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Some Inequalities for Chebyshev Polynomials

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Askey and Gasper (1976) proved a trigonometric inequality which improves another trigonometric inequality found by M. S. Robertson (1945). Here these inequalities are reformulated in terms of the Chebyshev polynomial of the first kind T_n and then put into a one-parametric family of inequalities. The extreme value of the parameter is found for which these inequalities hold true. As a step towards the proof of this result we establish the following complement to the finite increment theorem specialized to T_n':

$$T_n'(1) - T_n'(x) \geq (1-x)\, T_n''(x)\,, \qquad x \in [0,1]\,.$$

By a known expansion formula, this property is extended for the class of ultraspherical polynomials $P_n^{(\lambda)}$, $\lambda \geq 1$.

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1. Introduction and Statement of the Results

Positive trigonometric sums play an important role in Harmonic Analysis, Orthogonal Polynomials, Approximation Theory and many other branches of mathematics. In this note we discuss two trigonometric inequalities which appear in the book of Askey [1]. The first one is inequality (1.29) in [1], which reads as

$$\frac{\sin(n-1)\theta}{(n-1)\sin\theta} - \frac{\sin(n+1)\theta}{(n+1)\sin\theta} \leq \frac{4n}{n^2-1}\Big(1 - \frac{\sin n\theta}{n\sin\theta}\Big)\,, \qquad 0 \leq \theta \leq \pi \qquad (1)$$

(actually, (1.29) appears in [1] as a strict inequality and under the assumption $0 < \theta < \pi$). It was proved by Robertson [5] while studying the coefficients of

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univalent functions. The second one was proved by Askey and Gasper [2] and reads as

$$\frac{\sin(n-1)\theta}{(n-1)\sin\theta}-\frac{\sin(n+1)\theta}{(n+1)\sin\theta}\le\frac{(3+\cos\theta)n}{n^2-1}\Big(1-\frac{\sin n\theta}{n\sin\theta}\Big)\,,\qquad 0\le\theta\le\pi\,. \quad (2)$$

This is inequality (8.17) in [1], and as Askey wrote, it is sharper than (1). For more information on positive trigonometric sums and positive finite linear combinations of classical orthogonal polynomials we refer to [1, 3, 4] and the references therein.

We find it more convenient to reformulate inequalities (1) and (2) in terms of the Chebyshev polynomials of the first kind. Let us recall that the m-th Chebyshev polynomial of the first kind T_m, $m\in\mathbb{N}_0$, is defined by

$$T_m(x)=\cos m\theta,\qquad x=\cos\theta\in[-1,1],\quad \theta\in[0,\pi],$$

and its derivative is

$$T_m'(x)=m\,\frac{\sin m\theta}{\sin\theta},\qquad x=\cos\theta\,.$$

Using

$$\begin{aligned}\frac{\sin(n-1)\theta}{\sin\theta}&=\frac{\sin n\theta\cos\theta-\cos n\theta\sin\theta}{\sin\theta}=\frac{xT_n'(x)}{n}-T_n(x)\,,\\ \frac{\sin(n+1)\theta}{\sin\theta}&=\frac{\sin n\theta\cos\theta+\cos n\theta\sin\theta}{\sin\theta}=\frac{xT_n'(x)}{n}+T_n(x)\,,\end{aligned}$$

we find that inequality (1) of Robertson is equivalent to the inequality

$$f_1(x):=T_n(x)+2-\frac{x+2}{n^2}\,T_n'(x)\ge 0\,,\qquad n\ge 2,\quad x\in[-1,1]\,, \quad (3)$$

while the Askey-Gasper inequality (2) is equivalent to the inequality

$$f_2(x):=T_n(x)+\frac{x+3}{2}-\frac{3(x+1)}{2n^2}\,T_n'(x)\ge 0\,,\qquad n\ge 2,\quad x\in[-1,1]\,. \quad (4)$$

Since $\max_{x\in[-1,1]}T_n'(x)=T_n'(1)=n^2$, we have

$$f_1(x)-f_2(x)=\frac{1-x}{2n^2}\left(n^2-T_n'(x)\right)\ge 0,\qquad x\in[-1,1]\,,$$

hence inequality (3) is a consequence of inequality (4). Thus we naturally arrive at the following

Problem 1. *Find the largest constant $a=a(n)\ge 0$, $n\ge 2$, such that*

$$g_n(a;x):=T_n(x)+2-\frac{x+2}{n^2}\,T_n'(x)-a\,\frac{1-x}{n^2}\left(n^2-T_n'(x)\right)\ge 0,\quad x\in[-1,1]\,.$$

The cases $n = 2,\ 3$ are trivial: from $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$ one finds

$$g_2(a;x) = (1-a)(1-x)^2\,, \qquad g_3(a;x) = \frac{4}{3}\,(2-a)(1-x)^2(1+x)\,,$$

whence $a(2) = 1$ and $a(3) = 2$.

