CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2019
(B. Draganov, K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 147-155
Prof. Marin Drinov Publishing House of BAS, Sofia, 2020

Further Results on the Zeros of the Derivative of Oscillating Polynomials with Laguerre Weight *

Lozko Milev and Nikola Naidenov

Denote by $\mathcal{V}_n(\lambda)$ the set of all weighted polynomials of the form $f(x) = e^{-\lambda x}p(x)$ $(\lambda > 0)$, where p is an algebraic polynomial of degree n which has n simple real zeros. Given $f \in \mathcal{V}_n(\lambda)$, let $x_1 < \cdots < x_n$ and $t_1 < \cdots < t_n$ be the zeros of f and f', respectively.

In a recent paper we proved sharp estimates of the form

$$x_k + c_k h_k \le t_k \le x_{k+1} - d_k h_k, \qquad k = 1, \ldots, n-1, \tag{1}$$

where $h_k := x_{k+1} - x_k$, $k = 1, \ldots, n-1$, and $\{c_k\}_{k=1}^{n-1}$ $(\{d_k\}_{k=1}^{n-1})$ are explicit expressions, depending on λ and h_k.

Here we present improvements of (1) of the form

$$x_k + c_k^* h_k \le t_k \le x_{k+1} - d_k^* h_k, \qquad k = 1, \ldots, n-1,$$

where $\{c_k^*\}_{k=1}^{n-1}$ $(\{d_k^*\}_{k=1}^{n-1})$ depend on $\lambda, x_1, x_k, x_{k+1}$ $(\lambda, x_k, x_{k+1}, x_n)$.

We also give numerical examples and comparisons.

Keywords and Phrases: Weighted oscillating polynomials, zeros of polynomials.

Mathematics Subject Classification 2010: Primary 26C10; Secondary 41A50, 65H04.

1. Introduction and Statement of the Results

Denote by π_n the set of all real algebraic polynomials of degree at most n. Let $\mathcal{P}_n$ be the subset of π_n which consists of the oscillating polynomials, i.e. polynomials from π_n having n simple real zeros.

*This research was supported by the Sofia University Research Grant No. 80-10-17/2019.

Various extremal problems, concerning estimation of a derivative of a function from a given class of oscillating functions were studied in the papers Bojanov and Rahman [5], Milev [8], Bojanov [3], Bojanov and Naidenov [4], Milev and Naidenov [9, 10, 12].

A natural problem related to oscillating polynomials is to find bounds for the zeros of the derivative in terms of the zeros of the polynomial. Sharp estimates of this type for algebraic polynomials were established by Sz. Nagy [14] (see also [15, Corollary 6.5.6]).

In [13] we proved a generalization of the result of Sz. Nagy for the class of oscillating polynomials with Laguerre weight:

$$\mathcal{V}_n(\lambda) := \{e^{-\lambda x} p(x) : \ p \in \mathcal{P}_n\}, \qquad \lambda \neq 0.$$

Our approach is based on some monotonicity properties of the zeros of the polynomials from $\mathcal{V}_n(\lambda)$. In our opinion, the methods used in the case of algebraic polynomials cannot be applied for the class $\mathcal{V}_n(\lambda)$.

Next we formulate the main result from [13]. Without loss of generality we can suppose that $\lambda > 0$.

Theorem A. *Let $f \in \mathcal{V}_n(\lambda)$, $\lambda > 0$, has zeros $x_1 < \cdots < x_n$ and let $t_1 < \cdots < t_n$ be the zeros of f'. Then the following estimates hold true:*

$$x_k + c_k h_k \leq t_k \leq x_{k+1} - d_k h_k, \qquad k = 1, \ldots, n-1, \tag{2}$$

