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Weighted Approximation with the Bernstein-Chlodovsky Operators

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In 1937, I. Chlodovsky modified the Bernstein polynomial operators for use in approximation of functions in $C[0,\infty)$ and produced several interesting results on convergence, none of which addressed weighted approximation. This article investigates weighted approximation by operators of Bernstein-Chlodovsky type in the following two contexts:

First, with $W(x)=e^{-x^\alpha}$, let $C_0^W[0,\infty)$ be the space of all functions $f\in C[0,\infty)$ such that $W(x)f(x)\to 0$ as $x\to\infty$. The weighted approximation of functions $f\in C_0^W[0,\infty)$, will be by similarly weighted polynomials which are constructed using the Chlodovsky operators.

Second, with $W(x)=e^{-|x|^\alpha}$, let $C_0^W(-\infty,\infty)$ be the space of all functions $f\in C(-\infty,\infty)$ with $W(x)f(x)\to 0$ as $|x|\to\infty$. Similar questions relating to weighted convergence are then considered for functions in $C_0^W(-\infty,\infty)$, using similarly constructed operators.

Recent progress will be presented, and some unsolved open problems will be discussed.

1. Introduction

In 1912, Bernstein [3] introduced the operators

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}f\Big(\frac{k}{n}\Big)x^k(1-x)^{n-k}, \tag{1}$$

in which for each n the output of B_n is a polynomial of degree at most n, and then showed that $B_n(f;x)$ converges uniformly to any $f\in C[0,1]$. The proof then applies if $f\in C[a,b]$, where $[a,b]$ is any other interval, via an affine transformation. Bernstein's proof depended upon the uniform continuity of the function f on a closed and bounded interval; upon the fact that the operators B_n are monotone operators; upon $B_n(1;x)=1$ and $B_n(x;x)=x$, which properties are preserved under affine transformations to other intervals; and upon $B_n(x^2;x) = \frac{n-1}{n}x^2 + \frac{1}{n}x$ which is not thus preserved, but takes another form which depends upon the underlying interval.

The objective here is to describe and present some results which extend Bernstein's approximation operator to continuous functions defined on either of the two unbounded intervals $[0,\infty)$ or $(-\infty,\infty)$ and specifically to deal with weighted approximation, where the weight is exponentially decaying. The archetypical such weight function is of the form $W(x) = e^{-|x|^\alpha}$, which will be the weight function used here.

In this investigation, the author records a debt of gratitude to Katalin Balázs (08/13/1949 – 09/07/2016), who is well known for her work on monotone rational approximation operators [1] and [2]. During the summer of 2016, she shared her knowledge of previous work on Bernstein-type approximation operators on infinite intervals. Those discussions were instrumental in formulating the problems and questions discussed here, which are related to using operators similar to (1) on unbounded intervals in the presence of a weighted norm.

Insofar as the author is aware, extension of the Bernstein operator to infinite intervals began with an article of Chlodovsky [4], in 1937. Wishing to approximate functions in $C[0,\infty)$, he proposed a sequence of operators B_n^* in which the underlying interval for the operator of index n is of increasing length, of the form $[0,b_n]$, with $b_n \to \infty$ as $n \to \infty$. Specifically,

$$B_n^*(f;x) = \sum_{k=0}^{n} \binom{n}{k} f\Big(\frac{kb_n}{n}\Big) b_n^{-n} x^k (b_n - x)^{n-k}. \tag{2}$$

One may easily verify that, as these operators are derived from Bernstein's operators by affine transformations, it follows that $B_n^*(1;x) = 1$ and that $B_n^*(x;x) = x$.

Chlodovsky also observed quite accurately that his operators B_n^* could fail to converge to a continuous function f even in a pointwise sense unless the sequence of right endpoints $\{b_n\}_{n=1}^\infty$ were also to satisfy $b_n = o(n)$. One obvious reason why this requirement must be imposed is that the points $\frac{kb_n}{n}$, $k = 0,\ldots,n$, which are used in (2) for sampling the function f to be approximated, form a partition of the interval $[0,b_n]$ and if the mesh of this partition does not go to 0 then surely nothing good could occur. But Chlodovsky also gave two counterexamples. One of them is the quite ordinary function $f(x) = x^2$, for which

$$B_n^*(x^2;x) = \frac{n-1}{n}x^2 + \frac{b_n}{n}x, \tag{3}$$

and therefore $B_n^*(x^2;x) \to x^2$ as $n \to \infty$ clearly does not occur unless $\frac{b_n}{n} \to 0$.