We assume henceforth that $n \geq 4$. By the Askey-Gasper inequality (2), $g_n(1/2, x) = f_2(x) \geq 0$, $x \in [-1,1]$, and therefore $a(n) \geq 1/2$. We show below that $a(n)$ cannot be essentially larger than $1/2$. Indeed, let

$$x_k = \cos\frac{k\pi}{n}, \qquad k = 1,\ldots,n-1,$$

be the zeros of T_n'. Then $T_n(x_k) = (-1)^k$ and

$$g_n(a;x_k) = 2 + (-1)^k - a(1-x_k), \qquad k = 1,\ldots,n-1\,.$$

The condition that $g_n(a;x_{n-1}) \geq 0$ in the case of even n and $g_n(a;x_{n-2}) \geq 0$ in the case of odd n implies respectively

$$a \leq \frac{1}{1-x_{n-1}} = \frac{1}{1+x_1} = \frac{1}{1+\cos\frac{\pi}{n}}, \qquad \text{if } n \text{ is even,}$$

and

$$a \leq \frac{1}{1-x_{n-2}} = \frac{1}{1+x_2} = \frac{1}{1+\cos\frac{2\pi}{n}}, \qquad \text{if } n \text{ is odd.}$$

Since both upper bounds for a tend to $1/2$ as n grows, it follows that $a = 1/2$ is the best possible (the largest) absolute constant, ensuring that $g_n(a;x) \geq 0$ for every $x \in [-1,1]$ and all $n \in \mathbb{N}$, $n \geq 2$. In this sense, the Askey-Gasper inequality (4) is the best possible.

It turns out that the upper bounds for $a(n)$ found above actually provide the solution to Problem 1. Specifically, we prove the following statement.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 4$. Then*

$$T_n(x) + 2 - \frac{x+2}{n^2}\,T_n'(x) - a(n)\,\frac{1-x}{n^2}\,\big(n^2 - T_n'(x)\big) \geq 0\,, \quad x \in [-1,1]\,, \tag{5}$$

where

$$a(n) = \begin{cases} \dfrac{1}{1+\cos\frac{\pi}{n}}, & \text{if } n \text{ is even,} \\[2mm] \dfrac{1}{1+\cos\frac{2\pi}{n}}, & \text{if } n \text{ is odd.} \end{cases}$$

The constant $a(n)$ is the best possible in the sense that (5) fails for any larger constant. The equality in (5) is attained only at $x = 1$ and $x = -\cos\frac{\pi}{n}$ if n is even, and at $x = \pm 1$ and $x = -\cos\frac{2\pi}{n}$ if n is odd.

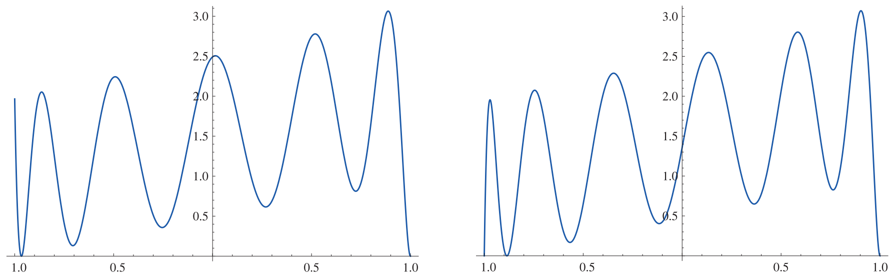

Figure 1. The graphs of $T_n(x)+2-\frac{x+2}{n^2}\,T_n'(x)-a(n)\,\frac{1-x}{n^2}\,\big(n^2-T_n'(x)\big)$ for $n=12$ (left) and $n=13$ (right).

The typical behavior of the function in (5) in the cases of even and odd n is shown in Figure 1. The graphs suggest that in the interval $[0,1]$ this function could be non-negative for a larger constant a than the one specified in Theorem 1. We show that this is indeed the case by proving that the non-negativeness in $[0,1]$ persists with $a=1$.