where

$$\begin{aligned} h_k &= x_{k+1} - x_k, \\ c_k &= \frac{2}{n-k+1+\lambda h_k + \sqrt{h_k^2\lambda^2 + 2\lambda(n-k-1)h_k + (n-k+1)^2}}, \\ d_k &= \frac{2}{\sqrt{(k+1-\lambda h_k)^2 + 4\lambda h_k} + k + 1 - \lambda h_k}, \end{aligned}$$

and

$$x_n + c_n h_{n-1} \leq t_n \leq x_n + d_n h_{n-1}, \tag{3}$$

where

$$\begin{aligned} c_n &= \frac{2}{\sqrt{h_{n-1}^2\lambda^2 + 4} + \lambda h_{n-1} - 2}, \\ d_n &= \frac{2}{\sqrt{(n-\lambda h_{n-1})^2 + 4\lambda h_{n-1}} + \lambda h_{n-1} - n}. \end{aligned}$$

In addition, the inequalities (2) *and* (3) *are sharp.*

In the present paper we obtain sharper estimates of the type (2) for t_k, $k = 1, \ldots, n-1$ taking into account the dependence of the factors c_k and d_k on the extreme zeros of a polynomial $f \in \mathcal{V}_n(\lambda)$.

Theorem 1. *Let $f \in \mathcal{V}_n(\lambda)$, $\lambda > 0$, has zeros $x_1 < \cdots < x_n$ and let $t_1 < \cdots < t_n$ be the zeros of f'. Then the following estimates hold true:*

$$x_k + c_k^* h_k \le t_k \le x_{k+1} - d_k^* h_k, \qquad k = 1, \ldots, n-1, \tag{4}$$

where $h_k = x_{k+1} - x_k$, c_1^* (d_{n-1}^*) coincides with c_1 (d_{n-1}) from Theorem A, $c_k^* = c_k^*(\lambda, x_1, x_k, x_{k+1})$, $k = 2, \ldots, n-1$ is such that $\underline{t}_k = x_k + c_k^* h_k$ is the intermediate root of the equation $p_k(x) = 0$, where

$$\begin{aligned} p_k(x) &:= -\lambda(x - x_1)(x - x_k)(x - x_{k+1}) + (k-1)(x - x_k)(x - x_{k+1}) \\ &\quad + (x - x_1)(x - x_{k+1}) + (n-k)(x - x_1)(x - x_k), \end{aligned}$$

and $d_k^* = d_k^*(\lambda, x_k, x_{k+1}, x_n)$, $k = 1, \ldots, n-2$ is such that $\bar{t}_k = x_{k+1} - d_k^* h_k$ is the smallest root of the equation $q_k(x) = 0$, where

$$\begin{aligned} q_k(x) &:= -\lambda(x - x_k)(x - x_{k+1})(x - x_n) + k(x - x_{k+1})(x - x_n) \\ &\quad + (x - x_k)(x - x_n) + (n-k-1)(x - x_k)(x - x_{k+1}). \end{aligned}$$

We also give numerical results illustrating the advantage of the new estimates for different distributions of the zeros. In addition, we study numerically the sharpness of the estimates depending on λ.

2. Proof of Theorem 1

We shall use the following lemmas from [13].

Let $X := \{(x_1, \ldots, x_n) : \; x_1 < \cdots < x_n\}$.

Lemma 1. *Let $f \in \mathcal{V}_n(\lambda)$, $\lambda > 0$, has zeros $\overline{x} = (x_1, \ldots, x_n) \in X$. Denote by $t_i(\overline{x}) \in (x_i, x_{i+1})$, $i = 1, \ldots, n$ ($x_{n+1} := +\infty$) the zeros of f'. Then for all $i, j \in \{1, \ldots, n\}$, $t_i(\overline{x})$ is a strictly increasing function of x_j in the domain X.*

Let $\overline{X} := \{(x_1, \ldots, x_n) : \; x_1 \le \cdots \le x_n\}$.

