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Worst Case and Average Case Cardinality of Strictly Acute Stencils for Two Dimensional Anisotropic Fast Marching

Jean-Marie Mirebeau and François Desquilbet *

We study a one dimensional approximation-like problem arising in the discretization of a class of Partial Differential Equations, providing worst case and average case complexity results. The analysis is based on the Stern-Brocot tree of rationals, and on a non-Euclidean notion of angles. The presented results generalize and improve on earlier work [10].

1. Introduction

This paper is devoted to the analysis of an approximation-like problem arising in the discretization of a class of Partial Differential Equations (PDE): eikonal equations, defined with respect to a possibly strongly anisotropic Finslerian metric. The results presented are related with the numerical solution of this equation on two dimensional cartesian grids, and their extension to higher dimension and/or to unstructured domains remains an open question. The unique viscosity solution to such an equation is a distance map, whose computation has numerous applications [14] in domains as varied as motion planning, seismic traveltime tomography [7], image processing [3], etc. The construction studied in this paper is designed is to achieve a geometrical property – strict acuteness with respect to a given asymmetric norm – ensuring that the resulting numerical scheme is strictly causal [6, 15, 1, 10, 9]. This in turn enables efficient algorithms for solving the numerical scheme, in a single pass over the domain, with linear complexity, and possibly in parallel [16, 13]. In order to better focus on the problem of interest, further discussion of the addressed PDE and of its discretization is postponed to § A.

We study in this paper a one dimensional approximation-like problem, involved in the construction of local stencils of minimal cardinality for a numerical solver of eikonal PDEs, see Definition 1.3 for a formal statement. The efficiency of the procedure is directly tied to the complexity of the numerical

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scheme. A few properties of this problem deviate from the common settings in approximation theory, and deserve to be discussed here.

- The main function $\varphi_F : \mathbb{R} \to]-\pi/2, \pi/2[$ considered benefits from regularity and integrability properties, derived from its geometrical interpretation § 2.2. However these are fairly uncommon: $-\varphi_F$ is one-sided Lipschitz, and $\tan(\varphi_F)$ is bounded in the $L^1([0, 2\pi])$ norm.

- The approximation-like problem involves an interval subdivision procedure, that is reminiscent of e.g. dyadic splitting in non-linear approximation based on the Haar system [5]. However, subdivision is here governed by the Stern-Brocot tree, and breaks the interval $[0, 2\pi]$ into unequal parts whose endpoints have rational tangents, see § 3.

- We present a uniform "worst case" complexity result, but also an "average case" result under random shifts, see Theorem 1.1. Because of the peculiarities of the approximation procedure, a more favorable estimate is obtained in the average case.

In the rest of this introduction, we introduce the notations and concepts necessary to state our main result. Our first step is to equip the Euclidean space $\mathbb{R}^2$ with the anisotropic geometry defined by a (possibly) asymmetric norm. Here and below, all asymmetric norms are on $\mathbb{R}^2$.

Definition 1.1. An asymmetric norm is a function $F : \mathbb{R}^2 \to \mathbb{R}_+$ which is 1-positively homogeneous, obeys the triangular inequality, and vanishes only at the origin:

$$F(\lambda u) = \lambda F(u), \qquad F(u+v) \le F(u) + F(v), \qquad F(u) = 0 \ \Leftrightarrow \ u = 0,$$

for all $u, v \in \mathbb{R}^2$, $\lambda \ge 0$. The anisotropy ratio of F is defined as

$$\mu(F) := \max_{|u|=|v|=1} \frac{F(u)}{F(v)}.$$

Note that an asymmetric norm is always a continuous and convex function. We denote by $\measuredangle(u, v) \in [0, \pi]$ the *unoriented* Euclidean angle between two vectors $u, v \in \mathbb{R}^2 \setminus \{0\}$, which is characterized by the identity

$$\cos \measuredangle(u, v) = \frac{\langle u, v \rangle}{\|u\| \|v\|}.$$

The next definition introduces a generalized measure of angle, associated with an asymmetric norm. We only consider acute angles, since obtuse angles will not be needed, and because their definition raises issues. The notion of F-acute angle is similarly defined in [10, 17], but the related angular measure is new.

Definition 1.2. Let F be an asymmetric norm, which is differentiable except at the origin, and let $u, v \in \mathbb{R}^2 \setminus \{0\}$. We say that u, v form an F-acute angle iff $\langle \nabla F(u), v \rangle \geq 0$. We define the F-angle $\measuredangle_F(u, v) \in [0, \pi/2] \cup \{\infty\}$ by

$$\cos \measuredangle_F(u, v) := \frac{\langle \nabla F(u), v \rangle}{F(v)} \tag{1.1}$$

if u, v form an F-acute angle. Otherwise we let $\measuredangle_F(u, v) := +\infty$.

We show in Lemma 2.1 that the r.h.s. of (1.1) is at most 1, so that $\measuredangle_F(u, v)$ is well defined, with equality if $u = v$, so that $\measuredangle_F(u, u) = 0$. If F is the Euclidean norm, then one easily checks that the F-angle coincides with the usual Euclidean angle, when the latter is acute. More generally, if $F(u) = \|Au\|$ for some invertible linear map A, then $\measuredangle_F(u, v) = \measuredangle(Au, Av)$, when the latter is acute. In general however, one has $\measuredangle_F(u, v) \neq \measuredangle_F(v, u)$, and F-acuteness is not a symmetric relation. The differentiability assumption in Definition 1.2 can be removed, see Definition 2.1.

The following definition introduces (F, α)-acute stencils, which are at the foundation of our numerical scheme, see Figure A.1 on page 177. Their cardinality is directly proportional to the algorithmic complexity of our eikonal PDE solver, see § A, hence it is important to choose them as small as possible. When $\alpha = \pi/2$ one recovers the F-acute stencils of [10], and closely related concepts are considered in [6, 15, 17, 1].

Definition 1.3. A *stencil* is a finite sequence of pairwise distinct vectors $u_1, \ldots, u_n \in \mathbb{Z}^2$, $n \geq 4$, such that

$$\det(u, v) = 1, \qquad \langle u, v \rangle \geq 0,$$

for all $u = u_i$, $v = u_{i+1}$, $1 \leq i \leq n$, with the convention $u_{n+1} := u_n$. It is said (F, α)-acute, where F is an asymmetric norm and $\alpha \in]0, \pi/2]$, iff with the same notations one has

$$\measuredangle_F(u, v) \leq \alpha, \qquad \measuredangle_F(v, u) \leq \alpha. \tag{1.2}$$

We let $N(F, \alpha)$ denote the minimal cardinality of an (F, α)-acute stencil.

We provide in § 3.2 a simple and efficient algorithm, based on a recursive refinement procedure and which is effectively used in our numerical implementation, for producing an (F, α)-acute stencil of minimal cardinality $N(F, \alpha)$. A similar method appears in [10] when $\alpha = \pi/2$. The main result of this paper is the following estimate of $N(F, \alpha)$, both in the worst case and in the average case over random rotations of the asymmetric norm F. The average case makes sense in view of our application to PDE discretizations § A, since the orientation of the grid can be set and modified arbitrarily.

Theorem 1.1. *For any asymmetric norm F and any $\alpha \in]0, \pi/2]$, one has*

$$N(F, \alpha) \leq C \frac{\mu}{\alpha^2} \ln\Big(\frac{\ln \mu}{\alpha^2}\Big), \quad \int_0^{2\pi} N(F \circ R_\theta, \alpha)\, \mathrm{d}\theta \leq C \frac{\ln \mu}{\alpha^2} \ln\Big(\frac{\mu}{\alpha^2}\Big), \quad (1.3)$$

where $\mu = \max\{\mu(F), 12\}$, R_θ denotes the rotation of angle $\theta \in \mathbb{R}$, and C is an absolute constant.

In the intended applications, one typically has $\mu(F) \lesssim 100$. The most pronounced anisotropies $\mu(F) \approx 100$ are often encountered in image processing methods [3, 9], and this bound is large enough that the asymptotic behavior of (1.3) w.r.t. μ is meaningful to our use cases. In contrast, we do confess that it seems pointless to let $\alpha \to 0$ in our applications (typically we set $\alpha = \pi/3$). If one fixes $\alpha_0 \in]0, \pi/2]$ then

$$N(F, \alpha_0) \leq C\mu \ln \ln \mu, \qquad \int_0^{2\pi} N(F \circ R_\theta, \alpha_0)\, \mathrm{d}\theta \leq C \ln^2 \mu, \qquad (1.4)$$

uniformly w.r.t. μ. This improves on [10], whose arguments are limited to the case $\alpha_0 = \pi/2$, and where the sub-optimal bounds $\mu \ln \mu$ (resp. $\ln^3 \mu$) are obtained for (1.4, left) (resp. right).

Outline. The notion of F-acute angle, see Definition 1.2, is described in more detail § 2, where related tools are introduced. The Stern-Brocot tree, an arithmetic structure underlying concept of stencil in Definition 1.3, is discussed in § 3. We conclude in § 4 the proof of Theorem 1.1. Some context on the intended applications of the presented results is given in § A.

2. Anisotropic Angle

This section is devoted to the study of the anisotropic measure of angle $\measuredangle_F(u, v)$ of Definition 1.2, where $u, v \in \mathbb{R}^2 \setminus \{0\}$ and F is an asymmetric norm. Some elementary comparison properties, with the Euclidean angle $\measuredangle(u, v)$ or with another angle $\measuredangle_F(u, w)$, are presented § 2.1. We prepare in § 2.2 (resp. § 2.3) the proof of the average case (resp. worst case) estimate of Theorem 1.1, by introducing a helper function φ_F (resp. $\psi_F^\pm$) for which we show a $L^1([0, 2\pi])$ norm estimate and a comparison principle with $\measuredangle_F$.