Chlodovsky was concerned with pointwise approximation and convergence on compact subsets of $[0,\infty)$ and thus for him the two requirements $b_n \to \infty$ and $b_n = o(n)$ sufficed. However, the goal here is weighted approximation, which requires attention to the entire unbounded interval. Conceivably, the weighted approximation can fail on the unbounded portions $[b_n, \infty)$ of the interval $[0, \infty)$ even while converging uniformly on the bounded subsets $[0, b_n]$.

Further to investigate this potential difficulty, let $f \in C[0, \infty)$ be uniformly continuous and bounded, or be continuous on $[0, \infty)$ and also satisfy $f(x) \to 0$ as $x \to \infty$ (which then makes f uniformly continuous and bounded), and assume that $\sup_x |f(x)| = 1$.

Then there is no problem in producing the successive polynomial approximants $B_n^*(f; x)$. And, with minor variations of Bernstein's original proof (cf. Kilgore [5, Lemma 5], for details used in a particular case, which are also applied in the proof of Theorem 1, below) it is easy to prove that, given $\epsilon > 0$, there is N such that for all $n \geq N$, one has $\sup_{x \in [0, b_n]} W(x)|f(x) - B_n^*(f; x)| < \epsilon$.

However, if one considers what can occur on the intervals $[b_n, \infty)$ there is the serious problem that, while each B_n^* is a monotone operator on $[0, b_n]$, it is not monotone on $[b_n, \infty)$. Indeed, if $|f(x)| \leq 1$ for $0 \leq x < \infty$, then when $x \geq b_n$ the best estimate available is

$$|B_n^*(f; x)| \leq \sum_{k=0}^{n} \binom{n}{k} \left| f\left(\frac{kb_n}{n}\right) \right| b_n^{-n} x^k |b_n - x|^{n-k} \leq b_n^{-n}(2x - b_n)^n$$

and clearly all three of these quantities can potentially be equal. Thus, in order to ensure the uniform boundedness of $W(x)B_n^*(f; x)$ as $n \to \infty$ one must ensure that $b_n^{-n} W(x)(2x - b_n)^n = b_n^{-n} e^{-x^\alpha}(2x - b_n)^n$ is uniformly bounded in n.

To assume that this expression is in fact maximized when $x = b_n$ leads to

$$b_n = \left(\frac{2n}{\alpha}\right)^{1/\alpha} \tag{4}$$

which suffices to guarantee the needed uniform boundedness. Of course, the uniform boundedness on $[b_n, \infty)$ also holds *a fortiori* if

$$b_n \geq \left(\frac{2n}{\alpha}\right)^{1/\alpha}.$$

Now, unless $\alpha > 1$ this choice of b_n conflicts with the requirement (3) which (as noted already by Chlodovsky) is necessary for convergence even on the intervals $[0, b_n]$. And the presence or absence of $W(x)$ has no influence here.

Therefore, it will be assumed here that b_n is given by (4), and that $\alpha > 1$.

Remark 1. While the above discussion shows that when $\alpha \leq 1$ and b_n is given as in (4) there is no hope of showing that $e^{-x^\alpha}\big(B_n^*(f; x) - f(x)\big) \to 0$ as $n \to \infty$, it nevertheless remains true even under these conditions that the uniform boundedness of the weighted operators $e^{-x^\alpha} B_n^*$ is unaffected, provided only that $\alpha > 0$.

2. Results on $[0, \infty)$

Theorem 1. *If $f \in C[0, \infty)$ satisfies $f(x) \to 0$ as $x \to \infty$ or is uniformly continuous and bounded on $C[0, \infty)$ and satisfies $\|f\| = M$ then $\left\|e^{-x^\alpha} B_n^*(f; x)\right\| \leq M$ for all n, provided that $b_n = \left(\frac{2n}{\alpha}\right)^{1/\alpha}$ and $\alpha > 1$. And then $\left\|e^{-x^\alpha}\left(B_n^*(f; x) - f(x)\right)\right\| \to 0$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given.