Theorem 2. *Let $n\in\mathbb{N}$, $n\ge 3$. Then*

$$T_n(x)+x+1-\frac{2x+1}{n^2}\,T_n'(x)\ge 0\,,\qquad x\in[0,1]\,, \tag{6}$$

or, equivalently,

$$T_n'(1)-T_n'(x)\ge(1-x)\,T_n''(x)\,,\qquad x\in[0,1]\,. \tag{7}$$

The equality in (6)–(7) *occurs only for $x=1$ and if $n\equiv 2 \pmod 4$, for $x=0$.*

For $n=0,\,1,\,2$, (7) becomes an identity. Inequality (7) provides an interesting complement to the finite increment formula $T_n'(1)-T_n'(x)=(1-x)\,T_n''(\xi)$, $\xi\in(x,1)$. Moreover, (7) implies a similar property for the ultraspherical polynomials $P_n^{(\lambda)}$, $\lambda\ge 1$.

Corollary 1. *For every ultraspherical polynomial $P_n^{(\lambda)}$, $\lambda\ge 1$, there holds*

$$P_n^{(\lambda)}(1)-P_n^{(\lambda)}(x)\ge\frac{d}{dx}\left\{P_n^{(\lambda)}(x)\right\}(1-x)\,,\qquad x\in[0,1]\,.$$

Corollary 1 easily follows from $T_n'=n\,P_{n-1}^{(1)}$ and the fact that if $\mu\ge\lambda$, then $P_n^{(\mu)}$ is represented as a linear combination of $\{P_m^{(\lambda)}\}_{m=0}^n$ with non-negative coefficients. There are examples showing that Corollary 1 is not true if $\lambda<1$, the case $\lambda=0$ is particularly easy to verify. A challenging problem is to characterize all pairs of parameters (α,β) ensuring similar inequality for the Jacobi polynomials $P_n^{(\alpha,\beta)}$.

The paper is organized as follows. In the next section we propose a short elementary proof of the Askey-Gasper inequality (4) (and thereby of the Robertson inequality (3)). In Section 3 we present a proof of Theorem 2. The proof of Theorem 1 is given in Section 4.

2. Proof of the Askey–Gasper Inequality (1.4)

We prove the following statement:

Proposition 1. *Let $n \in \mathbb{N}$, $n \geq 2$. Then*

$$f_2(x) = T_n(x) + \frac{x+3}{2} - \frac{3(x+1)}{2n^2}\, T_n'(x) \geq 0, \qquad x \in [-1,1]. \tag{8}$$

The equality in (8) *is attained only at $x = 1$ and if n is odd, at $x = -1$.*

Proof. From $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$ we find

$$f_2(x) = \begin{cases} \frac{1}{2}\,(1-x)^2, & \text{if } n = 2, \\ 2(1-x)^2(1+x), & \text{if } n = 3, \end{cases}$$

hence Proposition 1 is true for $n = 2$, 3, and we assume henceforth $n \geq 4$.

We shall use in this and in the next section the differential equation satisfied by T_n,

$$(1-x^2)\, T_n''(x) - x\, T_n'(x) + n^2\, T_n(x) = 0\,, \tag{9}$$

as well as the identity

$$n^2\, \big[T_n(x)\big]^2 + (1-x^2)\, \big[T_n'(x)\big]^2 = n^2\,. \tag{10}$$

Denote by τ the largest zero of T_n''.

Case 1: $x \in (\tau, 1]$. Using (9), we rewrite f_2 in the form

$$f_2(x) = \frac{x+3}{2n^2}\,\big(T_n'(1) - T_n'(x)\big) - \frac{1}{n^2}\,(1-x^2)T_n''(x)\,.$$

We observe that in this case the inequality $f_2(x) \geq 0$ is equivalent to

$$\frac{x+3}{2(x+1)}\,\frac{1}{1-x}\int_x^1 T_n''(u)\,du \geq T_n''(x)\,, \qquad x \in (\tau, 1]\,. \tag{11}$$

Since T_n'' is positive and monotonically increasing in $(\tau, 1]$,

$$\frac{1}{1-x}\int_x^1 T_n''(u)\,du \geq T_n''(x)$$

and (11) is a consequence of the inequality

$$\frac{x+3}{2(x+1)} \geq 1\,,$$

which is obviously true for $x \in (\tau, 1]$. Notice in the last two inequalities the equality holds only when $x = 1$. Hence, $f_2(x) \geq 0$ for $x \in (\tau, 1]$, and the equality is attained only for $x = 1$.