In the rest of this section, we fix an asymmetric norm F, assumed to be continuously differentiable on $\mathbb{R}^2 \setminus \{0\}$. That is with the exception of the following definition and proposition, where we briefly consider the case of non-differentiable norms, and show that the smoothness assumption holds without loss of generality. Closely related arguments are found in Lemma 2.11 of [10].

Definition 2.1. (Generalization of $\measuredangle_F(u, v)$ with no differentiability assumption). Let F be an asymmetric norm, and let $u, v \in \mathbb{R}^2 \setminus \{0\}$. We say that u, v form an F-acute angle iff $F(u + \delta v) \geq F(u)$ for all $\delta \geq 0$. In that case we let $\alpha = \measuredangle_F(u, v) \in [0, \pi/2]$ denote the smallest value such that

$$F(u + \delta v) \geq F(u) + \delta \cos(\alpha) F(v), \tag{2.1}$$

for all $\delta \geq 0$. If u, v do not form an F-acute angle, then we let $\measuredangle_F(u, v) := \infty$.

Proposition 2.1. *Definitions 1.2 and 2.1 agree on differentiable norms. Also, if $F_n \to F$ locally uniformly as $n \to \infty$, where $(F_n)_{n \geq 0}$ and F are asymmetric norms, and $u, v \in \mathbb{R}^2 \setminus \{0\}$, then*

$$\measuredangle_F(u, v) \leq \liminf_{n \to \infty} \measuredangle_{F_n}(u, v). \tag{2.2}$$

If Theorem 1.1 holds under the additional assumption $F \in C^1(\mathbb{R}^2 \setminus \{0\})$, then it does without it.

Proof. Under the assumptions of Definition 1.2 one has $F(u + \delta v) = F(u) + \delta \langle \nabla F(u), v \rangle + o(\delta)$ by differentiability of F at u, and $F(u + \delta v) \geq F(u) + \delta \langle \nabla F(u), v \rangle$ by convexity of F, for any $\delta \geq 0$ and any $v \in \mathbb{R}^2 \setminus \{0\}$. Thus Definitions 2.1 and 1.2 agree. The lower semi-continuity property (2.2) follows from the fact that (2.1) is closed under uniform convergence. Therefore if a given stencil is (F_n, α)-acute for all $n \geq 0$, then it is also (F, α)-acute, see Definition 1.3. Thus $N(F, \alpha) \leq \liminf_{n \to \infty} N(F_n, \alpha)$, and likewise for the l.h.s. of (1.3, right). Finally, we observe that any asymmetric norm F is the locally uniform limit of a sequence of asymmetric norms $F_n \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, $n \geq 1$, defined as

$$F_n(u) := \int_{\mathbb{R}} F(R_\theta u) \, \rho_n(\theta) \, \mathrm{d}\theta,$$

where $\rho_n(\theta) := n\rho(n\theta)$, and the mollifier ρ is smooth, non-negative, compactly supported, and has unit integral. The statement regarding Theorem 1.1 follows, which concludes the proof. $\square$

2.1. Elementary Comparison Properties

This subsection is devoted to elementary comparisons between $\measuredangle_F(u, v)$ and the angle between other vectors, see Lemma 2.2, or the Euclidean angle $\measuredangle(u, v)$, see Proposition 2.2, where $u, v \in \mathbb{R}^2 \setminus \{0\}$. In addition, Lemma 2.1 below was announced and used in the introduction to show that $\measuredangle_F(u, v)$ is well defined, and that $\measuredangle_F(u, u) = 0$. Throughout this subsection, F denotes a fixed asymmetric norm, assumed to be differentiable except at the origin.

Lemma 2.1. *For any $u, v \in \mathbb{R}^2$, with $u \neq 0$, one has*

$$\langle \nabla F(u), u \rangle = F(u), \qquad \langle \nabla F(u), v \rangle \leq F(v). \tag{2.3}$$

Proof. Euler's identity for the 1-homogeneous function F yields (2.3, left), whereas the triangular inequality $F(u+\delta v) \leq F(u)+\delta F(v)$ for all $\delta \geq 0$ yields (2.3, right). $\square$

The next lemma shows that the F-angle is non-increasing when an angular sector is split.

Lemma 2.2. *Let u, v form an F-acute angle, and let $w := \alpha u + \beta v$ for some $\alpha, \beta > 0$. Then*

$$\max\{\measuredangle_F(u,w), \measuredangle_F(w,v)\} \leq \measuredangle_F(u,v).$$

Proof. Assume w.l.o.g. that $\alpha = 1$, and denote $\lambda := \cos \measuredangle_F(u,v)$. By convexity of F one has

$$\begin{aligned}\langle \nabla F(w), v\rangle &= \langle \nabla F(u+\beta v), v\rangle \\ &= \langle \nabla F(u+\beta v) - \nabla F(u), v\rangle + \langle \nabla F(u), v\rangle \geq 0 + \lambda F(v).\end{aligned}$$

On the other hand, one obtains noting that $\lambda \in [0,1]$ by assumption

$$\begin{aligned}\langle \nabla F(u), w\rangle &= \langle \nabla F(u), u+\beta v\rangle \geq F(u) + \lambda\beta F(v) \\ &\geq \lambda(F(u)+\beta F(v)) \geq \lambda F(u+\beta v) = \lambda F(w).\end{aligned}$$ $\square$

The last proposition of this subsection is an upper bound on the F-angle in terms of the Euclidean angle and of the anisotropy ratio $\mu(F)$ of the asymmetric norm. This upper bound grows non-linearly and perhaps more quickly than one may expect, namely as the square root of the Euclidean angle, because we do not make any quantitative assumption on the smoothness of F. Here and below we denote $u^\perp := (-b, a)$ for any $u = (a,b) \in \mathbb{R}^2$.

Proposition 2.2. *For any $u, v \in \mathbb{R}^2 \setminus \{0\}$, assuming $\mu(F)\measuredangle(u,v) \leq 1/2$ one has*

$$\measuredangle_F(u,v) \leq \sqrt{5\mu(F)\measuredangle(u,v)}. \tag{2.4}$$

Proof. Denote $\theta := \measuredangle(u,v)$, $\alpha := \measuredangle_F(u,v)$, and $\mu := \mu(F)$. Assume w.l.o.g. that $v = u + \tan(\theta)\, u^\perp$. Then

$$\begin{aligned}\langle \nabla F(u), v\rangle &= \langle \nabla F(u), u\rangle + \tan(\theta)\,\langle \nabla F(u), u^\perp\rangle \geq F(u) - \tan(\theta)\, F(-u^\perp), \\ F(v) &= F(u+\tan(\theta)\, u^\perp) \leq F(u) + \tan(\theta)\, F(u^\perp).\end{aligned}$$

Observing that $F(u^\perp) \leq \mu F(u)$ and $F(-u^\perp) \leq \mu F(u)$, we obtain

$$\begin{aligned}\frac{1-\mu\tan\theta}{1+\mu\tan\theta} &\leq \frac{F(u) - F(-u^\perp)\tan\theta}{F(u)+F(u^\perp)\tan\theta} \\ &\leq \frac{\langle \nabla F(u), v\rangle}{F(v)} = \cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}.\end{aligned} \tag{2.5}$$

This implies $\tan^2(\alpha/2) \leq \mu \tan\theta$. We conclude the proof of (2.4) observing that $\tan(\alpha/2) \geq \alpha/2$, and $\tan\theta \leq (5/4)\theta$, both estimates by convexity of tan on $[0, \pi/2[$ and since $\theta \leq 1/2$. Note also that $\mu \tan\theta \leq (5/4)\mu\theta \leq 5/8 < 1$ by assumption, which shows that the l.h.s. of (2.5) is positive, and thus excludes the case where $\measuredangle_F(u, v) = \infty$, see Definition 1.2. □

2.2. Gradient Deviation

We describe and study a function φ_F attached to the asymmetric norm F of interest, introduced in [10] and used in the proof of the average case estimate in Theorem 1.1. More precisely, the quantity $\varphi_F(u)$ is the *oriented* Euclidean angle between a given vector $u \in \mathbb{R}^2 \setminus \{0\}$ and the gradient $\nabla F(u)$. Note that these two vectors are aligned if F is proportional to the Euclidean norm. The main results of this section are an L^1 estimate of $\tan\varphi_F$, see Corollary 2.1, and a comparison with the F-angle, see Proposition 2.4.

Definition 2.2. For each $u \in \mathbb{R}^2 \setminus \{0\}$, define a signed angle $\varphi_F(u) \in]-\pi/2, \pi/2[$ by

$$\langle u^\perp, \nabla F(u)\rangle = F(u) \tan\varphi_F(u). \tag{2.6}$$

For $\theta \in \mathbb{R}$, we abusively denote $\varphi_F(\theta) := \varphi_F(\,(\cos\theta, \sin\theta)\,)$.

The next lemma shows, as announced, that $|\varphi_F(u)|$ is the Euclidean angle between the given vector u and its image by the gradient of F, and establishes a uniform upper bound for φ_F.

Lemma 2.3. *For any $u \in \mathbb{R}^2 \setminus \{0\}$, one has*

$$|\varphi_F(u)| = \measuredangle(u, \nabla F(u)), \qquad |\tan\varphi_F(u)| \leq \mu(F). \tag{2.7}$$

Proof. Equality (2.7, left) follows from Euler's identity (2.3, left) and the definition (2.6). Estimate (2.7, right) follows from $-F(-u^\perp) \leq \langle u^\perp, \nabla F(u)\rangle \leq F(u^\perp)$ see (2.3, right), and from the upper bound $F(\pm u^\perp) \leq \mu(F)F(u)$ which holds by Definition 1.1 of the anisotropy ratio. □

We recall in the next proposition, without proof, two key properties of the function φ_F established in [10]: a one-sided regularity property, and an upper bound on the integral of $\tan(\varphi_F)$ on any interval. See the plots of φ_F in Figure A.1 on page 177.