As a first step, one may show that there exists N_1 such that for all $x \in [0, b_n]$ one has

$$W(x)|f(x) - B_n^*(f; x)| < \epsilon, \tag{5}$$

provided that $n \geq N_1$.

To show this, one needs to start by assuming that n is a fixed but arbitrary integer because the interval of definition $[0, b_n]$ also depends upon n. Other than that observation, the proof of (5) follows a very standard line.

Let x be any fixed but arbitrary $x \in [0, b_n]$, and let $t \in [0, b_n]$ be any other point. Then, using the uniform continuity of f and, using the linearity and monotonicity of B_n^*, one may proceed.

First, there is $\delta > 0$ which is independent of n, of x, and of t such that $|t - x| < \delta$ implies $|f(t) - f(x)| < \frac{\epsilon}{2}$. And, if $|t - x| \geq \delta$ then one may estimate $|f(t) - f(x)| \leq \frac{2M(t-x)^2}{\delta^2}$.

Therefore, for any $t \in [0, b_n]$ one has

$$|f(t) - f(x)| \leq \frac{\epsilon}{2} + \frac{2M(t - x)^2}{\delta^2}.$$

Regarding t as the variable and x and therefore $f(x)$ as constants, one may apply B_n^*, obtaining

$$\left|B_n^*(f(t); t) - f(x)\right| \leq \frac{\epsilon}{2}\, B_n^*(1; t) + \frac{2M}{\delta^2}\, B_n^*((t - x)^2; t)$$

in which, since t is the variable, not x,

$$B_n^*\left((t - x)^2; t\right) = B_n^*(t^2; t) - 2xt + x^2.$$

Now, what is above also holds in particular when $t = x$. And then using (3) and multiplying both sides by $W(x)$, one obtains

$$W(x)\left|B_n^*(f(x); x) - f(x)\right| \leq W(x)\,\frac{\epsilon}{2} + \frac{2M}{\delta^2}\, W(x)\,\frac{-x^2 + b_n x}{n}.$$

Since the above is true for every n and for every $x \in [0, b_n]$, the existence of N_1 clearly follows.

The second step is to consider what happens on the intervals $[b_n, \infty)$, on which

$$W(x)|B_n^*(f; x)| \leq W(x) b_n^{-n}(2x - b_n)^n.$$

It follows from (4) that for each n and for every x such that $x \geq b_n$, one has

$$W(x) b_n^{-n} (2x - b_n)^n \leq W(b_n)$$

and that

$$W(b_n) \to 0 \quad \text{as } n \to \infty.$$

Since it is also the case that

$$W(x)|f(x)| \to 0 \quad \text{as } x \to \infty,$$

it follows that there is N_2 such that for all $n \geq N_2$ and for all $x \geq b_n$, one has

$$W(x)|f(x) - B_n^*(f;x)| < \epsilon. \tag{6}$$

Now, for an arbitrary $x \in [0, \infty)$, if $n \geq N = \max\{N_1, N_2\}$, either (5) or (6) must be applicable at x. In either situation, it follows that

$$W(x)|f(x) - B_n^*(f;x)| < \epsilon,$$

and the proof of Theorem 1 is completed. □

At first sight, Theorem 1 seemingly applies only to a very restricted set of functions. Nevertheless, it has consequences which are in no way trivial. Two examples follow.

Theorem 2 (Kilgore [5]). *Let $W_1(x) = e^{-Q(x)}$ be a weight function on $[0, \infty)$ for which there exists $\alpha > 1$ such that*

$$\frac{Q(x)}{x^\alpha} \to \infty \text{ as } x \to \infty.$$

Let $\epsilon > 0$ be given. Then, for every function $f \in C_0^{W_1}[0, \infty)$ there exists a polynomial P obtainable by an operator of Bernstein-Chlodovsky type such that $\|W_1(x)(f(x) - P(x))\| < \epsilon$.