Lemma 2. *Given $\overline{x} \in \overline{X}$ and $\lambda > 0$, let $f(\overline{x}; s) = e^{-\lambda s}(s - x_1) \cdots (s - x_n)$ and $t_1(\overline{x}) \le \cdots \le t_n(\overline{x})$ be the zeros of $f'(\overline{x}; \cdot)$. Then for every $i = 1, \ldots, n$, $t_i(\overline{x})$ is a continuous function in $\overline{X}$.*

Lemma 3. *Let f and g be two polynomials from $\mathcal{V}_n(\lambda)$, $\lambda > 0$, with zeros $\overline{x}$ and $\overline{y}$, respectively, which satisfy the conditions: $x_1 \le \cdots \le x_n$, $y_1 \le \cdots \le y_n$, and $x_i \le y_i$, for $i = 1, \ldots, n$. Let $t_1(\overline{x}) \le \cdots \le t_n(\overline{x})$ and $t_1(\overline{y}) \le \cdots \le t_n(\overline{y})$ be the zeros of f' and g'. Then we have $t_i(\overline{x}) \le t_i(\overline{y})$, for $i = 1, \ldots, n$.*

Proof of Theorem 1. If $n = 2$ the estimates (4) coincide with (2) from Theorem A. Thus we can suppose that $n \ge 3$.

First we shall prove the *upper estimate* for t_k, $k = 1, \dots, n-2$. Recall that for $k = n-1$ the estimate is the same as that of Theorem A.

If $n = 3$ we have $f'(x) = e^{-\lambda x} q_1(x)$ hence $t_1 = \bar{t}_1$ and (4) holds as an equality.

Let us suppose that $n \geq 4$. We fix $k \in \{1, \dots, n-2\}$ and consider the polynomials

$$g_k(\overline{y}; x) = e^{-\lambda x}(x - y_1) \cdots (x - y_n),$$

where $y_1 < \cdots < y_n$ satisfy the conditions

$$\begin{aligned} & y_i \nearrow x_k, \quad i = 1, \dots, k-1, \quad y_i \in [x_i, x_k), \\ & y_i = x_i, \quad i = k, k+1, \\ & y_i \nearrow x_n, \quad i = k+2, \dots, n-1, \quad y_i \in [x_i, x_n), \\ & y_n = x_n. \end{aligned} \tag{5}$$

Here the notation $x \nearrow c$ means that x is strictly increasing and tends to c. Let us denote the zeros of $g_k'(\overline{y}; x)$ by $\tau_{1,k}(\overline{y}) < \cdots < \tau_{n,k}(\overline{y})$. It follows from Lemma 1 that $\tau_{i,k}(\overline{y})$, $i = 1, \dots, n$ are strictly increasing functions when $\overline{y}$ changes from $\overline{x} = (x_1, \dots, x_n)$ to $\overline{z} := ((x_k, k), x_{k+1}, (x_n, n-k-1))$ as in (5). Let $\bar{t}_{1,k} \leq \cdots \leq \bar{t}_{n,k}$ be the zeros of the derivative of

$$\overline{g}_k(x) := g_k(\overline{z}; x) = e^{-\lambda x}(x - x_k)^k (x - x_{k+1})(x - x_n)^{n-k-1}.$$

By Lemma 2, $\tau_{i,k}(\overline{y}) \to \bar{t}_{i,k}$, $i = 1, \dots, n$. In particular,

$$t_k = \tau_{k,k}(\overline{x}) < \bar{t}_{k,k}. \tag{6}$$

In view of (6), it remains to prove that $\bar{t}_k = \bar{t}_{k,k}$. It can be verified that

$$\overline{g}_k'(x) = \overline{g}_k(x) \overline{h}_k(x),$$

where

$$\overline{h}_k(x) := -\lambda + \frac{k}{x - x_k} + \frac{1}{x - x_{k+1}} + \frac{n-k-1}{x - x_n}.$$