Proposition 2.3 (Proposition 3.6 in [10]). *The function $\varphi_F : \mathbb{R} \to]-\pi/2, \pi/2[$ obeys:*

- *(Regularity) For all $\theta \in \mathbb{R}$, one has $\varphi_F'(\theta) \geq -1$.*
- *(Integral bound) One has $|\int_{\theta_*}^{\theta^*} \tan\varphi_F(\theta)\,\mathrm{d}\theta| \leq \ln\mu(F)$ for all $\theta_*, \theta^* \in \mathbb{R}$.*

Combining the one-sided regularity property and the integral bound, one obtains an L^1 estimate of $\tan(\varphi_F)$, as shown in the next lemma, which turns out to be a key ingredient of the proof of the average case estimate (1.3, right), see § 4.2.

Corollary 2.1 (L^1 estimate of $\tan\varphi_F$). *One has with $C = 2\pi\sqrt{3}$,*

$$\int_0^{2\pi} |\tan\varphi_F(\theta)|\,\mathrm{d}\theta \le C(1+\ln\mu(F)). \tag{2.8}$$

Proof. In view of Proposition 2.3 (Integral bound), of the continuity of φ_F, and of its 2π-periodicity, there exists $\alpha_0 \in \mathbb{R}$ such that $\varphi_F(\alpha_0) = 0$. Then inductively for $n \ge 0$ let

- β_n be the smallest $\beta \ge \alpha_n$ such that $|\varphi_F(\beta)| = \pi/3$,
- α_{n+1} be the smallest $\alpha \ge \beta_n$ such that $\varphi_F(\alpha) = 0$.

The sequences $(\alpha_n, \beta_n)_{n\ge 0}$ are well defined, thanks to the periodicity of φ_F, except if $|\varphi_F| < \pi/3$ uniformly, but in that case the announced result (2.8) clearly holds. If $\varphi_F(\beta_n) = \pi/3$ for some $n \ge 0$ then $\alpha_{n+1} - \beta_n \ge \pi/3$, whereas if $\varphi_F(\beta_n) = -\pi/3$ one has $\beta_n - \alpha_n \ge \pi/3$, by Proposition 2.3 (Regularity). Therefore $\alpha_{n+1} - \alpha_n \ge \pi/3$ for all $n \ge 0$, thus $\alpha_6 \ge \alpha_0 + 2\pi$, which implies

$$\int_0^{2\pi} |\tan\varphi_F(\theta)|\,\mathrm{d}\theta \le \int_0^{2\pi} \tan(\pi/3)\,\mathrm{d}\theta + 6\ln\mu(F) \le 2\pi\sqrt{3} + 6\ln\mu(F).$$

On each interval $[\alpha_n, \beta_n] \cap [0, 2\pi]$ we used the upper bound $|\varphi_F(\theta)| \le \pi/3$, which holds by definition of β_n. On each interval $[\beta_n, \alpha_{n+1}] \cap [0, 2\pi]$ we used Proposition 2.3 (Integral bound) and the fact that φ_F does not change sign, which holds by definition of α_{n+1}. □

The last result of this subsection can be regarded as a refinement of Proposition 2.2.

Proposition 2.4 (Estimate of $\measuredangle_F$ in terms of φ_F). *Let $u, v \neq 0$ be such that $\measuredangle(u,v) \le \pi/3$. Then one has, with $C = 32$,*

$$\min\left\{\measuredangle_F(u,v), 2\right\}^2 \le C\measuredangle(u,v)\max\left\{\measuredangle(u,v), |\tan\varphi_F(u)|, |\tan\varphi_F(v)|\right\}. \tag{2.9}$$

Proof. Denote $\theta := \measuredangle(u,v)$ and $\alpha := \measuredangle_F(u,v)$. Assuming w.l.o.g. that $\|u\| = \|v\| = 1$ and $\det(u,v) > 0$ one has

$$v = (u + u^\perp \tan\theta)\cos\theta, \qquad \text{and} \qquad u = (v - v^\perp \tan\theta)\cos\theta.$$

Using linearity in the first line, and convexity in the second line, we obtain

$$\begin{aligned}
\langle v, \nabla F(u)\rangle &= \langle u + u^\perp \tan\theta, \nabla F(u)\rangle \cos\theta \\
&= F(u)\big(1 + \tan\varphi_F(u)\tan\theta\big)\cos\theta \\
F(u) &= F(v - v^\perp\tan\theta)\cos\theta \ge \big(F(v) - \langle v^\perp, \nabla F(v)\rangle \tan\theta\big)\cos\theta \\
&= F(v)\big(1 - \tan\varphi_F(v)\tan\theta\big)\cos\theta.
\end{aligned} \tag{2.10}$$

Assume for a moment that $-\tan\varphi_F(u)\tan\theta \geq 1/2$. Recalling that $\theta \leq \pi/3$, thus $\tan\theta \leq 2\theta$, we obtain $-\tan\varphi_F(u)\theta \geq 1/4$ and the announced result (2.9) is proved. Likewise if $\tan\varphi_F(v)\tan\theta \geq 1/2$. In particular, if $\alpha = +\infty$ then $\langle \nabla F(u), v\rangle \leq 0$ by Definition 1.2, and therefore $-\tan\varphi_F(u)\tan\theta \geq 1$ by (2.10), so that the result is proved.

In the following, we let $t_u := \tan\varphi_F(u)$, $t_v := \tan\varphi_F(v)$. Based on the previous argument we assume w.l.o.g. that $t_u\tan\theta \geq -1/2$, $t_v\tan\theta \leq 1/2$ and $\alpha \neq \infty$. We obtain from (2.10)

$$\cos\alpha = \frac{\langle v, \nabla F(u)\rangle}{F(v)} = \frac{\langle v, \nabla F(u)\rangle}{F(u)} \times \frac{F(u)}{F(v)} \geq (1 + t_u\tan\theta)(1 - t_v\tan\theta)\cos^2\theta.$$

Taking logarithms yields with $t := \max\{0, -t_u, t_v\}$,

$$-\ln\cos\alpha \leq -2\ln(1 - t\tan\theta) - 2\ln\cos\theta. \tag{2.11}$$

An elementary function analysis shows that $-\ln\cos\alpha \geq \alpha^2/2$ for $\alpha \in [0, \pi/2[$, and $-\ln\cos\theta \leq \theta^2$ for $\theta \in [0, \pi/3]$. In addition $\tan\theta \leq 2\theta$ for $\theta \in [0, \pi/3]$, and $-\ln(1-x) \leq 2x$ for $x \in [0, 1/2]$. Inserting these bounds in (2.11) yields the announced result

$$\alpha^2/2 \leq -\ln\cos\alpha \leq 4\max\{-\ln(1 - t\tan\theta), -\ln\cos\theta\} \leq 4\max\{4t\theta, 2\theta^2\}.$$

$\square$

2.3. Regularized Gradient Deviation

We consider in this subsection two 1-Lipschitz regularizations ψ_F^+ and ψ_F^- of the gradient deviation φ_F. See the plots of $\psi_F^\pm$ in Figure A.1 on page 177. Note that $-\varphi_F$ is already (but also only) one-sided 1-Lipschitz, see Proposition 2.3 (Regularity). We extend to $\psi_F^\pm$ some of the results of § 2.2, namely the L^1-norm estimate in Corollary 2.2 and the comparison with the F-angle in Proposition 2.4, which are used § 4.1 in the proof of the worst case estimate in Theorem 1.1. We recall that $\varphi_F : \mathbb{R} \to]-\pi/2, \pi/2[$ is 2π-periodic.

Definition 2.3. Define for any $\theta \in \mathbb{R}$,

$$\psi_F^+(\theta) := \max_{\eta \geq 0} \varphi_F(\theta + \eta) - \eta, \qquad \psi_F^-(\theta) := \min_{\eta \geq 0} \varphi_F(\theta - \eta) + \eta. \tag{2.12}$$

The functions ψ_F^- and ψ_F^+ define an upper and lower envelope of φ_F: for any $\theta \in \mathbb{R}$,

$$-\pi/2 < \inf_{\mathbb{R}} \varphi_F \leq \psi_F^-(\theta) \leq \varphi_F(\theta) \leq \psi_F^+(\theta) \leq \sup_{\mathbb{R}} \varphi_F < \pi/2.$$

They play symmetrical roles, up to replacing φ_F with $\theta \mapsto -\varphi_F(-\theta)$, which amounts to reversing the orientation of the plane $\mathbb{R}^2$. Hence results established for ψ_F^+ automatically extend to ψ_F^-.

Lemma 2.4. *The map $\psi_F^+ : \mathbb{R} \to]-\pi/2, \pi/2[$ is 1-Lipschitz.*

Proof. By design (2.12, left) the function ψ_F^+ is one-sided 1-Lipschitz: for all $\theta \in \mathbb{R}$, $h \geq 0$,

$$\psi_F^+(\theta + h) = \sup_{\eta \geq h} \varphi_F(\theta + \eta) - (\eta - h) \leq \psi_F^+(\theta) + h.$$

On the other hand one has $\psi_F^+(\theta - h) \leq \psi_F^+(\theta) + h$, for all $h \geq 0$, as follows from the same property of the function φ_F, see Proposition 2.3 (Regularity). Combining these two estimates, we obtain $\psi_F^+(\theta+h) \leq \psi_F^+(\theta)+|h|$, for all $\theta \in \mathbb{R}$ and all $h \in \mathbb{R}$ (positive or negative), hence ψ_F^+ is 1-Lipschitz as announced. $\square$

The next lemma and corollary are devoted to estimating the $L^1([0, 2\pi])$ norm of ψ_F^+. We denote by $|A|$ the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}$.

