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Explicit Solution, for $n = 6$, to a Markov-type Extremal Problem Initiated by Schur

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We apply our recent result from [16] to the solution found by Erdős and Szegő [4, p. 452] respectively by Shadrin [20, p. 1193] to a A. A. Markov-type [7], respectively to a V. A. Markov-type [8] extremal problem which was initiated by Schur [18, p. 275] a hundred years ago. In particular, the said problem is to determine, for algebraic polynomials P_n of a given degree n ($n \geq 4$), and for each $k \in \{1, 2, \ldots, n-2\}$, the extremum

$$N_{n,k,x_0} = \sup_{x_0 \in \boldsymbol{I}} \sup_{P_n \in \boldsymbol{B}_n, P_n^{(k+1)}(x_0)=0} |P_n^{(k)}(x_0)|,$$

where $\boldsymbol{I} = [-1, 1]$ and $\boldsymbol{B}_n = \{P_n : |P_n(x)| \leq 1 \text{ for } x \in \boldsymbol{I}\}$. The said solutions as given in [4], [20] are of a general character. Concrete solutions are known for $n \in \{4, 5\}$, see [14], [15]. We consider here the next higher degree $n = 6$ and determine explicitly, for $k \in \{1, 2, 3, 4\}$, the sought-for extremum N_{6,k,x_0} as well as the sextic extremizer polynomials (which are, except for $k = 4$, normalized proper Zolotarev polynomials with certain optimal parameter values). To this end we deploy *Mathematica* software [21] and root objects of dedicated integer polynomials of degree ≤ 18. Finally, we compare (with T_6 denoting the 6-th Chebyshev polynomial from $\boldsymbol{B}_6$) the concrete constants $M_{6,k} = N_{6,k,1}/T_6^{(k)}(1)$ for $k \in \{1, 2, 3, 4\}$ to the corresponding upper bounds as provided in [20, p. 1198], and in particular we compare the constant $M_{6,1}$ to the so-called (asymptotic) Zolotarev-Schur constant $M_{\infty,1}$, see [4], [5, Section 3.9], [10].

Keywords and Phrases: Chebyshev polynomial, extremal problem of Schur, Markov-type inequality, Zolotarev polynomial, Zolotarev-Schur constant.

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1. Introduction and Historical Remarks

A. A. Markov's famous inequality [7] asserts a sharp estimate on the first derivative of a bounded algebraic polynomial on a given closed interval. We

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restate it in a normalized form as an equation (with the artificial factor 1 added deliberately) for the purpose of comparison with related results to follow below.

Theorem 1 (A. A. Markov, 1889).

$$\sup_{x \in \boldsymbol{I}} \sup_{P_n \in \boldsymbol{B}_n} |P_n^{(1)}(x)| = T_n^{(1)}(1) \cdot 1 = n^2 \cdot 1, \tag{1}$$

where

P_n = *real algebraic polynomial of degree n with* $P_n(x) = \sum_{j=0}^{n} b_j x^j$,
$\boldsymbol{B}_n = \{P_n : |P_n(x)| \le 1 \text{ for } x \in \boldsymbol{I}\}$ *(unit ball),*
$\boldsymbol{I} = [-1, 1]$ *(unit interval),*
T_n = *n-th Chebyshev polynomial from $\boldsymbol{B}_n$ with known coefficients.*

We confine ourselves, here and in what follows, to specify only one extremal polynomial and one point in $\boldsymbol{I}$ such that the double supremum in question will be attained.

As the title of [7] indicates, (1) has its origin in a question posed by the chemist D. I. Mendeleev, see [13] for historical details. Inspired by (1), Schur [18, Section 2] provided the following twofold theorem for polynomials belonging, for $x_0 \in \boldsymbol{I}$, to the subset $\boldsymbol{B}_{n,x_0,2} = \{P_n \in \boldsymbol{B}_n : P_n^{(2)}(x_0) = 0\}$ of $\boldsymbol{B}_n$:

Theorem 2 (Schur, 1919). *Let $n \ge 3$, $x_0 \in \boldsymbol{I}$ and $P_n \in \boldsymbol{B}_{n,x_0,2}$, then*

$$|P_n^{(1)}(x_0)| < n^2 \cdot \frac{1}{2}; \tag{2}$$

if $M_{n,1}$ denotes the least positive constant such that for all $P_n \in \boldsymbol{B}_{n,x_0,2}$ and all $x_0 \in \boldsymbol{I}$

$$|P_n^{(1)}(x_0)| \le n^2 \cdot M_{n,1} = \sup_{x_0 \in \boldsymbol{I}} \sup_{P_n \in \boldsymbol{B}_{n,x_0,2}} |P_n^{(1)}(x_0)|, \tag{3}$$

then

$$0.217\ldots \le \mu = \limsup_{n \to \infty} M_{n,1} \le 0.465\ldots\ . \tag{4}$$

Inspired by Theorem 2, Erdős and Szegő [4, p. 452] provided the following strengthening: They determined both the extremal polynomials and the optimal point $x_0 \in \boldsymbol{I}$, for which equality is attained in (3), and they determined the constant μ.