Case 2: $x \in [-1, \tau]$. In this case we rewrite inequality $f_2(x) \ge 0$ in the form

$$1 + T_n(x) + \frac{1+x}{2} \ge \frac{3(1+x)}{2n^2}\, T_n'(x), \qquad x \in [-1, \tau]\,. \tag{12}$$

The left-hand side of (12) is non-negative for $x \in [-1, \tau]$. On the other hand, the right-hand side of (12) is non-positive for $x \in \big[\cos\frac{2\pi}{n}, \cos\frac{\pi}{n}\big]$ as $T_n'(x) \le 0$ therein. Since $\tau \in \big(\cos\frac{2\pi}{n}, \cos\frac{\pi}{n}\big)$, (12) will be proved if we show that

$$1 + T_n(x) + \frac{1+x}{2} \ge \frac{3(1+x)}{2n^2}\, T_n'(x), \qquad x \in \Big[-1, \cos\frac{2\pi}{n}\Big]\,,$$

and it suffices to prove the inequality

$$\Big(1 + T_n(x) + \frac{1+x}{2}\Big)^2 \ge \frac{9(1+x)^2}{4n^4}\,\big[T_n'(x)\big]^2, \qquad x \in \Big[-1, \cos\frac{2\pi}{n}\Big]\,. \tag{13}$$

We estimate the left-hand side of (13) by the arithmetic mean – geometric mean inequality:

$$\Big(1 + T_n(x) + \frac{1+x}{2}\Big)^2 \ge 2(1+x)(1+T_n(x))\,, \tag{14}$$

and apply identity (10) to express $\big[T_n'(x)\big]^2$ in the right-hand side of (13),

$$\big[T_n'(x)\big]^2 = \frac{n^2(1-T_n(x))(1+T_n(x))}{1-x^2}\,.$$

It follows from (14) that (13) will hold for $x \in \big[-1, \cos\frac{2\pi}{n}\big]$ if

$$2(1+x)(1+T_n(x)) \ge \frac{9(1+x)}{4n^2(1-x)}\,(1+T_n(x))(1-T_n(x))\,. \tag{15}$$

Since $(1+x)(1+T_n(x)) \ge 0$ and $1 - T_n(x) \le 2$, the above inequality will be certainly true if

$$n^2(1-x) \ge \frac{9}{4}\,, \qquad x \in \Big[-1, \cos\frac{2\pi}{n}\Big]\,.$$

To see that the above inequality is true, we make use of $\sin\alpha > \frac{2}{\pi}\alpha$, $\alpha \in (0, \pi/2)$. For $x \in \big[-1, \cos\frac{2\pi}{n}\big]$,

$$n^2(1-x) \ge n^2\Big(1 - \cos\frac{2\pi}{n}\Big) = 2n^2\sin^2\frac{\pi}{n} > 2n^2\Big(\frac{2}{\pi}\,\frac{\pi}{n}\Big)^2 = 8 > \frac{9}{4}\,.$$

Thus, the inequality $f_2(x) \ge 0$ is proved in *Case 2*, too, and it remains to check when the equality is attained. Tracing backward our proof, we see that the equality in (15) holds only if either $x = -1$ or $T_n(x) + 1 = 0$, while the equality in (14) holds only if $T_n(x) + 1 = (1+x)/2$. Both conditions imply that $x = -1$ and $T_n(-1) = -1$, and the latter holds if and only if n is odd. □

3. Proof of Theorem 2

Let us set

$$\varphi_n(x) := T_n(x) + x + 1 - \frac{2x+1}{n^2}\,T_n'(x)\,,$$
$$\psi_n(x) := T_n'(1) - T_n'(x) - (1-x)\,T_n''(x)\,.$$

From $T_n'(1) = n^2$ and the differential equation (9) we have

$$\begin{aligned}\varphi_n(x) &= \frac{x+1}{n^2}\big(n^2 - T_n'(x)\big) - \frac{1}{n^2}\big(x\,T_n'(x) - n^2\,T_n(x)\big)\\ &= \frac{x+1}{n^2}\big(T_n'(1) - T_n'(x)\big) - \frac{1-x^2}{n^2}\,T_n''(x)\\ &= \frac{x+1}{n^2}\,\psi_n(x)\,,\end{aligned}$$

therefore inequalities (6) and (7), i.e., $\varphi_n(x) \ge 0$ and $\psi_n(x) \ge 0$, $x \in [0,1]$, are equivalent.