By the definition of $\overline{g}_k(x)$ and the Rolle's theorem it is seen that $\overline{h}_k(x)$ has exactly three real zeros: $\bar{t}_{k,k} \in (x_k, x_{k+1})$, $\bar{t}_{k+1,k} \in (x_{k+1}, x_n)$ and $\bar{t}_{n,k} \in (x_n, +\infty)$. Since

$$\overline{h}_k(x) = \frac{q_k(x)}{(x - x_k)(x - x_{k+1})(x - x_n)},$$

the upper bound $\bar{t}_{k,k}$ is the smallest root of the equation $q_k(x) = 0$, i.e. $\bar{t}_{k,k} = \bar{t}_k$. The proof of the upper bound in (4) is completed.

Now we shall prove the *lower estimate* for t_k, $k = 2, \dots, n-1$ and $n \geq 4$. For $k = 1$ the estimate follows from Theorem A, while for $n = 3$ and $k = 2$ we have $f'(x) = e^{-\lambda x} p_2(x)$.

We define the polynomials

$$g_k(\overline{y};x) = e^{-\lambda x}(x-y_1)\cdots(x-y_n),$$

where $y_1 < \cdots < y_n$ satisfy the conditions

$$\begin{aligned} &y_1 = x_1 \\ &y_i \searrow x_1, \quad i=2,\dots,k-1, \quad y_i \in (x_1, x_i], \\ &y_i = x_i, \quad i = k, k+1, \\ &y_i \searrow x_{k+1}, \quad i = k+2,\dots,n, \quad y_i \in (x_{k+1}, x_i]. \end{aligned} \tag{7}$$

The notation $x \searrow c$ means that x strictly decreases and tends to c. Lemma 1 implies that the zeros $\tau_{1,k}(\overline{y}) < \cdots < \tau_{n,k}(\overline{y})$ of $g_k'(\overline{y};x)$ are strictly decreasing when $\overline{y} \to \overline{z} := ((x_1, k-1), x_k, (x_{k+1}, n-k))$ as in (7). By Lemma 2, $\tau_{i,k}(\overline{y}) \to \underline{t}_{i,k}$, $i = 1,\dots,n$, where $\underline{t}_{1,k} \le \cdots \le \underline{t}_{n,k}$ are the zeros of the derivative of

$$\underline{g}_k(x) := g_k(\overline{z};x) = e^{-\lambda x}(x-x_1)^{k-1}(x-x_k)(x-x_{k+1})^{n-k}.$$

Since $\tau_{k,k}(\overline{y})$ strictly decreases from t_k to $\underline{t}_{k,k}$, we conclude that

$$\underline{t}_{k,k} < \tau_{k,k}(\overline{x}) = t_k. \tag{8}$$

Furthermore, we have

$$\underline{g}_k'(x) = \underline{g}_k(x)\underline{h}_k(x),$$

where

$$\underline{h}_k(x) := -\lambda + \frac{k-1}{x-x_1} + \frac{1}{x-x_k} + \frac{n-k}{x-x_{k+1}}.$$

Note that $\underline{h}_k(x)$ has exactly three real zeros which are $\underline{t}_{k-1,k} \in (x_1, x_k)$, $\underline{t}_{k,k} \in (x_k, x_{k+1})$ and $\underline{t}_{n,k} \in (x_{k+1}, +\infty)$. The equality

$$\underline{h}_k(x) = \frac{p_k(x)}{(x-x_1)(x-x_k)(x-x_{k+1})},$$

shows that the lower bound $\underline{t}_{k,k}$ is the intermediate root of the equation $p_k(x) = 0$, i.e. $\underline{t}_{k,k} = \underline{t}_k$. In view of (8), the proof of the lower bound in (4) is completed. Theorem 1 is proved. □

3. Numerical Examples and Comparisons

1. Let $\lambda = 1$, $n = 10$ and $p(x) = L_{10}(x)$, where $L_n(x)$ denotes the Laguerre polynomial of degree n. Recall that $L_n(x)$ is orthogonal to the polynomials of degree not exceeding $n-1$ with weight $\mu(x) = e^{-x}$ on the interval $(0,\infty)$.