Theorem 3 (Erdős & Szegő, 1942). *Let $n \ge 4$, then the extremum $n^2 \cdot M_{n,1}$ in (3) is attained for $x_0 = 1$ and for the proper Zolotarev polynomial $Z_{n,t} \in \boldsymbol{B}_n$ with $Z_{n,t}(-1) = (-1)^{n-1}$, where t denotes a real parameter the optimal value of which, $t = t^*$, unfolds from the condition $Z_{n,t} \in \boldsymbol{B}_{n,1,2}$, so that $\left|Z_{n,t^*}^{(1)}(1)\right| = n^2 \cdot M_{n,1}$. Furthermore, $\lim_{n \to \infty} M_{n,1} = \mu$ exists (and can be explicitly expressed with the aid of elliptic integrals) with numerical value*

$$\mu = 0.31\ldots\ . \tag{5}$$

The paper [4] is considered a masterpiece of classical analysis. It has been referenced to by several authors, e.g. [2], [5, Section 3.9], [11, Section 5 d], [20]. We focus here on the first part of Theorem 3, and in particular on the special degree $n = 6$. The reason is that for low-degree polynomials there is still room for refinement of Theorem 3 as far as the explicit representation of the extremizer polynomials Z_{n,t^*} and of the extrema $n^2 \cdot M_{n,1}$ is concerned. Therefore, we address the following issues which remain open in Theorem 3. Their resolution (at least for small values of n) would give a fuller and more concrete picture of the first part of Theorem 3.

Open Issues: *Try to determine explicitly*

(i) the parameterized coefficients of $Z_{n,t}$ *in the algebraic power form;*

(ii) the optimal parameter $t = t^*$*;*

(iii) the optimal coefficients of the extremizer Z_{n,t^*} *in the power form;*

(iv) the extremum $\sup_{x_0 \in \boldsymbol{I}} \sup_{P_n \in \boldsymbol{B}_{n,x_0,2}} |P_n^{(1)}(x_0)| = |Z_{n,t^*}^{(1)}(1)| = n^2 \cdot M_{n,1}$,

and hence the "diminishing factor" $M_{n,1}$.

The key issue one has to resolve first is (i). Let us consider this issue for the initial values of n.

An explicit representation, up to the sign, of $Z_{n,t}$ ($n = 4$) as an algebraic power form with parameterized coefficients has been provided by the first-named author [12, p. 357] and by Shadrin [19, p. 10]. However, Shadrin points out that $Z_{4,t}$ has already been given, in a somewhat roundabout way, by V. A. Markov back in 1892 [8, p. 73] but it has been omitted in the abridged German translation of [8]. The above Open Issues (ii)–(iv) for $n = 4$ have been resolved in [14]. There has been no progress for more than 100 years in finding an explicit representation of $Z_{n,t}$ ($n > 4$) as an algebraic power form with parameterized coefficients. As Shadrin put it in 2014 [20, p. 1185]: "There is no explicit expression for [proper] Zolotarev polynomials of degree $n > 4$."

Eventually, in 2017 the open issue (i) has been resolved (up to the sign and reflection) for $n = 5$ by Grasegger and Vo [6, p. 178]. Based on this information, the remaining Open Issues for $n = 5$ have been resolved in [15]. In contrast to the case $n = 4$, where the parametrization of the coefficients of $Z_{n,t}$ is a rational one, the parametrization is a radical one if $n = 5$. Therefore, the results for $n = 5$ are not so smooth as for $n = 4$. For $n = 6$ the Open Issue (i) has been resolved (up to the sign) by the present authors in 2019 [16, p. 45]. Regrettably, the solution for $n = 6$ as given in [6, Section 4.5-4.6] is flawed, see also [16, Section 4.1]. The parametrization of the coefficients of $Z_{6,t}$ is again a radical one. To resolve the remaining Open Issues (ii)–(iv) for $n = 6$ is the goal which we address in Section 2 below.

For an alternative approach see Remark 5. To the best of our knowledge, an explicit representation of the proper $Z_{n,t}$ as an algebraic power form with parameterized coefficients is not known if $n > 6$. But we point to our paper [17] where explicit algebraic constructions are given for $7 \le n \le 12$ to solve Zolotarev's so-called "First Problem" [22] which is referred to e.g. in [1], [3], [9]. This "First Problem", which asks for least deviating polynomials of degree n on $\boldsymbol{I}$ whose first two leading coefficients are prescribed, had been posed to Zolotarev by Chebyshev himself [22, p. 2]. Zolotarev gave a transcendental solution for $n \ge 2$ in terms of elliptic integrals. Trying to recover from that solution an explicit algebraic power form representation for $Z_{n,t}$ turns out to be unexpectedly complicated, even for the first reasonable polynomial degree $n = 2$, see [3, Section 3] for details.

2. Resolving the Open Issues (i)–(iv) for $n = 6$

A proper sextic Zolotarev polynomial $Z_{6,t} \in \boldsymbol{B}_6$ in the variable x with $Z_{6,t}(-1) = -1$, which appears in Theorem 3 when $n = 6$, can be readily adopted from [16, Theorem 3.1] by putting

$$Z_{6,t}(x) = \sum_{j=0}^{6} -b_{j,6}(t)x^j. \tag{6}$$

The explicit form of the parameterized coefficients $b_{j,6}(t)$ is provided in [16] as well as the range of the parameter t:

$$t \in I_6 = (\theta, 0), \qquad \text{with } \theta = \frac{1}{2}(5 - 3\sqrt{3}) = -0.0980762113\ldots\ .$$

In the following we shall make use of root objects $\texttt{Root}[f, \ell]$ which represent the ℓ-th root of the polynomial equation $f(x) = 0$, where x is an appropriate variable, see *Mathematica* [21] for details. Root objects are a concise way of expressing algebraic numbers via the minimal integer polynomial they satisfy, along with an ordering of the finitely many (complex) roots of f, specified by the second argument, ℓ. Operations can be used on root objects, and they can be determined numerically with arbitrary precision.