Lemma 2.5. *Let $\theta_0, \theta_1 \in \mathbb{R}$ be such that $\psi_F^+(\theta_0) = \psi_F^+(\theta_1)$. Then*

$$\left|\left\{\theta \in [\theta_0, \theta_1] :\ \psi_F^+(\theta) > \varphi_F(\theta)\right\}\right| \leq \left|\left\{\theta \in [\theta_0, \theta_1] :\ \psi_F^+(\theta) = \varphi_F(\theta)\right\}\right|. \quad (2.13)$$

Proof. Denote by A_0 (resp. A_1) the set appearing in (2.13, left) (resp. (2.13, right)). Then

$$0 = \psi_F^+(\theta_1) - \psi_F^+(\theta_0) = \int_{\theta_0}^{\theta_1} \frac{\mathrm{d}}{\mathrm{d}\theta}\,\psi_F^+ = \int_{A_0} \frac{\mathrm{d}}{\mathrm{d}\theta}\,\psi_F^+ + \int_{A_1} \frac{\mathrm{d}}{\mathrm{d}\theta}\,\psi_F^+ \geq |A_0| - |A_1|,$$

where we used the observation that $\frac{\mathrm{d}}{\mathrm{d}\theta}\,\psi_F^+(\theta) = 1$ for all $\theta \in A_0$, whereas $\frac{\mathrm{d}}{\mathrm{d}\theta}\psi_F^+(\theta) \geq -1$ for a.e. $\theta \in A_1$. The result follows. $\square$

Corollary 2.2 (L^1 estimate of ψ_F^+).

$$\int_0^{2\pi} \max\{0, \tan\psi_F^+ - 1\} \leq 2\int_0^{2\pi} \max\{0, \tan\varphi_F - 1\}.$$

Proof. As observed in the proof of Corollary 2.1 there exists $\theta_0 \in \mathbb{R}$ such that $\varphi_F(\theta_0) = 0$. Thus $\varphi_F(\theta) \leq \pi/4$ for all $\theta \in [\theta_0 - \pi/4, \theta_0]$, and therefore $\psi_F^+(\theta_0 - \pi/4) \leq \pi/4$. As a result, the level sets

$$\Psi(\lambda) := \{\theta \in \mathbb{R} :\ \psi_F^+(\theta) > \lambda\}, \qquad \Phi(\lambda) := \{\theta \in \mathbb{R} :\ \varphi_F(\theta) > \lambda\},$$

are strict subsets of $\mathbb{R}$ for any $\lambda \geq \pi/4$. They are also 2π-periodic sets, and for that reason we denote $\widetilde{\Phi}(\lambda) := \Phi(\lambda) \cap [0, 2\pi[$ and $\widetilde{\Psi}(\lambda) := \Psi(\lambda) \cap [0, 2\pi[$. Applying Lemma 2.5 to the closure $[\theta_0, \theta_1]$ of each connected component of

$\Psi(\lambda)$, and using periodicity, we obtain $|\widetilde{\Psi}(\lambda) \setminus \widetilde{\Phi}(\lambda)| \leq |\widetilde{\Phi}(\lambda)|$. Thus $|\widetilde{\Psi}(\lambda)| \leq 2|\widetilde{\Phi}(\lambda)|$, and therefore, as announced

$$\int_0^{2\pi} \max\{0, \tan\psi_F^+ - 1\} = \int_{\pi/4}^{\pi/2} |\widetilde{\Psi}(\lambda)| \tan\lambda \,\mathrm{d}\lambda$$
$$\leq 2\int_{\pi/4}^{\pi/2} |\widetilde{\Phi}(\lambda)| \tan\lambda \,\mathrm{d}\lambda = 2\int_0^{2\pi} \max\{0, \tan\varphi_F - 1\}. \qquad \square$$

From Corollaries 2.1 and 2.2 we obtain

$$\int_0^{2\pi} \max\{1, \tan\psi_F^+\} \leq C \ln\mu, \tag{2.14}$$

where $\mu := \max\{2, \mu(F)\}$ and C is an absolute constant. The same result holds for $\max\{1, -\tan\psi_F^-\}$, by a similar argument, see the comment after Definition 2.3. Finally, we compare the F-angle of two vectors with such integral quantities.

Proposition 2.5 (Estimate of $\measuredangle_F$ in terms of $\psi_F^\pm$). *Let $u, v \in \mathbb{R}^2 \setminus \{0\}$, with $\measuredangle(u,v) \leq \pi/3$. Let $\Theta \subseteq \mathbb{R}$ be a corresponding angular sector, with $|\Theta| = \measuredangle(u,v)$. Then*

$$\min\{\measuredangle_F(u,v), 2\}^2 \leq C' \int_\Theta \max\{1,\, \tan\psi_F^+,\, -\tan\psi_F^-\}, \tag{2.15}$$

where $C' = 4C$ and C is the constant from Proposition 2.4.

Proof. By angular sector, we mean that up to exchanging u and v one has $\Theta = [\theta_u, \theta_v[$ where u (resp. v) is positively proportional to $(\cos\theta_u, \sin\theta_u)$ (resp. $(\cos\theta_v, \sin\theta_v)$). By Proposition 2.4 one has

$$\measuredangle_F(u,v)^2 \leq C \max\{|\Theta|^2,\, |\Theta|\,|\tan\varphi_F(\theta_u)|,\, |\Theta|\,|\tan\varphi_F(\theta_v)|\}.$$

If $\measuredangle_F(u,v)^2 \leq C|\Theta|^2$ then the announced result (2.15) is proved, since $|\Theta| \leq \pi/3$. Otherwise we may assume w.l.o.g that $\measuredangle_F(u,v)^2 \leq |\Theta|\,|\tan\varphi_F(\theta_u)|$. Denoting $\psi_F := \max\{\psi_F^+, -\psi_F^-\}$ one has $\psi_F(\theta_u + h) \geq \varphi_* - h$ for all $h \geq 0$, where $\varphi_* := |\varphi_F(\theta_u)|$. We conclude by case elimination:

- If $\varphi_* \leq \pi/3$, then $\measuredangle_F(u,v)^2 \leq |\Theta| \tan(\pi/3)$, hence (2.15) holds as announced.
- Otherwise if $\varphi_* + \measuredangle(u,v) \geq \pi/2$, we obtain

  $$\int_\Theta |\tan\psi_F| \geq \int_0^{\frac{\pi}{2}-\varphi_*} \tan(\varphi_* - h)\,\mathrm{d}h$$
  $$= \ln\Big(\frac{\sin(2\varphi_*)}{\cos\varphi_*}\Big) = \ln(2\sin\varphi_*) \geq \ln(2\sin(\pi/3)) = \frac{\ln 3}{2}.$$

  Thus the r.h.s. of (2.15) is bounded below by $C\ln(3)/2 \geq 2^2$, hence (2.15) holds as announced.

- Otherwise if $\varphi_* + \measuredangle(u,v) \leq \pi/2$, then we obtain for all $\theta \in \Theta$

$$\begin{aligned}\tan\psi_F(\theta) &\geq \tan(\varphi_* - (\pi/2 - \varphi_*)) \\ &= -\cot(2\varphi_*) = \frac{1}{2}(\tan\varphi_* - \cot\varphi_*) \geq \frac{1}{4}\tan\varphi_*,\end{aligned}$$

using that $\varphi_* \geq \pi/3$ in the last inequality. Therefore $\int_\Theta \tan\psi_F \geq \frac{1}{4}|\Theta|\tan(\varphi_*)$, which implies (2.15) and concludes the proof. $\square$

3. The Stern-Brocot Tree

We describe a variant of the Stern-Brocot tree [12], an arithmetic structure which allows to effectively construct and study the minimal (F,α)-acute stencil considered in Definition 1.3. We formally introduce the Stern-Brocot tree in this introduction, and then we relate it in § 3.2 with the stencils of Definition 1.3. We estimate in § 3.3 the cardinality of a subtree, based on the number of its inner leaves and on a measure of their depth, for use in the proof § 4.1 of the worst case estimate of Theorem 1.1.

Let $\mathcal{Z}$ collect all elements of $\mathbb{Z}^2$ whose coordinates are co-prime, and $\mathcal{T}$ all elements of $\mathcal{Z}^2$ with unit determinant and a non-negative scalar product:

$$\begin{aligned}\mathcal{Z} &:= \{(a,b) \in \mathbb{Z}^2 \setminus \{0\} : \gcd(a,b) = 1\}, \\ \mathcal{T} &:= \{(u,v) \in \mathcal{Z}^2 : \langle u,v\rangle \geq 0, \det(u,v) = 1\}.\end{aligned}$$

We often denote $T = (u,v)$ the elements of the set $\mathcal{T}$.

Definition 3.1. For any $T = (u,v) \in \mathcal{T}$, we refer to $T' = (u, u+v) \in \mathcal{T}$ and $T'' = (u+v, v) \in \mathcal{T}$ as its children, and we denote this relation by $T \lhd T'$ and $T \lhd T''$. We also let

$$S(T) = \langle u,v\rangle, \qquad \Delta(T) = \min\{\|u\|^2, \|v\|^2\}.$$

By construction one has for any $T \lhd T' \in \mathcal{T}$

$$S(T) \geq 0, \quad \Delta(T) \geq 1, \quad S(T') \geq S(T) + \Delta(T), \quad \Delta(T') \geq \Delta(T). \tag{3.1}$$

Definition 3.2. A chain in $\mathcal{T}$ is a finite sequence $T_0 \lhd \cdots \lhd T_n$, where $n \geq 0$. We write $T_* \preceq T^*$ iff there exists a chain $T_* = T_0 \lhd \cdots \lhd T_n = T^*$ in $\mathcal{T}$ for some $n \geq 0$.