Note that the zeros of L_{10} are:

$$x_1 = 0.137793,\ x_2 = 0.729455,\ x_3 = 1.80834,\ x_4 = 3.40143,\ x_5 = 5.55250,$$
$$x_6 = 8.33015,\ x_7 = 11.8438,\ x_8 = 16.2793,\ x_9 = 21.9966,\ x_{10} = 29.9207.$$

Table 1 contains the values of the critical points and the corresponding estimates from Theorem A and Theorem 1 for $f(x) = e^{-x}L_{10}(x) \in \mathcal{V}_{10}(1)$.

k	$x_k + c_k h_k$	$x_k + c_k^* h_k$	t_k	$x_{k+1} - d_k^* h_k$	$x_{k+1} - d_k h_k$
1	0.193952	0.193952	0.334529	0.379853	0.390784
2	0.837661	0.85457	1.12825	1.32949	1.35414
3	1.97739	2.00987	2.39587	2.82409	2.86093
4	3.64286	3.69645	4.16684	4.8846	4.93242
5	5.88126	5.96451	6.48735	7.56453	7.62324
6	8.76508	8.89089	9.42835	10.9562	11.0272
7	12.4072	12.5943	13.1017	15.2155	15.3025
8	16.9937	17.2654	17.6965	20.6294	20.7377
9	22.8723	23.2412	23.5778	27.9584	27.9584

Table 1. Critical points and their estimates for $f(x) = e^{-x}L_{10}(x)$.

In order to compare the estimates we also consider the corresponding relative errors for the above weighted polynomial. The results are presented in Table 2.

k	$\frac{t_k-(x_k+c_k h_k)}{h_k}$	$\frac{t_k-(x_k+c_k^* h_k)}{h_k}$	$\frac{(x_{k+1}-d_k^* h_k)-t_k}{h_k}$	$\frac{(x_{k+1}-d_k h_k)-t_k}{h_k}$
1	0.237596	0.237596	0.0766055	0.09508
2	0.269345	0.253671	0.186521	0.209365
3	0.262685	0.242293	0.268797	0.291923
4	0.243589	0.218679	0.333675	0.355906
5	0.218205	0.188231	0.387801	0.408936
6	0.188772	0.152966	0.434828	0.455046
7	0.156576	0.114404	0.476566	0.496187
8	0.122929	0.0753967	0.51298	0.531928
9	0.0890258	0.0424728	0.552821	0.552821

Table 2. Relative errors of the estimates for the critical points of $f(x) = e^{-x}L_{10}(x)$.

2. Now we consider the polynomial $f(x) \in \mathcal{V}_{10}(1)$ which has equally spaced zeros in the interval $[0, 1]$. The estimates for the critical points and the relative errors are given in Tables 3 and 4.

The following trend can be observed:

- the lower estimates from Theorem 1 improve more significantly these from Theorem A for the larger critical points;
- the upper estimates from Theorem 1 improve more significantly these from Theorem A for the smaller critical points.

k	$x_k + c_k h_k$	$x_k + c_k^* h_k$	t_k	$x_{k+1} - d_k^* h_k$	$x_{k+1} - d_k h_k$
1	0.109911	0.109911	0.128562	0.137903	0.148751
2	0.211002	0.212044	0.235246	0.258204	0.265918
3	0.312365	0.313738	0.340283	0.369419	0.374525
4	0.414113	0.41591	0.444719	0.476172	0.479676
5	0.516438	0.518846	0.548945	0.580586	0.583099
6	0.619684	0.623035	0.653205	0.683668	0.685538
7	0.724537	0.72943	0.757752	0.785938	0.787362
8	0.832601	0.840106	0.862994	0.887702	0.888778
9	0.948751	0.959966	0.970046	0.989909	0.989909