Proposition 1. *The optimal parameter $t = t^* \in I_6$ of the extremizer Z_{6,t^*} in Theorem 3 (when $n = 6$) is given by*

$$t^* = \texttt{Root}[P_{12}, 2] = -0.0584562201\ldots, \tag{7}$$

where

$$\begin{aligned} P_{12}(x) = 100 &+ 807x - 10041x^2 + 66803x^3 - 347670x^4 + 1344726x^5 \\ &- 4143576x^6 + 10413324x^7 - 21339144x^8 + 33885896x^9 \\ &- 35284272x^{10} + 23462592x^{11} + 5236480x^{12}. \end{aligned}$$

Proof. The condition $Z_{6,t} \in \mathbf{B}_{6,1,2}$, that is $Z_{6,t}^{(2)}(1) = 0$ leads to, see [16, Theorem 3.1],

$$
\begin{aligned}
0 = 2880 &+ \frac{320\sqrt{3}(1-4t)t(1+6t+12t^2+116t^3)}{(-1+t)^2\sqrt{(-1+t)t^3(1+t+7t^2)}} \\
&+ \frac{288(7-25t+66t^2-146t^3-64t^4+2580t^5-6800t^6+26252t^7)}{(-1+t)^5(1+t+7t^2)} \\
&- \frac{24\sqrt{3}t^4(1+2t)(-1+4t)}{(-1+t)^6\big((-1+t)t^3(1+t+7t^2)\big)^{3/2}} \\
&\times (5+3t-6t^2+564t^3-3408t^4+13296t^5-35136t^6+107976t^7 \\
&\quad -130416t^8+243952t^9) \\
&+ \frac{12}{(-1+t)^{10}(1+t+7t^2)} \\
&\times (13-102t+390t^2-880t^3-288t^4+19296t^5-102792t^6+390816t^7 \\
&\quad -939024t^8+1167536t^9-258720t^{10}-339888t^{11}+2720848t^{12}).
\end{aligned} \tag{8}
$$

Solving equation (8) with the *Mathematica*-call `Solve` and imposing the condition $t < 0$ gives (7).

An alternative proof would be to solve equation (2.17) in [4], see also the identical equation (5.20) in [11], which is equivalent to the condition $Z_{6,t}^{(2)}(1) = 0$. We restate that equation for $n = 6$ using the notation defined in [16, Eqs. (18)–(20)]:

$$\frac{36(\gamma(t)-1)^2}{(\alpha(t)-1)(\beta(t)-1)} = 1 + 2\Big(\frac{2}{\gamma(t)-1} - \frac{1}{\alpha(t)-1} - \frac{1}{\beta(t)-1}\Big). \tag{9}$$

It turns out that the solution $t \in I_6$ of (9) coincides, as can be checked with the aid of *Mathematica*, with $t = t^*$ as given in (7). □

Proposition 2. *The optimal coefficients of the extremizer polynomial* Z_{6,t^*} *in Theorem 3 (when* $n = 6$*) can be expressed with the aid of root objects in a closed form as follows, where the indicated polynomials* $Q_{12,j}$ $(j = 0, \ldots, 6)$ *are defined in Section 5:*

$$Z_{6,t^*}(x) = \sum_{j=0}^{6} -b_{j,6}(t^*)x^j \tag{10}$$

with

$$
\begin{aligned}
-b_{0,6}(t^*) &= 0.7554514270\ldots = \texttt{Root}[Q_{12,0}, 1], \\
-b_{1,6}(t^*) &= 3.5019958196\ldots = \texttt{Root}[Q_{12,1}, 2], \\
-b_{2,6}(t^*) &= -10.9528678404\ldots = \texttt{Root}[Q_{12,2}, 1], \\
-b_{3,6}(t^*) &= -15.1732709430\ldots = \texttt{Root}[Q_{12,3}, 1], \\
-b_{4,6}(t^*) &= 24.8001463072\ldots = \texttt{Root}[Q_{12,4}, 2], \\
-b_{5,6}(t^*) &= 12.6712751233\ldots = \texttt{Root}[Q_{12,5}, 2], \\
-b_{6,6}(t^*) &= -14.6027298938\ldots = \texttt{Root}[Q_{12,6}, 1].
\end{aligned} \tag{11}
$$

Proof. Insert $t = t^*$ from (7) into the coefficients $-b_{6,j}$ in (6). Simplifying by the *Mathematica*-call `RootReduce`, which reduces a term into a single root object, yields the optimal coefficients $-b_{j,6}(t^*)$ as given in (11). □

Figure 1 displays the extremizer Z_{6,t^*} (see (10)) on the interval $[-1, 1.35]$.

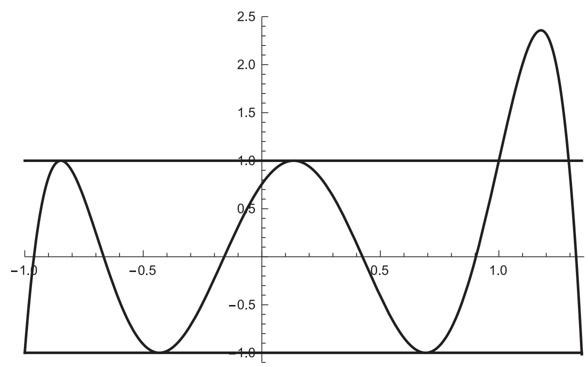

Figure 1. Z_{6,t^*}

Proposition 3. *For $n = 6$, a closed form for the extremum*

$$\sup_{x_0 \in \boldsymbol{I}} \sup_{P_6 \in \boldsymbol{B}_{6,x_0,2}} |P_6^{(1)}(x_0)| = |Z_{6,t^*}^{(1)}(1)| = 36 \cdot M_{6,1}$$

can be given with the aid of a root object as follows:

$$36 \cdot M_{6,1} = \texttt{Root}[Q_{12}, 2] = 11.0170287924\ldots, \tag{12}$$

where

$$\begin{aligned} Q_{12}(x) = &-31701690482688000 - 41907873613086720x - 17030850570878976x^2 \\ &+ 157312823721984x^3 + 1918287790661632x^4 + 452851936267520x^5 \\ &- 8322884811393x^6 - 18067969617221x^7 - 2600900581531x^8 \\ &- 63885008695x^9 + 16222621400x^{10} + 1493400000x^{11} + 40000000x^{12}. \end{aligned}$$