Theorem 2 is the Weierstrass theorem for $f \in C_0^{W_1}[0, \infty)$. The effect of the conditions upon the weight W_1 are such as to imply that there exists a weight of the form $W(x) = e^{-x^\alpha}$ (with $\alpha > 1$) and a constant $K > 0$ such that $W_1(x) \leq K\, W(x)$ on $[0, \infty)$. But no other conditions upon W_1 are imposed, including in particular conditions relating to differentiability, strict monotonicity of decrease, or even continuity. The proof of this theorem then proceeds as follows:

First, let $\epsilon > 0$ and a function f be given which meets the hypotheses. Then there exists some B (which can depend upon f), such that

$$\sup_{x \geq B} W_1(x)|f(x)| < \frac{\epsilon}{2K}.$$

This being the case, define a function $g(x)$ by the rule $g = f$ on $[0, B]$ and $g(x) = f(B)$ for $x \in [B, \infty)$. Then use Theorem 1 to find a polynomial P such that $\|W(x)(g(x) - P(x))\| < \frac{\epsilon}{2K}$. Then one also has $\|W(x)(f(x) - P(x))\| < \frac{\epsilon}{K}$. And since $W_1(x) \leq K\,W(x)$, the result follows.

Since the conclusions of Theorem 2 were essentially already known, the theorem is not exactly a surprise. However, the proof described here is entirely new, is completely independent both in method and in detail from previous proofs, and furthermore follows easily from Theorem 1 and from the fact that the functions to which Theorem 1 applies are a dense subspace in $C_0^{W_1}[0, \infty)$.

Now, let us consider a second application of Theorem 1. An attempt to approximate directly an arbitrary function $f \in C_0^W[0, \infty)$ using $B_n^* f$ can fail. For, in Theorem 5 below it will be shown that there do exist functions $f \in C_0^W[0, \infty)$ for which $\|W(x)B_n^*(f; x)\|$ fails to be uniformly bounded. Nevertheless, one may apply Theorem 1 instead to the function $W(x)f(x)$ which *does* meet its hypotheses, obtaining

Theorem 3. *Let* $W(x) = e^{-x^\alpha}$ *with* $\alpha > 1$, *and let* b_n *be as in* (4). *Then for* $f \in C_0^W[0, \infty)$ *it follows that* $W(x)B_n^*(Wf; x) \to W(x)f(x)$ *uniformly as* $n \to \infty$.

One may note that this result is in fact completely analogous to approximation results obtained by using a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ which are orthogonal with respect to the weight $W^2(x)$ (the same as saying that the functions $W(x)p_n(x)$ are pairwise orthogonal), and the coefficients in the Fourier expansion of the function $f(x)$ are computed by integrating $W^2(x)f(x)p_n(x)$ (the same as integrating the product of $W(x)f(x)$ and $W(x)p_n(x)$). For, what is being done both in that context and in Theorem 3 is to approximate the function $f \in C_0^W[0, \infty)$, not by directly approximating f itself but rather by constructing an approximation for Wf by a *similarly weighted polynomial*, which in Theorem 3 is given by $W(x)B_n^*(Wf; x)$.

These observations both agree with and extend the observations of Lorentz [7, Introduction, pp. 3–4], which point out that the Bernstein operator B_n can be represented using a Stieltjes integral, and also state that "…the theory of the Bernstein polynomials, as well as the theory of Fourier series, is a chapter of the theory of singular integrals. (If one would wish to compare these theories, the Bernstein polynomial $B_n(x)$ would correspond rather to the Fejér's mean $\sigma_n(x)$ than to the partial sum of the Fourier series $s_n(x)$.)"

3. Convergence and Divergence Results in $C_0^W[0, \infty)$

Here, in contrast to what was done in Theorem 3, direct application of B_n^* to a function $f \in C_0^W[0, \infty)$ is considered. With $W(x) = e^{-x^\alpha}$ and $\alpha \geq 1$, growth conditions on the function f are found, which guarantee that

$\sup_n \|W(x)B_n^*(f;x)\|$ is finite (Theorem 4), or respectively is infinite (Theorem 5). Theorem 6 then addresses weighted convergence. These three theorems and their complete proofs have appeared in the article of Kilgore and Szabados [6].

A new weight function

$$W_\lambda(x) = e^{-(\lambda x)^\alpha}, \text{ with } 0 < \lambda < 1$$

is introduced here, and it will be assumed that the function f lies in $C_0^{W_\lambda}[0,\infty)$, which is a proper subspace of $C_0^W[0,\infty)$.