Table 3. Critical points and their estimates for $f(x) = e^{-x} \prod_{k=1}^{10}(x - \frac{k}{10})$.

k	$\frac{t_k-(x_k+c_k h_k)}{h_k}$	$\frac{t_k-(x_k+c_k^* h_k)}{h_k}$	$\frac{(x_{k+1}-d_k^* h_k)-t_k}{h_k}$	$\frac{(x_{k+1}-d_k h_k)-t_k}{h_k}$
1	0.186511	0.186511	0.0934074	0.201889
2	0.242439	0.232023	0.229574	0.306716
3	0.279183	0.265449	0.291366	0.342425
4	0.306069	0.288099	0.314526	0.349567
5	0.325068	0.300984	0.316412	0.341547
6	0.335208	0.301696	0.304631	0.323329
7	0.332144	0.283213	0.281861	0.296105
8	0.303934	0.228886	0.247075	0.257839
9	0.212953	0.100800	0.198632	0.198632

Table 4. Relative errors of the estimates for the critical points of $f(x) = e^{-x} \prod_{k=1}^{10}(x - \frac{k}{10})$.

The above observation is in accordance with the fact that $c_1^* = c_1$ and $d_{n-1}^* = d_{n-1}$, respectively.

Moreover, the relative errors form either a monotone sequence or a sequence having one change of the monotonicity.

3. Our next goal is to study numerically, depending on λ, for a given polynomial $f \in \mathcal{V}_n(\lambda)$ the quantities:

$$k_f^l(\lambda) = \frac{t_{n-1} - (x_{n-1} + c_{n-1}h_{n-1})}{t_{n-1} - (x_{n-1} + c_{n-1}^* h_{n-1})} \qquad \text{and} \qquad k_f^u(\lambda) = \frac{(x_2 - d_1 h_1) - t_1}{(x_2 - d_1^* h_1) - t_1}.$$

Here $k_f^l(\lambda)$ ($k_f^u(\lambda)$) is the ratio of the relative errors in the lower (upper) estimates, where the improvement is maximal.

$n = 10$	$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 5$	$\lambda = 10$	$\lambda = 25$
$k_f^l(\lambda)$	5.306	2.364	2.096	2.405	2.471	2.512
$k_f^u(\lambda)$	1.223	1.232	1.241	1.257	1.249	1.239

Table 5. The quantities $k_f^l(\lambda)$ and $k_f^u(\lambda)$ for $f(x) = e^{-\lambda x} L_{10}(x)$.

$n = 5$	$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 5$	$\lambda = 10$	$\lambda = 25$
$k_f^l(\lambda)$	7.740	4.407	3.463	3.443	3.509	3.557
$k_f^u(\lambda)$	1.791	1.837	1.874	1.886	1.871	1.859

Table 6. The quantities $k_f^l(\lambda)$ and $k_f^u(\lambda)$ for $f(x) = e^{-\lambda x} L_5(x)$.

$n = 10$	$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 5$	$\lambda = 10$	$\lambda = 25$
$k_f^l(\lambda)$	2.135	2.125	2.112	1.996	1.839	1.531
$k_f^u(\lambda)$	2.140	2.149	2.161	2.234	2.281	2.264

Table 7. The quantities $k_f^l(\lambda)$ and $k_f^u(\lambda)$ for $f(x) = e^{-\lambda x} \prod_{k=1}^{10}(x - \frac{k}{10})$.

$n = 5$	$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 5$	$\lambda = 10$	$\lambda = 25$
$k_f^l(\lambda)$	3.241	3.204	3.157	2.745	2.372	2.120
$k_f^u(\lambda)$	3.259	3.293	3.333	3.539	3.591	3.487

Table 8. The quantities $k_f^l(\lambda)$ and $k_f^u(\lambda)$ for $f(x) = e^{-\lambda x} \prod_{k=1}^{5}(x - \frac{k}{5})$.