Hence the "diminishing factor" $M_{6,1}$ can be expressed in closed form as

$$M_{6,1} = \frac{\texttt{Root}[Q_{12}, 2]}{36} = 0.3060285775\ldots\ . \tag{13}$$

Proof. Form the polynomial $Z_{6,t^*}(x) = \sum_{j=0}^{6} -b_{j,6}(t^*)x^j$ according to (10), differentiate it with respect to x, and evaluate the resulting term at $x = 1$. Simplifying (again using `RootReduce`) yields (12). Dividing in (12) both sides by 36 yields (13). □

We have thus obtained, based on our recent result [16], a refined version of the first part of Theorem 3 (when $n = 6$). It supplements the results for $n = 4$ and $n = 5$ in [14], [15].

3. Extensions, for $n = 6$, to k-th Derivatives ($k = 2, 3, 4$)

We next turn to the question of extending the previous result concerning the extremal first derivative to higher derivatives. To this end, we start with a well-known generalization of A. A. Markov's inequality, from the first derivative to k-th derivatives, $k \geq 2$, which is due to his half-brother, V. A. Markov [8].

Theorem 4 (V. A. Markov, 1892).

$$\sup_{x\in \boldsymbol{I}} \sup_{P_n\in \boldsymbol{B}_n} |P_n^{(k)}(x)| = T_n^{(k)}(1)\cdot 1 = \prod_{\tau=0}^{k-1} \frac{n^2-\tau^2}{2\tau+1}\cdot 1, \qquad (2\leq k\leq n).$$

This theorem suggests to extend the first part of Schur's Markov-type extremal problem, as amplified by Erdős and Szegő, to higher derivatives. Such an extension was proposed, and solved, by Shadrin [20, Proposition 4.4]. We put it in the following way:

Theorem 5 (Shadrin, 2014). *Let $n \geq 4$ and $k \in \{2, \ldots, n-2\}$ and set, for $x_0 \in \boldsymbol{I}$, $\boldsymbol{B}_{n,x_0,k+1} = \{P_n \in \boldsymbol{B}_n : P_n^{(k+1)}(x_0) = 0\}$. Then, the extremum*

$$\sup_{x_0\in \boldsymbol{I}} \sup_{P_n\in \boldsymbol{B}_{n,x_0,k+1}} |P_n^{(k)}(x_0)| = T_n^{(k)}(1)\cdot M_{n,k}, \tag{14}$$

with $M_{n,k} \in (0,1)$ unspecified, is attained either for

(i) $P_n = T_n$ and $x_0 = \omega_{n,k}$ = the rightmost zero of $T_n^{(k+1)}$, or for

(ii) $P_n = Z_{n,t} \in \boldsymbol{B}_n$ (proper Zolotarev polynomial) and $x_0 = 1$, where the optimal parameter of $Z_{n,t}$, $t = t_k^*$, unfolds from $Z_{n,t}^{(k+1)}(1) = 0$, or for

(iii) $P_n = Z_{n,\sigma} \in \boldsymbol{B}_n$ (improper Zolotarev polynomial with $Z_{n,\sigma}(x) = -T_n\left(\frac{x-\sigma}{\sigma+1}\right)$ and parameter $\sigma \in \left(0, \tan^2\left(\frac{\pi}{2n}\right)\right]$) and $x_0 = 1$, where the optimal parameter,

$\sigma = \sigma_k^*$, unfolds from $Z_{n,\sigma}^{(k+1)}(1) = 0$, so that there holds

$$T_n^{(k)}(1)\cdot M_{n,k} = \max\left\{|T_n^{(k)}(\omega_{n,k})|, |Z_{n,t_k^*}^{(k)}(1)|, |Z_{n,\sigma_k^*}^{(k)}(1)|\right\}, \quad \textit{for } k\in\{2,\ldots,n-2\}. \tag{15}$$

In Theorem 5 we directly focus on the special degree $n = 6$. Our first observation is that then the polynomials in Theorem 5 (iii) cannot attain the extremum (14) and are therefore negligible, so for $n = 6$ (15) reduces to

$$\sup_{x_0\in \boldsymbol{I}} \sup_{P_6\in \boldsymbol{B}_{6,x_0,k+1}} |P_6^{(k)}(x_0)| = T_6^{(k)}(1)\cdot M_{6,k} = \max\left\{|T_6^{(k)}(\omega_{6,k})|, |Z_{6,t_k^*}^{(k)}(1)|\right\}, \quad \text{for } k\in\{2,3,4\}. \tag{16}$$

Lemma 1. *For $n = 6$ the polynomials P_6 in Theorem 5 (iii) are incompatible with the condition $P_6^{(k+1)}(1) = 0$ for $k \in \{2, 3, 4\}$.*

Proof. Consider $P_6(x) = Z_{6,\sigma}(x) = -T_6\left(\frac{x-\sigma}{\sigma+1}\right)$ with $0 < \sigma \le \tan^2\left(\frac{\pi}{12}\right) = 7 - 4\sqrt{3} = 0.0717967697\ldots$. The condition $P_6^{(k+1)}(1) = 0$ yields

$$\frac{384(1-\sigma)(7-26\sigma+7\sigma^2)}{(1+\sigma)^6} = 0, \quad \frac{1152(9-22\sigma+9\sigma^2)}{(1+\sigma)^6} = 0, \quad \frac{23040(1-\sigma)}{(1+\sigma)^6} = 0,$$

for $k = 2, 3, 4$, respectively. Obviously, these equations are not solvable for σ from the specified range. □

Remark 1. T_{n-1} can also be considered as an (exceptional) Zolotarev polynomial. It is readily verified by comparison with (17) below that for $n = 6$ and $k \in \{2, 3, 4\}$, T_5 cannot attain the extremum in (14).