The next lemma fully describes the graph $(\mathcal{T}, \lhd)$. For that purpose, denoting by (e_1, e_2) the canonical basis of $\mathbb{R}^2$ we let

$$\mathcal{T}_0 := \{(e_1,e_2), (e_2,-e_1), (-e_1,-e_2), (-e_2,e_1)\}.$$

Lemma 3.1 ([10, Lemma 2.3]).

- *Let $T = (u, v) \in \mathcal{T}$. The following are equivalent:*

  (i) $T \in \mathcal{T}_0$,
  (ii) $\|u\| = \|v\|$,
  (iii) $S(T) < \Delta(T)$,
  (iv) T *has no parent.*

- *The graph $(\mathcal{T}, \lhd)$ is the disjoint union of four complete infinite binary trees, whose roots lie in $\mathcal{T}_0$.*

The tree rooted in (e_1, e_2) is isomorphic to the classical Stern-Brocot tree [12], an infinite binary tree labeled with rationals, via the mapping $(u, v) \mapsto p/q$ where $(p, q) = u + v$. Each positive rational appears exactly once as a label, in its irreducible form, as follows from the first statement of the next proposition. See also [12].

Proposition 3.1. *For each $u \in \mathcal{Z}$ with $\|u\| > 1$, there exists a unique $(u_-, u_+) \in \mathcal{T}$ such that $u = u_- + u_+$. By convention we let $(u_-, u_+) := (-u^\perp, u^\perp)$ if $\|u\| = 1$. For any $u, v \in \mathcal{Z}$*

$$(u, v) \in \mathcal{T} \Leftrightarrow \exists\, k \geq 0,\ v = u_+ + ku, \qquad (v, u) \in \mathcal{T} \Leftrightarrow \exists\, k \geq 0,\ v = u_- + ku.$$

Furthermore, $\|u_\pm + ku\| > k\|u\|$ for all $k \geq 0$. Also, $\|u_\pm\| \leq \|u\|$ with equality iff $\|u\| = \|u_\pm\| = 1$.

Proof. See Proposition 1.2 in [11] for the existence and uniqueness of (u_-, u_+).

The announced properties are obvious if $\|u\| = 1$, hence w.l.o.g. we assume $\|u\| > 1$. One has $\|u\|^2 = \|u_+\|^2 + 2\langle u_+, u_-\rangle + \|u_-\|^2 \geq \|u_+\|^2 + 0 + 1$, hence $\|u\| > \|u_+\|$ as announced, and likewise for u_-. One has $\|u_+ + ku\|^2 = k^2\|u\|^2 + 2k\langle u, u_+\rangle + \|u_+\|^2 \geq k^2\|u\|^2 + 0 + 1$ for all $k \geq 0$, hence $\|u_+ + ku\| > k\|u\|$ and likewise for u_- as announced.

If $(u, v) \in \mathcal{T}$, then $\det(u, v) = \det(u, u_+)$, hence $v = u_+ + ku$ for some $k \in \mathbb{R}$. Since u_+, v have integer coordinates, and u has co-prime coordinates, one has $k \in \mathbb{Z}$. By definition $0 \leq \langle u, v\rangle = \langle u, u_+\rangle + k\|u\|^2 < (k + 1)\|u\|^2$, showing that $k \geq 0$ as announced. Likewise for u_-, and the reverse implication is obvious. □

3.1. Angular Partitions

To each element $T = (u, v)$ of (our variant of) the Stern-Brocot tree one can associate an angular sector, whose width and covering properties are the object of this short subsection.

Lemma 3.2. *For all* $(u,v) \in \mathcal{T}$ *one has*

$$(\|u\|\|v\|)^{-1} \leq \measuredangle(u,v) \leq \frac{\pi}{2}(\|u\|\|v\|)^{-1}.$$

Proof. One has $\sin(\measuredangle(u,v)) = \det(u,v)/(\|u\|\|v\|) = (\|u\|\|v\|)^{-1}$. Also, by concavity, one has $\frac{2}{\pi}\varphi \leq \sin\varphi \leq \varphi$ for all $\varphi \in [0,\pi/2]$, hence $t \leq \arcsin t \leq \frac{\pi}{2} t$ for all $t \in [0,1]$. $\square$

Definition 3.3. Given $T = (u,v) \in \mathcal{T}$ we let $\Theta(T) := [\theta_u, \theta_v[$, where u is positively proportional to $(\cos\theta_u, \sin\theta_u)$ and $\theta_u \in [0, 2\pi[$, and likewise for v and $\theta_v \in]0, 2\pi]$.

If $T \in \mathcal{T}_0$, then $\Theta(T) = [k\pi/2, (k+1)\pi/2[$ for some $0 \leq k \leq 3$. By construction, $\Theta(T) = \Theta(T') \sqcup \Theta(T'')$ if T' and T'' are the children of T, where $\sqcup$ denotes the disjoint union. In addition $|\Theta(T)| = \measuredangle(u,v)$ for all $T = (u,v) \in \mathcal{T}$.

Definition 3.4.

- A sub-forest is a set $\mathcal{T}_* \subseteq \mathcal{T}$ which contains the parent, if any, of each of its elements: for all $T \lhd T'$ with $T' \in \mathcal{T}_*$ one has $T \in \mathcal{T}_*$.
- An outer leaf of $\mathcal{T}_*$ is an element of the set $\mathcal{T} \setminus \mathcal{T}_*$ whose parent, if any, lies in $\mathcal{T}_*$. An inner leaf of $\mathcal{T}_*$ is an element of $\mathcal{T}_*$ whose two children lie outside $\mathcal{T}_*$. Their sets are respectively denoted

$$\mathcal{L}^o(\mathcal{T}_*) \subseteq \mathcal{T} \setminus \mathcal{T}_*, \qquad \mathcal{L}^i(\mathcal{T}_*) \subseteq \mathcal{T}_*.$$

Said otherwise, an element $T \in \mathcal{T} \setminus \mathcal{T}_*$ (resp. $T \in \mathcal{T}_*$) is an outer leaf (resp. inner leaf) of a sub-forest $\mathcal{T}_* \subseteq \mathcal{T}$, iff $\mathcal{T}_* \cup \{T\}$ (resp. $\mathcal{T}_* \setminus \{T\}$) also is a sub-forest. In addition one easily checks that the angular sectors associated with the outer leaves define a partition of the angular space $[0, 2\pi[$, and that the angular sectors associated with the inner leaves are pairwise disjoint:

$$\bigsqcup_{T \in \mathcal{L}^o(\mathcal{T}_*)} \Theta(T) = [0, 2\pi[\,, \qquad \bigsqcup_{T \in \mathcal{L}^i(\mathcal{T}_*)} \Theta(T) \subseteq [0, 2\pi[\,. \tag{3.2}$$

3.2. Stencil Construction

We show in Proposition 3.2 that *stencils* are in one to one correspondence with *finite sub-forests* of $\mathcal{T}$, see Definitions 1.3 and 3.4. This yields an efficient construction of stencils with minimal cardinality, and a way of counting their elements, see Corollary 3.1.

Proposition 3.2. *Let $(u_1, \dots, u_n)$, $n \geq 4$, be a stencil in the sense of Definition 1.3, and let*

$$\mathcal{L}_* := \{(u_i, u_{i+1}) : \ 1 \leq i \leq n\},$$

collect the pairs of consecutive elements, with $\alpha_{n+1} := \alpha_n$. Then $\mathcal{L}_*$ is the set of outer leaves of some finite sub-forest $\mathcal{T}_* \subseteq \mathcal{T}$, and in particular $\#\mathcal{L}_* = 4 + \#\mathcal{T}_*$. Any finite sub-forest $\mathcal{T}_*$ of $\mathcal{T}$ can be obtained in this way.

Proof. We proceed by induction on the cardinality of $\mathcal{L}_*$. For initialization, we note that $\#\mathcal{L}_* \geq 4$, with equality iff $\mathcal{L}_* = \mathcal{T}_0$, in which case it collects the outer leaves of the empty sub-forest $\mathcal{T}_* = \emptyset$. Otherwise denote $u = u_i$ the element of $\mathcal{L}_*$ with maximal norm, and observe that $u_{i+1} = u_+$ and $u_{i-1} = u_-$ by Proposition 3.1. Since $\mathcal{L}_* \subsetneq \mathcal{T}_0$ one has $\|u\| > 1$, and therefore $(u_{i-1}, u_{i+1}) = (u_-, u_+) \in \mathcal{T}$, showing that $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$ is also a stencil in the sense of Definition 1.3. Thus by induction $\mathcal{L}_* \cup \{(u_{i-1}, u_{i+1})\} \setminus \{(u_{i-1}, u_i), (u_i, u_{i+1})\} = \mathcal{L}^o(\mathcal{T}'_*)$ for some sub-forest $\mathcal{T}'_*$ of $\mathcal{T}$, and therefore $\mathcal{L}_* = \mathcal{L}^o(\mathcal{T}'_* \cup \{(u_{i-1}, u_{i+1})\})$ as announced.

Conversely, we observed in (3.2) that the set of outer leaves of a finite sub-forest of $\mathcal{T}$ defines a partition the angular space, and thus yields a stencil.