From

$$T_n(x) = \cos n\theta, \qquad T_n'(x) = n\,\frac{\sin n\theta}{\sin\theta}\,, \qquad x = \cos\theta\,,\quad \theta \in [0,\pi]\,,$$

we find

$$\varphi_n(0) = \begin{cases} 2, & n \equiv 0 \pmod 4\\ 1 - 1/n, & n \equiv 1 \pmod 4\\ 0, & n \equiv 2 \pmod 4\\ 1 + 1/n, & n \equiv 3 \pmod 4\,, \end{cases}$$

hence $\varphi_n(0) \ge 0$, with the equality holding only if $n = 4k+2$, and we may assume further $x \in (0,1]$.

If $\tau > 0$ is the largest zero of T_n'', then, by the same argument as in *Case 1* in the preceding section, we obtain

$$\psi_n(x) = \int_x^1 T_n''(u)\,du - (1-x)T_n''(x) \ge 0, \qquad x \in [\tau, 1]\,,$$

with the equality holding only for $x = 1$, so we may restrict our consideration to the case $x \in (0,\tau)$. Furthermore, $T_n'(x) \le 0$ for $x \in \big[\cos\frac{2\pi}{n}, \cos\frac{\pi}{n}\big]$, and then obviously $\varphi_n(x) > 0$ for $x \in \big[\cos\frac{2\pi}{n}, \cos\frac{\pi}{n}\big]$. Since τ lies in this interval, it remains to prove either of the inequalities $\varphi_n(x) > 0$ and $\psi_n(x) > 0$ when $x \in \big(0, \cos\frac{2\pi}{n}\big)$. There is nothing to prove if $n = 4$, so we assume $n \ge 5$. It suffices to show that $\varphi_n(t) > 0$ (or $\psi_n(t) > 0$) for every critical point t of ψ_n (i.e. zero of ψ_n') in the interval $\big(0, \cos\frac{2\pi}{n}\big)$. Since

$$\psi_n'(x) = (x-1)T_n'''(x)\,,$$

the critical points of ψ_n in $\big(0, \cos\frac{2\pi}{n}\big)$ are zeros of T_n'''.

Let $t \in \left(0, \cos\frac{2\pi}{n}\right)$ be a zero of T_n'''. From

$$\cos\frac{2\pi}{n} = 1 - 2\,\sin^2\frac{\pi}{n} < 1 - 2\left(\frac{2}{\pi}\,\frac{\pi}{n}\right)^2 = 1 - \frac{8}{n^2},$$

we conclude that

$$0 < t < 1 - \frac{8}{n^2}\,. \tag{16}$$

Since $y = T_n'(x)$ satisfies the differential equation

$$(1-x^2)y'' - 3x\,y' + (n^2-1)\,y = 0$$

(this can be seen e.g., by differentiating (9)), and $y''(t) = 0$, we have

$$T_n''(t) = \frac{n^2-1}{3t}\,T_n'(t)\,.$$

Now, (9) with $x = t$ yields

$$(1-t^2)\,\frac{n^2-1}{3t}\,T_n'(t) - t\,T_n'(t) + n^2\,T_n(t) = 0\,,$$

whence

$$\frac{T_n'(t)}{n^2} = -\frac{3t\,T_n(t)}{n^2-1-(n^2+2)t^2}\,. \tag{17}$$

Let us point out that (16) implies $n^2-1-(n^2+2)t^2 > 0$, since

$$t^2 < t < 1 - \frac{8}{n^2} < 1 - \frac{3}{n^2+2} = \frac{n^2-1}{n^2+2}\,.$$

Replacing $T_n'(t)/n^2$ with the right-hand side of (17) in $\varphi_n(t)$, we obtain

$$\begin{aligned}
\varphi_n(t) &= t + 1 + \left\{1 + \frac{3t(2t+1)}{n^2-1-(n^2+2)t^2}\right\} T_n(t) \\
&= (t+1)\left\{1 + \frac{n^2-1-(n^2-4)t}{n^2-1-(n^2+2)t^2}\,T_n(t)\right\} \\
&\ge (t+1)\left\{1 - \frac{n^2-1-(n^2-4)t}{n^2-1-(n^2+2)t^2}\right\}
\end{aligned}$$