As for possible open issues in Theorem 5 (when $n = 6$) we notice that the coefficients of the potential extremizer polynomials are explicitly known. Rather, in view of (16), we have now to determine in closed form the extrema $T_6^{(k)}(1) \cdot M_{6,k}$, and hence the "diminishing factors" $M_{6,k}$, for $k \in \{2, 3, 4\}$, as well as the zeros $\omega_{6,k}$, respectively the optimal parameters t_k^* and the optimal coefficients of Z_{6,t_k^*}, as the case may be.

We compute first the potential extremum $|T_6^{(k)}(\omega_{6,k})|$ in (16).

Lemma 2. *For $k \in \{2, 3, 4\}$ the numbers $\omega_{6,k}$ turn into*

$$\omega_{6,2} = \sqrt{\tfrac{3}{10}} = 0.5477225575\ldots, \quad \omega_{6,3} = \tfrac{1}{\sqrt{10}} = 0.3162277660\ldots, \quad \omega_{6,4} = 0$$

(and the numbers $T_6^{(k)}(1)$ turn into 420, 2688, 10368, respectively). Furthermore, the numbers $\gamma_k = |T_6^{(k)}(\omega_{6,k})|$ read

$$\gamma_2 = \frac{252}{5} = 50.4, \quad \gamma_3 = 384\sqrt{\tfrac{2}{5}} = 242.8629243009\ldots, \quad \gamma_4 = 1152. \tag{17}$$

Proof. The results follow from elementary calculations which are skipped. □

We compute next the potential extremum $|Z_{6,t_k^*}^{(k)}(1)|$ in (16) and state the results separately for each $k \in \{2, 3, 4\}$.

Proposition 4. *For $k = 2$ the optimal parameter $t = t_2^* \in I_6$ of the potential extremizer Z_{6,t_2^*} in (16) is given by*

$$t_2^* = \texttt{Root}[P_{18}, 2] = -0.0287786947\ldots, \tag{18}$$

where

$$\begin{aligned}P_{18}(x) = {} & 1225 + 24888x - 474909x^2 + 3992802x^3 - 25274670x^4 + 130292688x^5 \\ & - 556873092x^6 + 2039612664x^7 - 6489821256x^8 + 17961559552x^9 \\ & - 43178756880x^{10} + 89513877984x^{11} - 157496244000x^{12} \\ & + 228768698112x^{13} - 256699577280x^{14} + 188617173120x^{15} \\ & - 42305434752x^{16} - 66456864768x^{17} + 84916137472x^{18},\end{aligned}$$

and the potential extremum $|Z^{(2)}_{6,t_2^*}(1)|$ *in* (16) *can be given in a closed form as*

$$|Z^{(2)}_{6,t_2^*}(1)| = \texttt{Root}[Q_{18}, 2] = 77.9961703319\ldots, \tag{19}$$

where

$$\begin{aligned}Q_{18}(x) = {} & 272791650756043802250000000000 + 33077521213137508060800000000x \\ & + 526422537457950121474661043 75x^2 \\ & + 2744096409884598824837048526 0x^3 \\ & + 2100788061109367753676562831 8x^4 \\ & + 4722067208953141496722372752x^5 \\ & + 16804877919470540324958236 77x^6 + 1449198338028850455698990 48x^7 \\ & - 585063323448346092633627 54x^8 - 250899581009316548250546 8x^9 \\ & + 670342831875862334542487x^{10} - 202184519175981938838 16x^{11} \\ & + 1223680944762670387728x^{12} - 188166471057566641920x^{13} \\ & + 8964809376904673536x^{14} - 183902146449287168x^{15} \\ & + 1787642963095552x^{16} - 7870663884800x^{17} + 12845056000x^{18}.\end{aligned}$$

Proof. The condition $Z^{(3)}_{6,t}(1) = 0$ can be reduced to (see (6))

$$\begin{aligned}0 = {} & -24(-1+t)^2\sqrt{t^3(-1-6t^2+7t^3)} \\ & \times (-13 + 55t - 174t^2 + 554t^3 - 1364t^4 + 3900t^5 - 7480t^6 + 26392t^7) \\ & + \sqrt{3}t(-1+4t) \\ & \times (-35 - 67t - 3648t^3 + 16320t^4 - 45360t^5 + 137136t^6 - 240096t^7 \\ & \quad + 377136t^8 - 171280t^9 + 520384t^{10}).\end{aligned}$$

Solving this equation with the *Mathematica*-call `Solve` and imposing the condition $t < 0$ gives (18). Inserting then, coefficient-wise, into $|Z^{(2)}_{6,t}(x)|$ the values $t = t_2^*$ and $x = 1$ yields, after simplification with `RootReduce`, (19). □

At this stage we are able to state a closed-form solution to Theorem 5 for $n = 6$ and $k = 2$ since we are now in a position to compare the two numbers which are competing for the extremum in (16).

Proposition 5. *For $k = 2$ the extremum in (16) is, see (19),*

$$T_6^{(2)}(1) \cdot M_{6,2} = |Z_{6,t_2^*}^{(2)}(1)| = \mathtt{Root}[Q_{18}, 2] > \gamma_2,$$

and hence the diminishing factor at $T_6^{(2)}(1)$ is

$$M_{6,2} = \frac{\mathtt{Root}[Q_{18}, 2]}{420} = 0.1857051674\ldots\ .$$

Proof. In view of (16), the statements follow from Lemma 2 and Proposition 4. □

To give a full picture of the solution to Theorem 5, when $n = 6$ and $k = 2$, we provide next the coefficients of Z_{6,t_2^*} in closed form.