Recall that a finite complete rooted binary tree has one more outer leaf than inner nodes. Since $\mathcal{T}_*$ collects the inner nodes (possibly none) of four such trees, and $\mathcal{L}_*$ their leaves, one has $\#\mathcal{L}_* = 4 + \#\mathcal{T}_*$ as announced. □

Corollary 3.1. *Let F be an asymmetric norm, and let $\alpha \in]0, \pi/2]$. Define*

$$\mathcal{T}(F, \alpha) := \{(u, v) \in \mathcal{T} : \measuredangle_F(u, v) > \alpha \text{ or } \measuredangle_F(v, u) > \alpha\}. \tag{3.3}$$

Then $\mathcal{T}(F, \alpha)$ is a finite sub-forest of $\mathcal{T}$, and $N(F, \alpha) = 4 + \#\mathcal{T}(F, \alpha)$.

Proof. The set $\mathcal{T}(F, \alpha)$ is a sub-forest of $\mathcal{T}$ by Lemma 2.2, and is finite by Proposition 2.2. Denote by $\mathcal{L}^o(F, \alpha)$ the collection of its outer leaves, and by $u_1, \dots, u_n$ the corresponding stencil, see Proposition 3.2. One has $(u_i, u_{i+1}) \in \mathcal{L}^o(F, \alpha) \subseteq \mathcal{T} \setminus \mathcal{T}(F, \alpha)$, for any $1 \leq i \leq n$, implying the (F, α)-acuteness property (1.2) by definition of $\mathcal{T}(F, \alpha)$. This implies the upper bound $N(F, \alpha) \leq n = \#\mathcal{L}^o(F, \alpha) = 4 + \#\mathcal{T}(F, \alpha)$.

Conversely, let $u_1, \dots, u_n$ be an (F, α)-acute stencil with minimal cardinality, and let $\mathcal{L}_*$ and $\mathcal{T}_*$ be as in Proposition 3.2. By Lemma 2.2, and recalling Definition 3.2, all elements of the set

$$\mathcal{E} := \{T' \in \mathcal{T} : \exists T \in \mathcal{L}_*, \, T \preceq T'\}$$

obey the acuteness condition (1.2), hence $\mathcal{E} \subseteq \mathcal{T} \setminus \mathcal{T}(F, \alpha)$. On the other hand, one has $\mathcal{E} = \mathcal{T} \setminus \mathcal{T}_*$, hence $\mathcal{T}(F, \alpha) \subseteq \mathcal{T}_*$, which yields the lower bound $\#\mathcal{T}(F, \alpha) \leq \#\mathcal{T}_* = \#\mathcal{L}_* - 4 = N(F, \alpha)$. □

Thanks to the tree structure, the set $\mathcal{T}(F, \alpha)$ can easily be computed in practice, as well as the corresponding minimal (F, α)-acute stencil, by e.g. depth first search as in [10].

3.3. Cardinality of a Sub-forest

We estimate the cardinality of a sub-forest of $\mathcal{T}$ based on the number of inner leaves and on their depth as measured by the function S, see Corollary 3.2 and Definition 3.1. The proof is based on a decomposition of the sub-forest into a disjoint union of chains. We state, without proof, a lower bound on the depth of the last element of a chain, which immediate follows from (3.1).

Lemma 3.3. *If $T_0 \lhd \cdots \lhd T_n$ is a chain in $\mathcal{T}$, then $S(T_n) \geq n\Delta(T_0)$.*

Definition 3.5. Let $\mathcal{T}_*$ be a finite sub-forest of $\mathcal{T}$. Then $\mathcal{T}_*$ is the union of a finite family of chains $C_1, \ldots, C_I$, each denoted $C_i = \{T_0^i \lhd \cdots \lhd T_{n_i}^i\}$, and defined as follows:

- (Main loop, iteration variable: i the chain index) Choose an element T_0^i minimizing S in $\mathcal{T}_* \backslash C_0 \sqcup \cdots \sqcup C_{i-1}$. If this set is empty, then the algorithm ends.
- (Inner loop, iteration variable: k the chain element index) Consider the two children T', T'' of T_k^i. If both lie in $\mathcal{T}_*$, then define T_{k+1}^i as the one minimizing S (any in case of tie). If only one lies in $\mathcal{T}_*$, then define it as T_{k+1}^i. If none lies in $\mathcal{T}_*$ then the inner loop ends.

Lemma 3.4. *With the notations and assumptions of Definition 3.5, the chains are disjoint and their number I is also the number of inner leaves of $\mathcal{T}_*$. Denote by $(u_i, v_i) = T_0^i$, $1 \leq i \leq I$, the first element of each chain. Then the vectors $\{u_i : 1 \leq i \leq I, \|u_i\| < \|v_i\|\}$ are pairwise distinct, and likewise $\{v_i : 1 \leq i \leq I, \|u_i\| > \|v_i\|\}$.*

Proof. Assume for contradiction that $T_k^i = T_\ell^j$ for some $0 \leq i < j \leq I$, $k \leq n_i$, $\ell \leq n_j$, where (i, j, k, ℓ) is minimal for lexicographic ordering. By construction of the first element of each chain, one has $\ell \geq 1$. One has $k = 0$, since otherwise $T_{k-1}^i = T_{\ell-1}^j$ contradicting the minimality of (i, j, k, ℓ). Thus $S(T_0^j) < S(T_\ell^j) = S(T_0^i)$, contradicting the definition of T_0^i.

By construction, the chains exhaust $\mathcal{T}_*$, are disjoint as shown in the above paragraph, and each one ends at an inner leaf. Hence their number is the number of inner leaves, as announced.

Assume that (u, v_i) and (u, v_j) are the first element of the chains C_i and C_j, with $\|u\| < \min\{\|v_i\|, \|v_j\|\}$ and $i < j$. Then $v_i = u_+ + ku$ and $v_j = u_+ + \ell u$ for some $1 \leq k < \ell$, by Proposition 3.1. Since $(u, u_+) \lhd \cdots \lhd (u, u_+ + \ell u)$, one has $(u, u_+ + ru) \in \mathcal{T}_*$ for all $0 \leq r \leq \ell$. The two children of $T = (u, u_+ + ru)$ are $T' = (u, u_+ + (r+1)u)$ and $T'' = (u_+ + (r+1)u, u_+ + ru)$, and satisfy $S(T'') - S(T') = \langle u_+ + (r-1)u, u_+ + (r+1)u\rangle > 0$ for all $r \geq 1$. Hence $T_0^j \in C_i$, by construction of C_i, see the inner loop, which is a contradiction. The result follows. $\square$

Corollary 3.2. *Let $\mathcal{T}_*$ be a sub-forest of $\mathcal{T}$. Then for some absolute constant C,*

$$\#\mathcal{T}_* \leq C \Big(1 + \max_{T \in \mathcal{T}_*} S(T)\Big) \ln \big(\max \{2, \#\mathcal{L}^i(\mathcal{T}_*)\}\big).$$

Proof. Denote by $I := \#\mathcal{L}^i(\mathcal{T}_*)$ the number of inner leaves, and $s := \max\{S(T) : T \in \mathcal{T}_*\}$ the depth of $\mathcal{T}_*$ as measured by S. By Lemma 3.4, $\mathcal{T}_*$ is the disjoint union of I chains, with $n_1, \dots, n_i$ elements, and whose first element we denote $(u_1, v_1), \dots, (u_I, v_I)$. By Lemma 3.3 one has

$$\#\mathcal{T}_* = \sum_{i=1}^{I} n_i \leq \sum_{i=1}^{I} \frac{s+1}{\min\{\|u_i\|, \|v_i\|\}^2} \leq (s+1)\Big(4 + 2\sum_{i=1}^{I} \frac{1}{\|w_i\|^2}\Big), \tag{3.4}$$

where $(w_n)_{n \geq 1}$ is an enumeration of $\mathbb{Z}^2 \setminus \{0\}$ sorted by non-decreasing norm. In (3.4, r.h.s.) the constant 4 corresponds to the case $\|u_i\| = \|v_i\|$ and thus to chains rooted in $\mathcal{T}_0$ by Lemma 3.1. The sum comes from the cases $\|u_i\| < \|v_i\|$ or $\|u_i\| > \|v_i\|$ and from the injectivity property of Lemma 3.4. Observing that $\|w_I\| \leq C\sqrt{I}$ and using (4.1, left) below, we conclude the proof. $\square$

4. Complexity Estimates

This section concludes the proof of Theorem 1.1, dealing with the worst case and average case complexity estimates in § 4.1 and § 4.2 respectively. Most of the material has been prepared in § 2 and § 3. The following elementary estimate serves in several occasions.

Lemma 4.1 ([10, Lemma 2.7]). *For all $r \geq 2$, one has with C an absolute constant*

$$\sum_{\substack{0 < \|u\| \leq r \\ u \in \mathbb{Z}^2}} \frac{1}{\|u\|^2} \leq C \ln r. \tag{4.1}$$

Corollary 4.1. *For any $r \geq 2$, one has with C an absolute constant*

$$\#\{(u, v) \in \mathcal{T} :\, \|u\|\|v\| \leq r\} \leq Cr \ln r$$

Proof. We distinguish the cases $\|u\| = \|v\|$, $\|u\| < \|v\|$, and $\|u\| > \|v\|$. In the first case one has $(u, v) \in \mathcal{T}_0$, see Lemma 3.1, so that the contribution of these terms is 4. Otherwise, assuming w.l.o.g. that $\|u\| < \|v\|$, one has $v = u_+ + ku$ for some $k \geq 1$, see Proposition 3.1. Therefore $\|v\| \geq k\|u\|$, thus $k \leq r/\|u\|^2$, which is an upper bound for the number of possible choices of v for a given u. Eventually we conclude the proof using (4.1),

$$\#\{(u, v) \in \mathcal{T} : \|u\|\|v\| \leq r\} - 4 \leq 2 \sum_{\|u\| \in \mathcal{Z}} \Big\lfloor \frac{r}{\|u\|^2} \Big\rfloor \leq 2 \sum_{0 < \|u\| \leq \sqrt{r}} \frac{r}{\|u\|^2} \leq Cr \ln r.$$

$\square$

4.1. Worst Case

We establish the upper bound on the cardinality $N(F,\alpha)$ of a minimal (F,α)-acute stencil, announced in Theorem 1.1. The asymmetric norm F and parameter $\alpha \in]0,\pi/2]$ are fixed throughout this section.