(we have used that the factor in front of $T_n(t)$ in the curly brackets is positive and $T_n(t) \ge -1$). Hence, to prove $\varphi_n(t) > 0$ it suffices to show that

$$1 - \frac{n^2-1-(n^2-4)t}{n^2-1-(n^2+2)t^2} > 0\,,$$

which is equivalent to

$$\frac{t\left[n^2-4-(n^2+2)t\right]}{n^2-1-(n^2+2)t^2} > 0\,.$$

This inequality is true, since the numerator in the left-hand side is positive. Indeed, from (16) we have

$$t < 1 - \frac{8}{n^2} < 1 - \frac{6}{n^2+2} = \frac{n^2-4}{n^2+2}.$$

The proof of Theorem 2 is complete. □

4. Proof of Theorem 1

Recall that

$$f_1(x) = T_n(x) + 2 - \frac{x+2}{n^2}\, T_n'(x),$$

and set

$$f_3(x) := \frac{1}{n^2}\,(1-x)(n^2 - T_n'(x)),$$
$$F_a(x) := (1+a) f_1(x) - f_3(x)\,.$$

Clearly, Theorem 1 is equivalent to the following statement:

Theorem 3. *Let $n \in \mathbb{N}$, $n \geq 4$ and*

$$a = \begin{cases} \cos\frac{\pi}{n}, & \text{if } n \text{ is even},\\ \cos\frac{2\pi}{n}, & \text{if } n \text{ is odd}. \end{cases} \tag{18}$$

Then

$$F_a(x) \geq 0\,, \qquad x \in [-1,1]\,. \tag{19}$$

The equality in (19) *occurs only for $x = 1$ and $x = -\cos\frac{\pi}{n}$ if n is even, and for $x = \pm 1$ and $x = -\cos\frac{2\pi}{n}$ if n is odd. For every constant a, smaller than the one specified in* (18)*, inequality* (19) *fails to hold.*

Proof. According to Robertson's inequality, $f_1(x) \geq 0$ in $[-1,1]$, therefore if $a_1 > a_2$, then $F_{a_1}(x) \geq F_{a_2}(x)$ for every $x \in [-1,1]$. In particular, for every $a > 0$,

$$F_a(x) \geq F_0(x) = T_n(x) + x + 1 - \frac{2x+1}{n^2}\, T_n'(x) = \varphi_n(x)\,, \qquad x \in [-1,1]\,.$$

In view of Theorem 2, $\varphi_n(x) \geq 0$ for every $x \in [0,1]$, with the equality holding only for $x = 1$ and if $n = 4k+2$, for $x = 0$. Therefore, with a as given in (18), we have $F_a(x) \geq \varphi_n(x) > 0$ for every $x \in (0,1)$, and it remains to prove inequality (19) and clarify the cases of equality only when $x \in [-1,0]$.

Let us denote by

$$x_k = \cos\frac{k\pi}{n}, \qquad k = 0,\ldots,n\,,$$

the zeros of $(1-x^2)T_n'(x)$, and let $H(f;x)$ be the Hermite interpolating polynomial, with interpolation nodes $x_0, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}, x_n$, for a differentiable function f, i.e., $H(f;x)$ is determined by the conditions

$$\begin{aligned} H(f;x_k) &= f(x_k), \qquad k=0,1,\ldots,n,\\ H'(f;x_k) &= f'(x_k), \qquad k=1,\ldots,n-1\,. \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned} H(f;x) = {} & \frac{\big[T_n'(x)\big]^2}{2n^4}\,\big[(1+x)\,f(x_0)+(1-x)\,f(x_n)\big]\\ & + (1-x^2)\sum_{k=1}^{n-1}\frac{\ell_k^2(x)}{(1-x_k^2)^2}\,\mathcal{L}_k(f;x)\,, \end{aligned} \tag{20}$$

where, for $k=1,\ldots,n-1$, ℓ_k are the Lagrange basis polynomials for interpolation at the zeros of T_n',

$$\ell_k(x) = \frac{T_n'(x)}{(x-x_k)T_n''(x_k)},$$

and

$$\mathcal{L}_k(f;x) := (1-x_k\,x)f(x_k) + (1-x_k^2)(x-x_k)f'(x_k)\,. \tag{21}$$

It follows from the uniqueness of the Hermite interpolation polynomial that $H(f;\cdot)\equiv f(\cdot)$ whenever f is a polynomial of degree at most $2n-1$, in particular, $H(f_i;\cdot)\equiv f_i(\cdot)$ for $i=1,\,3$ and $H(F_a;\cdot)\equiv F_a(\cdot)$.