Proposition 6. *The optimal coefficients of the extremizer polynomial Z_{6,t_2^*} can be expressed with the aid of root objects as follows, where the indicated polynomials $R_{18,j}$ $(j = 0, \ldots, 6)$ are defined in Section 5:*

$$\begin{aligned}
-b_{0,6}(t_2^*) &= 0.5566741350\ldots = \mathtt{Root}[R_{18,0}, 1],\\
-b_{1,6}(t_2^*) &= 4.2942729690\ldots = \mathtt{Root}[R_{18,1}, 2],\\
-b_{2,6}(t_2^*) &= -7.6296272013\ldots = \mathtt{Root}[R_{18,2}, 2],\\
-b_{3,6}(t_2^*) &= -17.8150103785\ldots = \mathtt{Root}[R_{18,3}, 1],\\
-b_{4,6}(t_2^*) &= 16.8032140651\ldots = \mathtt{Root}[R_{18,4}, 2],\\
-b_{5,6}(t_2^*) &= 14.5207374094\ldots = \mathtt{Root}[R_{18,5}, 2],\\
-b_{6,6}(t_2^*) &= -9.7302609988\ldots = \mathtt{Root}[R_{18,6}, 1].
\end{aligned}$$

Proof. Insert t_2^* (see (18)) into the coefficients of $Z_{6,t}$ (see (6)) and simplify the resulting terms by applying `RootReduce`. □

We next turn to the case $k = 3$.

Proposition 7. *For $k = 3$ the optimal parameter $t = t_3^* \in I_6$ of the potential extremizer Z_{6,t_3^*} in (16) is given by*

$$t_3^* = \mathtt{Root}[P_{14}, 2] = -0.0108249663\ldots, \tag{20}$$

where

$$\begin{aligned}
P_{14}(x) = 100 &+ 8227x - 87224x^2 + 539195x^3 - 2661322x^4 + 10438004x^5\\
&- 33840984x^6 + 91647672x^7 - 201405936x^8 + 359738960x^9 - 480448768x^{10}\\
&+ 332936624x^{11} + 224814304x^{12} - 815121728x^{13} + 991739776x^{14},
\end{aligned}$$

and the potential extremum $|Z_{6,t_3^*}^{(3)}(1)|$ in (16) can be given in a closed form as

$$|Z_{6,t_3^*}^{(3)}(1)| = \mathtt{Root}[Q_{14}, 2] = 356.6856256173\ldots, \tag{21}$$

where

$$\begin{aligned}
Q_{14}(x) = &-18437877163098133401600000000000000 \\
&+ 25202046556957919495196556800000000x \\
&- 10295654795094951050943230460291 60x^2 \\
&+ 1091888613116184976203716541644 13x^3 \\
&- 17277244877523400059448524945 6x^4 \\
&- 13931845019504847976611974096 88x^5 \\
&+ 3558631486790113483930293513 6x^6 \\
&+ 73479517857693920177632188 0x^7 \\
&- 18129471741236756962472659 2x^8 + 50835924950011021807990 08x^9 \\
&- 8966932064926701992576 0x^{10} - 168062686853157087408x^{11} \\
&+ 942821098666444800x^{12} + 358805811456000x^{13} + 33177600000x^{14}.
\end{aligned}$$

Proof. The condition $Z_{6,t}^{(4)}(1) = 0$ can be reduced to (see (6))

$$\begin{aligned}
0 = 34560 &+ \frac{1920\sqrt{3}(1-4t)t(1+6t+12t^2+116t^3)}{(-1+t)^2\sqrt{(-1+t)t^3(1+t+7t^2)}} \\
&+ \frac{576(7-25t+66t^2-146t^3-64t^4+2580t^5-6800t^6+26252t^7)}{(-1+t)^5(1+t+7t^2)}.
\end{aligned}$$

Solving this equation with the *Mathematica*-call `Solve` and imposing the condition $t < 0$ gives (20). Inserting then, coefficient-wise, into $|Z_{6,t}^{(3)}(x)|$ the values $t = t_3^*$ and $x = 1$ yields, after simplification with `RootReduce`, (21). □

At this stage we are able to state a closed-form solution to Theorem 5 for $n = 6$ and $k = 3$ since we are now in a position to compare the two numbers which are competing for the extremum in (16).

Proposition 8. *For* $k = 3$ *the extremum in* (16) *is, see* (21),

$$T_6^{(3)}(1) \cdot M_{6,3} = |Z_{6,t_3^*}^{(3)}(1)| = \texttt{Root}[Q_{14}, 2] > \gamma_3,$$

and hence the diminishing factor at $T_6^{(3)}(1)$ *is*

$$M_{6,3} = \frac{\texttt{Root}[Q_{14}, 2]}{2688} = 0.1326955452\ldots\ .$$

Proof. In view of (16), the statements follow from Lemma 2 and Proposition 7. □

To give a full picture of the solution to Theorem 5, when $n = 6$ and $k = 3$, we provide next the coefficients of Z_{6,t_3^*} in closed form.