The numerical results in Tables 5–8 suggest the following conclusions.

The lower estimates from Theorem 1 give the largest improvement compared to these of the Theorem A for small positive λ.

For the upper estimates there exists a unique $\lambda = \lambda^* > 0$, for which $k_f^u(\lambda)$ attains its maximal value.

References

[1] B. Bojanov, Polynomial inequalities, *in* "Open Problems in Approximation Theory" (B. Bojanov, Ed.), pp. 25–42, SCT Publishing, Singapore, 1994.

[2] B. Bojanov, Markov interlacing property for perfect splines, *J. Approx. Theory* **100** (1999), 183–201.

[3] B. Bojanov, Markov-type inequalities for polynomials and splines, *in* "Approximation Theory X: Abstract and Classical Analysis" (C. K. Chui, L. L. Schumaker and J. Stöckler, Eds.), pp. 31–90, Vanderbilt University Press, Nashville, TN, 2002.

[4] B. Bojanov and N. Naidenov, Exact Markov-type inequalities for oscillating perfect splines, *Constr. Approx.* **18** (2002), 37–59.

[5] B. D. Bojanov and Q. I. Rahman, On certain extremal problems for polynomials, *J. Math. Anal. Appl.* **189** (1995), 781–800.

[6] M. Marden, “Geometry of Polynomials”, American Mathematical Society, Providence, Rhode Island, 1966.

[7] V. A. Markov, On the Functions of Least Deviation from Zero in a Given Interval, Izdat. Imp. Akad. Nauk, St. Petersburg, 1892 (in Russian).

[8] L. Milev, Weighted polynomial inequalities on infinite intervals, *East J. Approx.* **5** (1999), 449–465.

[9] L. Milev and N. Naidenov, Markov’s inequalities in integral norm for oscillating weighted polynomials, *in* “Approximation Theory: A Volume Dedicated to Borislav Bojanov” (D. K. Dimitrov, G. Nikolov and R. Uluchev, Eds.), pp. 176–185, Marin Drinov Academic Publishing House, Sofia, 2004.

[10] L. Milev and N. Naidenov, Monotonicity of the zeros and the extremal points of certain oscillating polynomials, *C. R. Acad. Bulg. Sci.* **60** (2007), 111–116.

[11] L. Milev and N. Naidenov, Markov interlacing property for exponential polynomials, *J. Math. Anal. Appl.* **367** (2010), 669–676.

[12] L. Milev and N. Naidenov, Markov type inequalities for oscillating exponential polynomials, *in* “Constructive Theory of Functions, Sozopol 2010: In memory of Borislav Bojanov” (G. Nikolov and R. Uluchev, Eds.), pp. 201–212, Prof. Marin Drinov Academic Publishing House, Sofia, 2012.

[13] L. Milev and N. Naidenov, Sharp estimates for the zeros of the derivative of oscillating polynomials with Laguerre weight, *Math. Inequal. Appl.* **23** (2020), 509–519.

[14] J. v. Sz. Nagy, Über algebraische Gleichungen mit lauter reelen Wurzeln, *Jahresbericht der Deutschen Math. Vereinigung* **27** (1918), 37–43.

[15] Q. I. Rahman and G. Schmeisser, “Analytic Theory of Polynomials”, Clarendon Press, Oxford, 2002.

[16] T. J. Rivlin, “Chebyshev Polynomials”, 2nd edition, John Wiley & Sons, Inc., New York, 1990.

Lozko Milev
Department of Mathematics and Informatics
University of Sofia
Boulevard James Bourchier 5
1164 Sofia
BULGARIA
E-mail: milev@fmi.uni-sofia.bg

Nikola Naidenov
Department of Mathematics and Informatics
University of Sofia
Boulevard James Bourchier 5
1164 Sofia
BULGARIA
E-mail: nikola@fmi.uni-sofia.bg