Lemma 4.2. *$\mathcal{T}(F,\alpha)$ has at most $C\ln(\mu)/\alpha^2$ inner leaves, each obeying $S(T) \leq 5\mu/\alpha^2$, where $\mu := \max\{2,\mu(F)\}$ and C is an absolute constant.*

Proof. The set $\mathcal{T}(F,\alpha)$ is introduced in Corollary 3.1, and the quantity $S(T)$ in Definition 3.1. Denoting by $\mathcal{L}^i(F,\alpha)$ the set of inner leaves of $\mathcal{T}(F,\alpha)$, see Definition 3.4, we obtain

$$\begin{aligned}\alpha^2 \#\mathcal{L}^i(F,\alpha) &\leq \sum_{\substack{T\in\mathcal{L}^i(F,\alpha)\\ T=(u,v)}} \max\{\measuredangle_F(u,v), \measuredangle_F(v,u)\}^2 \\ &\leq C\int_0^{2\pi} \max\left\{1, \tan\psi_F^+, -\tan\psi_F^-\right\} \leq C'\ln\mu.\end{aligned}$$

We successively used (i) the inclusion $\mathcal{L}^i(F,\alpha)\subseteq\mathcal{T}(F,\alpha)$ and definition (3.3) of $\mathcal{T}(F,\alpha)$, (ii) Proposition 2.5 and (3.2), (iii) the integral upper bound (2.14). The first announced point follows.

On the other hand, for each $T=(u,v)\in\mathcal{T}(F,\alpha)$ one has by (3.3) and Proposition 2.2

$$\alpha < \measuredangle_F(u,v) \leq \sqrt{5\mu\measuredangle(u,v)},$$

and therefore since $\det(u,v)=1$

$$\alpha^2/(5\mu) \leq \measuredangle(u,v) = \arctan(1/\langle u,v\rangle) \leq 1/\langle u,v\rangle,$$

implying as announced that $S(T) := \langle u,v\rangle \leq 5\mu/\alpha^2$. □

Corollary 4.2. *$\#\mathcal{T}(F,\alpha) \leq C\frac{\mu}{\alpha^2}\ln\frac{\ln\mu}{\alpha^2}$, with $\mu := \max\{12,\mu(F)\}$ and C an absolute constant.*

Proof. The announced estimate immediately follows from Lemma 4.2 and Corollary 3.2. Note that $\frac{\ln\mu}{\alpha^2} \geq \frac{\ln 12}{(\pi/2)^2} > 1$. □

4.2. Average Case

Throughout this section, we denote by F an asymmetric norm, which is continuously differentiable except at the origin. In the following, $\chi_{\geq 1} : \mathbb{R} \to \{0,1\}$ denotes the indicator function of the set $[1,\infty[$.

Recall that $\mathcal{T}(F,\alpha)$ is a family of pairs (u,v) of vectors, playing symmetrical roles, see (3.3). Our first lemma breaks this symmetry, and lets u (or v) play a preferred role through the introduction of auxiliary sets $\mathcal{Z}_\sigma(F,\delta,u)$, for suitable $\delta\geq 0$, $\sigma\in\{+,-\}$.

Definition 4.1. For each $u \in \mathcal{Z}$, $\sigma \in \{+,-\}$, $\delta > 0$, let $\mathcal{Z}_\sigma(F,\delta,u)$ collect all $v \in \mathcal{Z}$ such that

$$|\tan \varphi_F(u)| \geq \delta \|u\|\|v\|, \qquad \langle u, v\rangle \geq 0, \quad \det(u,v) = \sigma 1. \tag{4.2}$$

Lemma 4.3. *Let $\delta = \alpha^2/C$, where C is from Proposition 2.4. Then $\mathcal{T}(F,\alpha)$ is a subset of*

$$\begin{aligned}\{(u,v) \in \mathcal{T} : \alpha\|u\|\|v\| \leq C\} \cup \{(u,v) \in \mathcal{T} : v \in \mathcal{Z}_+(F,\delta,u)\}\\ \cup \{(u,v) \in \mathcal{T} : u \in \mathcal{Z}_-(F,\delta,v)\}.\end{aligned}$$

Therefore for some absolute constant C',

$$\#\mathcal{T}(F,\alpha) \leq \frac{C'}{\alpha}|\ln \alpha| + \sum_{u\in\mathcal{Z}} \sum_{\sigma\in\{+,-\}} \#\mathcal{Z}_\sigma(F,\delta,u). \tag{4.3}$$

Proof. Let $(u,v) \in \mathcal{T}(F,\alpha)$, so that $\measuredangle_F(u,v) > \alpha$ or $\measuredangle_F(v,u) > \alpha$, see (3.3). Then by Proposition 2.4, and recalling that $\|u\|\|v\|\measuredangle(u,v) \leq 1$, see Lemma 3.2, we obtain

$$\|u\|\|v\|\alpha^2 \leq C \max\Big\{\frac{1}{\|u\|\|v\|}, |\tan\varphi_F(u)|, |\tan\varphi_F(v)|\Big\}.$$

The announced inclusion follows, implying the cardinality estimate by Corollary 4.1. $\square$

The next lemma estimates the cardinality of each $\mathcal{Z}_\sigma(F,\delta,u)$ individually. Recall that $u_\pm$ is defined in Proposition 3.1.

Lemma 4.4. *For each $u \in \mathcal{Z}$, $\sigma \in \{+,-\}$, $\delta > 0$, one has $\mathcal{Z}_\sigma(F,\delta,u) = \emptyset$ if $\|u\| > \mu(F)/\delta$, and else*

$$\#\mathcal{Z}(F,\delta,u) \leq \frac{|\tan\varphi_F(u)|}{\delta\|u\|^2} + \chi_{\geq 1}\Big(\frac{|\tan\varphi_F(u)|}{\delta\|u\|\|u_\sigma\|}\Big). \tag{4.4}$$

Proof. From definition (4.2, right) we obtain $v = u_\sigma + ku$ for some $k \geq 0$. One has $\|u_\sigma + ku\| \geq \max\{\|u_\sigma\|, k\|u\|\}$, see Proposition 3.1, hence the inequality $k \leq (\tan\varphi_F(u))/(\delta\|u\|^2)$ which accounts for the first contribution in (4.4). The second contribution corresponds to the case $k = 0$.

Finally, if $\|u\| > \mu(F)/\delta$ then (4.2, left) yields $|\tan\varphi_F(u)| \geq \delta\|u\|\|v\| > \mu(F)$, since $\|v\| \geq 1$, in contradiction with $|\tan\varphi_F(u)| \leq \mu(F)$ see (2.7). This concludes the proof. $\square$

In view of (4.3) and towards the average case estimate of $\mathcal{T}(F \circ R_\theta)$, where R_θ denotes the rotation of angle $\theta \in [0, 2\pi]$, we consider the following integral.

Let $0 < \delta \le 1$ be fixed.

$$\sum_{u\in\mathcal{Z}} \int_0^{2\pi} \#\mathcal{Z}_\sigma(F\circ R_\theta, \delta, u)\,\mathrm{d}\theta$$
$$\le \sum_{\|u\|\le\mu(F)/\delta} \int_0^{2\pi} \frac{|\tan\varphi_F(R_\theta u)|}{\delta\|u\|^2} + \chi_{\ge 1}\Big(\frac{|\tan\varphi_F(R_\theta u)|}{\delta\|u\|\|u_\sigma\|}\Big)\,\mathrm{d}\theta \tag{4.5}$$
$$= \sum_{\|u\|\le\mu(F)/\delta} \int_0^{2\pi} \frac{|\tan\varphi_F(\theta)|}{\delta\|u\|^2} + \chi_{\ge 1}\Big(\frac{|\tan\varphi_F(\theta)|}{\delta\|u\|\|u_\sigma\|}\Big)\,\mathrm{d}\theta,$$

where implicitly $u \in \mathcal{Z}$ in each of the sums. Recall that φ_F is defined both on non-zero vectors and on reals, by taking the argument see Definition 2.2, and that on $\mathbb{R}$ it is 2π-periodic.

The first contribution of (4.5) is separable w.r.t. θ and u, hence can be bounded as follows:

$$\sum_{\|u\|\le\mu(F)/\delta} \int_0^{2\pi} \frac{|\tan\varphi_F(\theta)|}{\delta\|u\|^2}\mathrm{d}\theta = \frac{1}{\delta}\int_0^{2\pi} |\tan\varphi_F(\theta)|\mathrm{d}\theta \sum_{0<\|u\|\le\mu(F)/\delta} \frac{1}{\|u\|^2}$$
$$\le \frac{C}{\delta}\ln(\mu)\ln\Big(\frac{\mu}{\delta}\Big), \tag{4.6}$$

where $\mu := \max\{2, \mu(F)\}$. We used Corollary 2.1 to upper bound the integral w.r.t. θ, and Lemma 4.1 for the summation over u.

In contrast, the second contribution in (4.5) is non-separable, motivating the following lemma.