Proposition 9. *The optimal coefficients of the extremizer polynomial Z_{6,t_3^*} can be expressed with the aid of root objects as follows, where the indicated polynomials $S_{14,j}$ $(j = 0,\dots,6)$ are defined in Section 5:*

$$\begin{aligned}
-b_{0,6}(t_3^*) &= 0.3528636055\ldots = \texttt{Root}[S_{14,0}, 1],\\
-b_{1,6}(t_3^*) &= 4.7385613504\ldots = \texttt{Root}[S_{14,1}, 2],\\
-b_{2,6}(t_3^*) &= -4.6794790628\ldots = \texttt{Root}[S_{14,2}, 1],\\
-b_{3,6}(t_3^*) &= -19.2068591849\ldots = \texttt{Root}[S_{14,3}, 1],\\
-b_{4,6}(t_3^*) &= 10.1600515023\ldots = \texttt{Root}[S_{14,4}, 2],\\
-b_{5,6}(t_3^*) &= 15.4682978344\ldots = \texttt{Root}[S_{14,5}, 1],\\
-b_{6,6}(t_3^*) &= -5.8334360449\ldots = \texttt{Root}[S_{14,6}, 1].
\end{aligned}$$

Proof. Insert t_3^* (see (20)) into the coefficients of $Z_{6,t}$ (see (6)) and simplify the resulting terms by applying `RootReduce`. □

Finally, we consider the case $k = 4$.

Proposition 10. *For $k = 4$ the optimal parameter $t = t_4^* \in I_6$ of the potential extremizer Z_{6,t_4^*} in (16) is given by*

$$t_4^* = \texttt{Root}[P_8, 2] = -0.0022727265\ldots,$$

where

$$\begin{aligned}
P_8(x) = 1 &+ 436x - 1748x^2 + 5272x^3 - 15632x^4 + 24592x^5 - 12752x^6\\
&- 48416x^7 + 212272x^8,
\end{aligned}$$

and the potential extremum $|Z^{(4)}_{6,t_4^*}(1)|$ in (16) can be given in a closed form as

$$|Z^{(4)}_{6,t_4^*}(1)| = \texttt{Root}[Q_8, 1] = 1064.5483463044\ldots, \tag{22}$$

where

$$\begin{aligned}
Q_8(x) = {} & 328420417785644453125 + 749468822132812500 0x\\
&+ 3315302378241643 98x^2 - 20489444419134816x^3 + 131526600431067x^4\\
&+ 2002970341872x^5 + 3880894842x^6 - 5639400x^7 + 125x^8.
\end{aligned}$$

Proof. The proof for $k = 4$ is analogous to the proof for $k = 3$ of Proposition 7. □

A comparison of (22) with (17) shows in view of (16) that now T_6 with $T_6(x) = -1 + 18x^2 - 48x^4 + 32x^6$ is the extremizer in Theorem 5 if $n = 6$ and $k = 4$, so that we get

Proposition 11. *For $k = 4$ the extremum in* (16) *is*

$$T_6^{(4)}(1) \cdot M_{6,4} = |T_6^{(4)}(0)| = 1152 = \gamma_4 > \texttt{Root}[Q_8, 1],$$

and hence the diminishing factor at $T_6^{(4)}(1)$ is

$$M_{6,4} = \frac{1152}{10368} = \frac{1}{9} = 0.1111111111\ldots\ .$$

The extremizer polynomial is T_6.

4. Concluding Remarks

Remark 2. For the diminishing factors $M_{n,k}$ which appear in Theorems 2, 3, and 5, Shadrin [20, Theorem 7.1] has provided the upper bounds

$$M_{n,k} \leq \lambda_{n,k} = \frac{1}{k+1} \cdot \frac{n-1}{n-1+k} \quad (n \geq 4,\ 1 \leq k \leq n-2).$$

Using results from [14], [15] and from the present paper we now compare, for $n \in \{4, 5, 6\}$, $M_{n,k}$ to $\lambda_{n,k}$ and notice that even $M_{n,n-2} = \lambda_{n,n-2}$ holds for those values of n:

$$\begin{aligned}
M_{4,1} &= 0.2992279308\ldots < \lambda_{4,1} = \frac{3}{8} = 0.375, \\
M_{4,2} &= \lambda_{4,2} = \frac{1}{5} = 0.2, \\
M_{5,1} &= 0.3036993415\ldots < \lambda_{5,1} = \frac{2}{5} = 0.4, \\
M_{5,2} &= 0.1832314132\ldots < \lambda_{5,2} = \frac{2}{9} = 0.2222222222\ldots, \\
M_{5,3} &= \lambda_{5,3} = \frac{1}{7} = 0.1428571428\ldots, \\
M_{6,1} &= 0.3060285775\ldots < \lambda_{6,1} = \frac{5}{12} = 0.4166666666\ldots, \\
M_{6,2} &= 0.1857051674\ldots < \lambda_{6,2} = \frac{5}{21} = 0.2380952380\ldots, \\
M_{6,3} &= 0.1326955452\ldots < \lambda_{6,3} = \frac{5}{32} = 0.15625, \\
M_{6,4} &= \lambda_{6,4} = \frac{1}{9} = 0.1111111111\ldots\ .
\end{aligned} \tag{23}$$

Remark 3. The Markov-type extremal problem under consideration may thus be summarized, for $n = 6$ and $k = 1$, as follows: Given an arbitrary $P_n \in \boldsymbol{B}_n$ and an arbitrary $x_0 \in \boldsymbol{I}$, the value $\left|P_n^{(1)}(x_0)\right|$ cannot exceed $n^2 = 36$

(sharp upper bound (1)). Under the additional condition $P_n^{(2)}(x_0) = 0$ there hold the estimates

$$\left|P_n^{(1)}(x_0)\right| \le \begin{cases} n^2 \cdot \frac{1}{2} = 18 \quad \text{(upper bound, see (2))}, \\ n^2 \cdot \frac{5}{12} = 15 \quad \text{(improved upper bound, see (23))}, \\ n^2 \cdot M_{6,1} = 11.0170287924\ldots \\ \qquad\qquad = \texttt{Root}[Q_{12}, 2] \quad \text{(sharp upper bound, see (12))}. \end{cases}$$