From $T_n(x_k)=(-1)^k$, $0\le k\le n$, and $T_n'(x_k)=0$, $1\le k\le n-1$, $T_n'(x_0)=n^2$, $T_n'(x_n)=(-1)^n\,n^2$, we find

$$\begin{aligned} & f_1(x_0) = f_3(x_0) = 0,\\ & f_1(x_k) = 2+(-1)^k, \qquad f_3(x_k) = 1-x_k, \qquad 1\le k\le n-1,\\ & f_1(x_n) = f_3(x_n) = 2(1+(-1)^n)\,. \end{aligned} \tag{22}$$

In particular, (22) yields

$$F_a(x_0)=0\,, \qquad F_a(x_n) = 2a\big[1+(-1)^n\big]\,,$$

in agreement with the claim of Theorem 3 that $F_a(x)$ vanishes at x_0 and if n is odd, at x_n; in addition, $F_a(x_n)>0$ if n is even and $a>0$. Moreover, from $F_a(\cdot)\equiv H(F_a;\cdot)$, (20) and (21) we infer

$$F_a(x) = \frac{a\big[1+(-1)^n\big]}{n^4}\,(1-x)\big[T_n'(x)\big]^2 + (1-x^2)\sum_{k=1}^{n-1}\frac{\ell_k^2(x)}{(1-x_k^2)^2}\,\mathcal{L}_k(F_a;x)\,. \tag{23}$$

In order to find $\mathcal{L}_k(F_a;x)$, $1\le k\le n-1$, we firstly evaluate $\mathcal{L}_k(f_i;x)$, $i=1,\,3$.

Making use of the differential equation (9) and

$$f_1'(x)=\Big(1-\frac{1}{n^2}\Big)\,T_n'(x)-\frac{x+2}{n^2}\,T_n''(x),\quad f_3'(x)=\frac{1}{n^2}\,\big[T_n'(x)-(1-x)\,T_n''(x)\big]-1\,,$$

we find

$$f_1'(x_k)=(-1)^k\,\frac{x_k+2}{1-x_k^2}\,,\quad f_3'(x_k)=\frac{(-1)^k}{1+x_k}-1\,,\quad k=1,\dots,n-1\,. \tag{24}$$

From (21), (22) and (24) we obtain

$$\mathcal{L}_k(f_1;x)=(1-x_k\,x)\big(2+(-1)^k\big)+(-1)^k(x_k+2)(x-x_k)\,,$$

or, equivalently,

$$\mathcal{L}_k(f_1;x)=\begin{cases}(1-x_k)(3+2x+x_k), & \text{if } k \text{ is even,}\\ (1+x_k)(1+x_k-2x), & \text{if } k \text{ is odd.}\end{cases} \tag{25}$$

Likewise, we get

$$\mathcal{L}_k(f_3;x)=(1-x_k\,x)(1-x_k)+(x-x_k)\,\big[(-1)^k(1-x_k)-1+x_k^2\big]\,,$$

which simplifies to

$$\mathcal{L}_k(f_3;x)=\begin{cases}(1-x_k)(1+x_k^2-2x_k\,x), & \text{if } k \text{ is even,}\\ (1-x_k^2)(1+x_k-2x), & \text{if } k \text{ is odd.}\end{cases} \tag{26}$$

Now $\mathcal{L}_k(F_a;x)=(1+a)\,\mathcal{L}_k(f_1;x)-\mathcal{L}_k(f_3;x)$, (25) and (26) yield

Proposition 2. *For $1\le k\le n-1$, we have*

$$\mathcal{L}_k(F_a;x)=\begin{cases}(1-x_k)\big[2(1+x)(1+a+x_k)+(a-x_k)(1+x_k)\big], & \textit{if } k \textit{ is even,}\\ (a+x_k)(1+x_k)(1+x_k-2x), & \textit{if } k \textit{ is odd.}\end{cases}$$

Let us note that (23) and Proposition 2 hold for an arbitrary constant a. We show below that, with a as given in (18), $\mathcal{L}_k(F_a;x)>0$ for $x\in(-1,0]$, except for a specific index k. When k is even, we prove even more.

Proposition 3. *If $n\ge 4$ is even, k is even, $1<k<n-1$, and $a=x_1$, then*

$$\mathcal{L}_k(F_a;x)>0,\qquad -1\le x\le 1\,.$$

Proof. In this case $a-x_k=x_1-x_k>0$, and it follows from Proposition 2 that

$$\mathcal{L}_k(F_a;x)>2(1+x)(1-x_k)(1+a+x_k)\ge 0,\qquad -1\le x\le 1\,.$$

□

Proposition 4. *If $n \ge 4$, k is odd, $1 \le k \le n-1$, and a is given by* (18), *then*

$$\mathcal{L}_k(F_a;x) \begin{cases} > 0, & -1 < x \le 0, \quad 1 \le k < n-2\,, \\ \equiv 0, & \text{if} \;\; k = n-2 \;\; \text{and} \;\; n \;\; \text{is odd}, \\ \equiv 0, & \text{if} \;\; k = n-1 \;\; \text{and} \;\; n \;\; \text{is even}. \end{cases}$$