Lemma 4.5. *For all $r \ge 2$, $\sigma \in \{+,-\}$, one has with C an absolute constant*

$$\sum_{u\in\mathcal{Z}} \chi_{\ge 1}\Big(\frac{r}{\|u\|\|u_\sigma\|}\Big) \le C r \ln r. \tag{4.7}$$

Proof. For each $u \in \mathcal{Z}$ one has $(u, u_+) \in \mathcal{T}$ and $(u_-, u) \in \mathcal{T}$. Hence (4.7) is bounded by the cardinality of $\{(u,v) \in \mathcal{T} : \|u\|\|v\| \le r\}$, which is estimated in Corollary 4.1. $\square$

The second contribution of (4.5) is bounded as follows, denoting $r(\theta) := \max\{2, |\tan\varphi_F(\theta)|/\delta\}$,

$$\sum_{\|u\|\le\mu(F)/\delta} \int_0^{2\pi} \chi_{\ge 1}\Big(\frac{|\tan\varphi_F(\theta)|}{\delta\|u\|\|u_\sigma\|}\Big)\mathrm{d}\theta \le C\int_0^{2\pi} r(\theta)\ln r(\theta)\,\mathrm{d}\theta$$
$$\le C\int_0^{2\pi} \max\Big\{2, \frac{|\tan\varphi_F(\theta)|}{\delta}\Big\}\ \ln\Big(\frac{\mu}{\delta}\Big)\mathrm{d}\theta \le C\,\frac{\ln\mu}{\delta}\ \ln\Big(\frac{\mu}{\delta}\Big),$$

where we used successively (i) Lemma 4.5, (ii) the uniform upper bound $|\tan\varphi_F(\theta)| \le \mu(F)$ see Lemma 2.3, and (iii) the L^1 estimate of $|\tan\varphi_F|$ established in Corollary 2.1. Together with (4.6), this proves that (4.5) is bounded by $C\frac{\ln\mu}{\delta}\ln\frac{\mu}{\delta}$. In view of Lemma 4.3, this concludes the proof of Theorem 1.1.

Appendix

A. Semi-Lagrangian Discretization of Finslerian Eikonal Equations

We present an elementary introduction to numerical methods for the computation of generalized traveltimes and distance maps, focusing on single pass semi-Lagrangian methods [16, 6, 15, 4, 1, 10, 9], at the expense of alternative approaches such as [8, 4], which is the context underlying of the problem studied in this paper. An open source code implementing this method is available on the author's webpage `github.com/Mirebeau`.

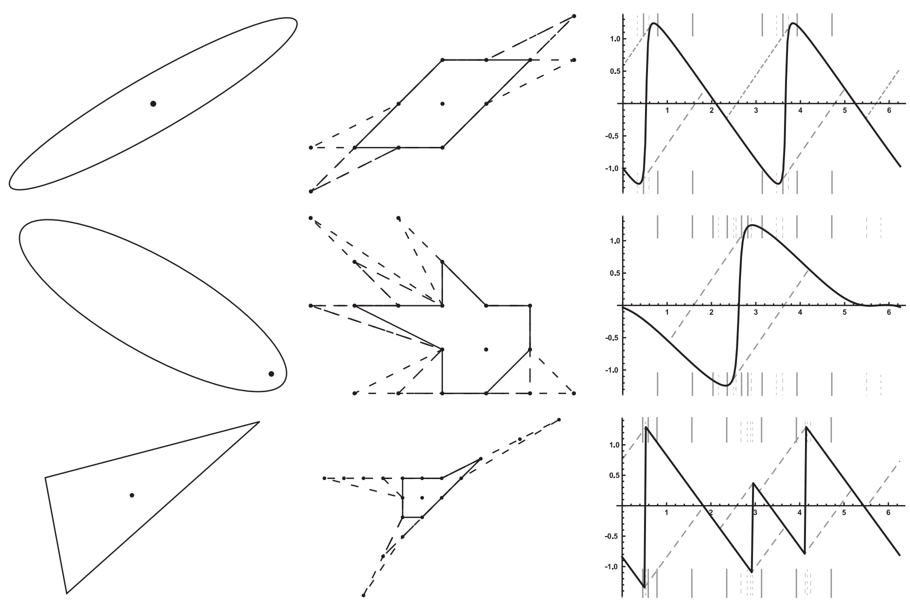

Figure A.1. Left: Unit sphere $\{F = 1\}$ of a norm F, which is asymmetric in the second and third row. The origin is marked with a point. Center: Minimal (F, α)-acute stencil for $\alpha = \pi/2$ (solid), $\pi/3$ (dashed), $\pi/4$ (dotted). Right: Function φ_F (solid), ψ_F^+ (dashed, above), ψ_F^- (dotted, below). Vertical bars correspond to the angles of the stencil points.

The main result of this section is Proposition A.1 known as *acuteness implies causality* [15]. It requires that the numerical method be based upon strictly acute stencils, in the sense of Definition 1.3 with $\alpha < \pi/2$. Under this condition, one can compute an approximate travel time $T_h(x)$, at a given discretization point $x \in \Omega_h$ where h is the grid scale, in terms of suitable neighbor values $T_h(x + hu_i)$ and $T_h(x + hu_{i+1})$ no greater than $T_h(x) - h\varepsilon$, where $\varepsilon > 0$ is uniform over the domain. As a result, T_h can be efficiently computed in a single pass over the domain using the fast-marching algorithm, similar to Dijkstra's method on graphs, which deals with vertices in the order of increasing values of T_h. In addition let us mention that uniform causality, a.k.a. $\varepsilon > 0$, is a stable property which is also satisfied by suitably small perturbations of the numerical scheme, such as those related to second order accuracy [14] and to source factorization [8].

Consider a bounded domain $\Omega \subseteq \mathbb{R}^2$, equipped with a Finslerian metric $\mathcal{F} : \overline{\Omega} \times \mathbb{R}^2$, $(x, u) \mapsto \mathcal{F}_x(u)$. In other words, $\mathcal{F}$ is a continuous mapping, and $\mathcal{F}_x(\cdot)$ is an asymmetric norm for each $x \in \overline{\Omega}$ in the sense of Definition 1.1. The Finslerian distance from x to $y \in \overline{\Omega}$ is defined as

$$\begin{aligned} d_{\mathcal{F}}(x,y) &:= \inf_{\gamma \in \Gamma_{x\to y}} \int_0^1 \mathcal{F}_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t, \\ \Gamma_{x\to y} &:= \big\{\gamma \in \mathrm{Lip}([0,1], \overline{\Omega}) :\, \gamma(0) = x,\, \gamma(1) = y\big\}. \end{aligned}$$

One is interested in the distance from the boundary, $T(x) := \min\{d_{\mathcal{F}}(x, y) : y \in \partial\Omega\}$ often referred to as the "escape time" from the domain, which under mild assumptions is the unique viscosity solution [2] to the following (generalized) eikonal PDE, written in Bellman form:

$$\inf_{u \in S^1} \mathcal{F}_x(u) + \langle \nabla T(x), u\rangle = 0, \quad \forall x \in \Omega, \qquad T(x) = 0, \quad \forall x \in \partial\Omega. \tag{A.1}$$

Note that the PDE remains equivalent if the unit circle S^1 is replaced with any curve enclosing the origin. In particular, we can consider the closed polygonal line defined by a stencil, see Definition 1.3, possibly depending on $x \in \Omega$ and denoted $u_1(x), \dots, u_{n(x)}(x)$ where $n(x) \geq 4$. In the following, the explicit dependency $u_i = u_i(x)$ w.r.t. the base point $x \in \Omega$ is often omitted readability, and by convention $u_{n(x)+1} := u_1$.

Consider a grid scale $h > 0$, and introduce the sets $\Omega_h := \Omega \cap h\mathbb{Z}^2$ and $\partial\Omega_h := (\mathbb{R}^2 \setminus \Omega) \cap h\mathbb{Z}^2$ devoted to the discretization of Ω and $\partial\Omega$. Semi-Lagrangian numerical schemes for the eikonal equation mimick (A.1) as follows: find $T_h : h\mathbb{Z}^2 \to \mathbb{R}$ such that

$$\begin{aligned} \min_{1 \leq i \leq n(x)} \min_{s \in [0,1]} &\mathcal{F}_x((1-s)u_i + s u_{i+1}) \\ &+ \frac{(1-s)T_h(x + hu_i) + s\,T_h(x + hu_{i+1}) - T_h(x)}{h} \end{aligned} \tag{A.2}$$

equals 0 for all $x \in \Omega_h$, with again the boundary condition $T_h(x) = 0$ for all $x \in \partial\Omega_h$.

Proposition A.1 (Acuteness implies causality [15]). *Assume that $u_1(x), \dots, u_{n(x)}(x)$ is an $(\mathcal{F}_x, \alpha)$-acute stencil, where $\alpha \in]0, \pi/2[$. Assume also that (A.2) vanishes, and that the minimum is attained for some $1 \leq i \leq n(x)$ and $s \in]0, 1[$. Then*

$$\begin{aligned} T_h(x) &\geq h\cos(\alpha)\mathcal{F}_x(u_i) + T_h(x + hu_i), \\ T_h(x) &\geq h\cos(\alpha)\mathcal{F}_x(u_{i+1}) + T_h(x + hu_{i+1}). \end{aligned}$$

Proof. A standard analysis based on Lagrange's optimality conditions shows that

$$hA^T \nabla \mathcal{F}_x((1-s)u_i + su_{i+1}) + \begin{pmatrix} T_h(x+hu_i) \\ T_h(x+hu_{i+1}) \end{pmatrix} = T_h(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where A is the matrix of columns u_i and u_{i+1}, see the Appendix of [15] or the Appendix of [10]. Considering this vector equality componentwise yields the announced result. □

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Jean-Marie Mirebeau

University Paris-Sud
CNRS University Paris-Saclay
91405 Orsay
FRANCE
E-mail: jean-marie.mirebeau@math.u-psud.fr

François Desquilbet

University Grenoble Alpes
University Paris-Saclay
91405 Orsay
FRANCE
E-mail: francois.desquilbet@ens.fr