The case $\alpha = 1$ is included in Theorem 4 as a "limiting case" since the operators B_n^* apparently can not then be so constructed as to converge. But the estimates concerning weighted uniform boundedness hold for this case, too.

Theorem 4. *We have*

$$\sup_n \|WB_n^*(f)\| \le 2\|W_\lambda f\|$$

in any of the following cases:

(a) $0 < \frac{\log(1-\lambda)}{\log\lambda} + 1 \le \alpha$*;*

(b) $0 < \lambda < \frac{\log 3}{2} = 0.54\ldots$ *for* $\alpha \ge 1$*;*

(c) $0 < \lambda \le \frac{1}{e-1} = 0.58\ldots$ *for* $\alpha \ge \frac{\log 2}{\log(e-1)} = 1.28\ldots$.

Theorem 5. *If* $\lambda > 2^{\alpha-2}\log\left(2e^{\frac{1}{2^{\alpha-1}}} - 1\right)$, $\alpha \ge 1$, *then there exists a function* f *satisfying* $\|W_\lambda f\| < \infty$ *and* $\sup_n \|WB_n^*(f)\| = \infty$.

Theorem 6. *Let* $\alpha > 1$, *and assume that* $f \in C[0,\infty)$ *satisfies the condition*

$$\lim_{x\to\infty} W_\lambda(x)f(x) = 0$$

with λ *satisfying any of the conditions* (a), (b), (c) *of Theorem 4. Then*

$$\lim_{n\to\infty} \|W(f - B_n^*(f))\| = 0.$$

Remark 2. For any $\alpha > 1$, if λ satisfies any one of the three conditions in Theorem 4, that will guarantee convergence. However, for every $\alpha > 1$ there is a gap between greatest of the three upper bounds found in Theorem 4, and the lower bound on λ obtained in Theorem 5, above which the uniform boundness fails. What occurs when λ lies in that gap remains unknown.

4. When the Interval is $(-\infty, \infty)$

First, one needs an operator similar to Chlodovsky's operator. Such was introduced in Kilgore [5]:

Let $\{c_n\}_{n=1}^{\infty}$ be any increasing sequence with $c_1 > 0$. Affine transformation of the Bernstein operator (1) to the interval $[-c_n, c_n]$ gives

$$\overline{B}_n(c_n, f; x) = \sum_{k=0}^{n} \binom{n}{k} f\Big(- c_n + \frac{2kc_n}{n}\Big)\Big(\frac{c_n + x}{2c_n}\Big)^k \Big(\frac{c_n - x}{2c_n}\Big)^{n-k}. \qquad (7)$$

The basic properties of $\overline{B}_n$, the restrictions upon the endpoints c_n which suffice to guarantee convergence, and the limitations on α when the weight function is $e^{-|x|^{\alpha}}$ are briefly discussed below.

Superficially, $\overline{B}_n$ is similar to the operator B_n^*. But there is an obvious difference, and also potential consequences of that obvious difference, as yet not fully explored. The obvious difference is that this new operator preserves the evenness and oddness properties of the input function f.

As the first example of what the preservation of evenness and oddness does, one can notice that

$$\overline{B}_n(c_n, x^2; x) = \frac{n-1}{n} x^2 + \frac{c_n^2}{n}. \qquad (8)$$

And then the obvious consequence of (8) is that the endpoint c_n must here satisfy $\frac{c^2}{n} \to 0$ as $n \to \infty$ before convergence can take place.

Now, as was noted above concerning the operator B_n^* which was used to approximate functions defined on $[0, \infty)$, the operator $\overline{B}_n$ is monotone inside its interval of support $[-c_n, c_n]$ but fails to be monotone outside of that interval. This issue must be addressed. To this end, let $f \in C(-\infty, \infty)$ be bounded and uniformly continuous, with $\|f\| = 1$. If it is assumed that $|x| \geq c_n$, then the outcome is simpler than what it was when the interval was $[0, \infty)$ and the operator was B_n^*. Here, it is easy to see that when $|x| \geq c_n$ one has the estimate

$$\overline{B}_n(c_n, f; x) \leq \left(\frac{|x|}{c_n}\right)^n.$$

Consequently, a sufficient condition for the weighted boundedness on the intervals $(-\infty, -c_n]$ and $[c_n, \infty)$ is that $|x|^n e^{-|x|^{\alpha}}$ should be maximized when $x = \pm c_n$. From this, one gets

$$c_n = \Big(\frac{n}{\alpha}\Big)^{1/\alpha}. \qquad (9)$$

Combining (9) with $\frac{c_n^2}{n} \to 0$ then requires $\alpha > 2$.