Proof. If n is even, then $a = x_1 = -x_{n-1}$, thus $a + x_{n-1} = 0$ and $\mathcal{L}_{n-1}(F_a;x) \equiv 0$ in view of Proposition 2. If n is odd, then $a = x_2 = -x_{n-2}$ and $a + x_{n-2} = 0$, which by Proposition 2 implies $\mathcal{L}_{n-2}(F_a;x) \equiv 0$. Finally, if $k < n-2$ is odd, then $a + x_k > a + x_{n-2} \ge x_2 + x_{n-2} = 0$, and from Proposition 2 we infer

$$\mathcal{L}_k(F_a;x) = (a+x_k)(1+x_k)(1+x_k-2x) > 0, \qquad -1 \le x < \frac{1+x_{n-3}}{2}.$$

Since $(1+x_{n-3})/2 > 0$, it follows that $\mathcal{L}_k(F_a;x) > 0$ for every $x \in [-1,0]$. $\square$

We are ready to accomplish the proof of the claim of Theorem 3 in the case $x \in [-1,0]$.

Assume first that $n \ge 4$ is even. If $a = x_1$, then (23) and Proposition 4 imply

$$F_{x_1}(x) = \frac{2x_1}{n^4}\,(1-x)\big[T_n'(x)\big]^2 + (1-x^2)\sum_{k=1}^{n-2}\frac{\ell_k^2(x)}{(1-x_k^2)^2}\,\mathcal{L}_k(F_a;x)\,. \tag{27}$$

Since $\ell_k(x_{n-1}) = 0$ for $1 \le k \le n-2$ and $T_n'(x_{n-1}) = 0$, it follows that $F_{x_1}(x_{n-1}) = F_{x_1}(-x_1) = 0$. If, on the other hand, $x \in (-1,0]$ and $x \ne x_{n-1}$, then all the summands in the sum in the right-hand side of (27) are non-negative, by virtue of Propositions 3 and 4, with at least one of them strictly positive, therefore $F_{x_1}(x) > 0$ in this case. If $a < x_1$, then from (23) and Proposition 2 we obtain

$$\begin{aligned} F_a(x_{n-1}) &= (1-x_{n-1}^2)\sum_{k=1}^{n-1}\frac{\ell_k^2(x_{n-1})}{(1-x_k^2)^2}\,\mathcal{L}_k(F_a;x_{n-1}) \\ &= \frac{\mathcal{L}_{n-1}(F_a;x_{n-1})}{1-x_{n-1}^2} = a + x_{n-1} < x_1 + x_{n-1} = 0\,, \end{aligned}$$

showing that the inequality $F_a(x) \ge 0$ fails to hold for $x = x_{n-1}$.

Now assume that $n \ge 5$ is odd. If $a = x_2$, then (23) and Proposition 4 imply

$$F_{x_2}(x) = (1-x^2)\Big[\sum_{k=1}^{n-3}\frac{\ell_k^2(x)}{(1-x_k^2)^2}\,\mathcal{L}_k(F_a;x) + \frac{\ell_{n-1}^2(x)}{(1-x_{n-1}^2)^2}\,\mathcal{L}_{n-1}(F_a;x)\Big]. \tag{28}$$

Since $\ell_k(x_{n-2}) = 0$ for $k \neq n-2$, we have $F_{x_2}(x_{n-2}) = 0$. If $x \in (-1, 0]$, $x \neq x_{n-2}$, then, in view of Propositions 3 and 4, all the summands in the brackets in (28) are non-negative, and at least one of them is strictly positive, therefore $F_{x_2}(x) > 0$. Finally, if $a < x_2$, then from (23) and Proposition 2 we find

$$\begin{aligned} F_a(x_{n-2}) &= (1 - x_{n-2}^2) \sum_{k=1}^{n-1} \frac{\ell_k^2(x_{n-2})}{(1 - x_k^2)^2} \mathcal{L}_k(F_a; x_{n-1}) \\ &= \frac{\mathcal{L}_{n-2}(F_a; x_{n-2})}{1 - x_{n-2}^2} = a + x_{n-2} < x_2 + x_{n-2} = 0\,, \end{aligned}$$

i.e., $F_a(x_{n-2}) < 0$ if $a < x_2$.

The proof of Theorem 1 is complete. □

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