 4.4], by suggesting that the zeros of the Laguerre polynomials L_n^{α} and $L_{n-m}^{\alpha+t}$ interlace for all $\alpha > -1$ if and only if $t \in \{0, 1, \dots, 2m\}$.

The Wilson, Continuous dual Hahn, Meixner-Pollaczek and Pseudo-Jacobi polynomials are examples of polynomial systems, where the polynomial $g_{n-m,k}$, obtained from making a parameter shift of k units, $k \in \{0, 1, \dots, m\}$, is orthogonal with respect to a polynomial of degree $2k$ times the weight function of p_n. Similar to Theorem 3, we can prove that, for $k \in \mathbb{N}_0$ and $m \in \{2, 3, \dots, n\}$ fixed, the polynomials p_n, p_{n-1} (orthogonal with respect to $w(x)$ on (a, b)) and $g_{n-m,k}$ (orthogonal with respect to $c_{2k}(x)w(x) > 0$ on (a, b)), satisfy the equation

$$c_{2k}(x)g_{n-m,k}(x) = R(x)p_n(x) - G(x)p_{n-1}(x), \qquad n \in \{2, 3, \dots\},$$

where $c_{2k}(x) = c_k(x)c_k(-x)$ and $R(x)$ and $G(x)$ are polynomials such that $\deg(R) = \max\{m-2, 2k-m\}$ and $\deg(G) = \max\{m-1, 2k-m-1\}$.

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