5. Results on $(-\infty, \infty)$

Obvious analogues of Theorems 1, 2, and 3, are true on $(-\infty, \infty)$, with very similar proofs.

Theorem 7. *If $f \in C(-\infty, \infty)$ satisfies $f(x) \to 0$ as $|x| \to \infty$ or is uniformly continuous and bounded on $(-\infty, \infty)$ and satisfies $\|f\| = M$ then $\left\| e^{-|x|^\alpha} \overline{B}_n(f;x) \right\| \le M$ for all n, provided that $c_n = \left(\frac{n}{\alpha}\right)^{1/\alpha}$ and $\alpha > 2$. And then $e^{-|x|^\alpha} \overline{B}_n(f;x) \to f(x)$ uniformly as $n \to \infty$.*

Theorem 8 (Kilgore [5]). *Let $W_2(x) = e^{-Q(x)}$ be a weight function on $(-\infty, \infty)$ for which there exists $\alpha > 2$ such that*

$$|x|^{-\alpha} Q(x) \to \infty \text{ as } |x| \to \infty.$$

Let $\epsilon > 0$ be given. Then, for every function $f \in C_0^{W_2}(-\infty, \infty)$ there exists a polynomial P obtainable by an operator of Bernstein-Chlodovsky type such that $\|W_2(x)(f(x) - P(x))\| < \epsilon$.

Theorem 8 is the Weierstrass theorem for $f \in C_0^{W_2}(-\infty, \infty)$. Again, the proof using operators of Bernstein-Chlodovsky type is completely new and follows easily from Theorem 7 and from the fact that the functions to which Theorem 7 applies comprise a dense subspace in $C_0^{W_2}(-\infty, \infty)$.

Finally, in order to approximate a function $f \in C_0^W(-\infty, \infty)$ one may apply Theorem 7 instead to the function $W(x)f(x)$ which does meet the hypotheses of Theorem 7, obtaining

Theorem 9. *Let $W(x) = e^{-|x|^\alpha}$ with $\alpha > 2$, and let c_n be as in (9). Then for $f \in C_0^W(-\infty, \infty)$ it follows that $W(x)\overline{B}_n(Wf;x) \to W(x)f(x)$ uniformly as $n \to \infty$.*

Again, this result is analogous to the results obtained using the very different approach using weighted orthogonal polynomials defined on $(-\infty, \infty)$, where the orthogonality is with respect to the weight W^2, with the effect that to approximate the function f in the weighted norm by a polynomial P one is in fact approximating Wf by WP. The remarks of Lorentz quoted above are very much pertinent here, too.

6. Unsolved Problems

- Are the sufficient conditions for b_n and c_n also necessary? In particular, is there a choice of b_n which gives convergence at least when $\alpha = 1$? And for c_n when $\alpha = 2$?
- If the answer above is in fact “no” is there anything else which could be done to get around these restrictions?

- Bernstein polynomial expansions on $[-1,1]$ and, more generally, on intervals $[-c,c]$ preserve evenness and oddness. Perhaps this phenomenon could be exploited to yield new identities for binomial expansions, starting with such basic things as adding together the kth and $n-k$th terms in $\overline{B}_n(c_n,1;x)$ with $c_n := 1$ for all n. Such results may exist, but a search has found essentially nothing.
- Continuing with the topic of evenness and oddness and new identities for the summation of the terms, might it be possible to get better convergence results for weighted approximation on $[0,\infty)$ by first computing $\overline{B}_n f$ for an even function f and then replacing x^2 everywhere by x? Or would this end up not making any difference?
- The operators described here all depend upon continuity, but an adaptation of Kantorovich type extends them to L^p. What can be said about weighted approximation in L^p?
- Both Bernstein and Chlodovsky established results about convergence of derivatives of their polynomial operators to corresponding derivatives of the function being approximated. To what extent might these properties apply or fail to apply in the context of weighted approximation?

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