Remark 4. Improving on Schur's limit result (4), Erdős and Szegő [4, p. 452] provided the exact formula for $\mu = \lim_{n\to\infty} M_{n,1}$ in terms of elliptic integrals. For the numerical value of μ we have given in (5) deliberately only two valid decimal digits, because Erdős and Szegő [4, p. 452] have given the value $\mu = 0.3124\ldots$ which, however, is biased due to roundings and table look-ups. Chopped after the tenth decimal digit the correct numerical value is

$$\mu = 0.3110788667\ldots \ .$$

More decimal figures as computed by means of the exact formula for μ (adjusted to the *Mathematica*-notation) are to be found in [10]. The values $M_{4,1}$, $M_{5,1}$, $M_{6,1}$ support the conjecture that the sequence $\{M_{n,1}\}_{n=4}^{\infty}$ converges monotonically to μ. The limit μ is coined "Zolotarev-Schur constant" in [10]. We propose to use the term "Erdős-Szegő and Schur constant" instead, because Zolotarev did not contribute to μ whereas Erdős and Szegő provided the exact formula for μ.

Remark 5. It is possible to obtain our results alternatively with Groebner-basis computations (as provided e.g. in *Mathematica*) utilizing the Abel-Pell differential equation for proper sextic Zolotarev polynomials, and we have used that approach to cross-check our results. But the present approach, which relies on [16], is more obvious and inherent.

5. Auxiliary Polynomials

In this Section we provide the bulky minimal polynomials which have been utilized in Proposition 2 for $k = 1$, Proposition 6 for $k = 2$, and Proposition 9 for $k = 3$ for the representation of the optimal coefficients of the respective extremizer polynomials Z_{6,t^*}, Z_{6,t_2^*}, Z_{6,t_3^*}.

The polynomials $Q_{12,j}$ $(j = 0, \ldots, 6)$ of degree 12, utilized in Proposition 2:

$$
\begin{aligned}
Q_{12,0}(x) = {} & 3561062321529505751661802890625000000 \\
& - 26901897649217604657422198845087500000x \\
& - 90347881674907502269048465353318239400x^{2} \\
& + 99214648383838540571325803108837361310x^{3} \\
& + 96286117302023422801549648079945139200x^{4} \\
& - 71155193950983199995179189240279449600x^{5} \\
& + 61365720299595130317176616180832010240x^{6} \\
& - 33250023370490071035659162861132840960x^{7} \\
& + 13060859144011028058251982295531520000x^{8} \\
& - 3591686080319180126952099938304000000x^{9} \\
& + 636598544035144833790115840000000000x^{10} \\
& - 64415788773475457433600000000000000x^{11} \\
& + 2814749767106560000000000000000000x^{12}
\end{aligned}
$$

$$
\begin{aligned}
Q_{12,1}(x) = {} & -1347010710736634197727049020754405 \\
& - 19009823801952096377640501263717229x \\
& + 55493489661258177384772525042160250x^{2} \\
& - 25350669443508486415575052279353030x^{3} \\
& - 32255932153700873448361286815709440x^{4} \\
& + 17850996969202641118929978221506560x^{5} \\
& - 3909505531418153322539272818851840x^{6} \\
& - 8130234358489447767267217611161600x^{7} \\
& + 9386309113738524405678263500800000x^{8} \\
& - 2844416328451923687967293440000000x^{9} \\
& + 514242441301386382540800000000000x^{10} \\
& - 46845110050095104000000000000000x^{11} \\
& + 1717986918400000000000000000000x^{12}
\end{aligned}
$$

$$
\begin{aligned}
Q_{12,2}(x) = {} & 74758354163549766117520654710937500000 \\
& + 883069090816615023040710271728716250000x \\
& + 3063518507986400893335042451485667703400x^{2} \\
& - 2299457732065990339400864804324855487300x^{3} \\
& - 3886168418220563598228390224630279859200x^{4} \\
& - 769368881424238421658457126135197696000x^{5} \\
& + 341598028525445919692822578014715904000x^{6} \\
& + 318571714344376464826804397584416768000x^{7} \\
& + 43384184437184804281854909281730560000x^{8} \\
& + 2247057035830221694284522638540800000x^{9} \\
& + 65963962941653345990868992000000000x^{10} \\
& + 788783853937293787136000000000000x^{11} \\
& + 4503599627370496000000000000000x^{12}
\end{aligned}
$$

$$
\begin{aligned}
Q_{12,3}(x) = & - 24211890636850099765625000000 \\
& + 590821775274689348749012500000x \\
& - 6715809987414915917683808854 0x^2 \\
& + 4854881758837532734052707100 3x^3 \\
& - 2143676900042240252875051884 8x^4 \\
& + 593841086095901206417943193 6x^5 \\
& - 132590415106590998059518771 2x^6 \\
& + 16490362744172960764546252 8x^7 \\
& - 2428787386227023114403840x^8 \\
& + 520081057543632348774400 0x^9 \\
& + 104060784476344549376000 0x^{10} \\
& + 56207693420953600000000x^{11} \\
& + 1073741824000000000000x^{12}
\end{aligned}
$$

$$
\begin{aligned}
Q_{12,4}(x) = & - 4479745163357398964484375000000000000 \\
& - 186864769454320373865799010625000000000x \\
& + 225192382371501516482842133741552479 20x^2 \\
& - 170437674907997678181733370416335979 71x^3 \\
& + 44933830117301296226042052457570992 64x^4 \\
& + 156477346046073716160243694178745548 8x^5 \\
& - 151799523661847221753696843425670758 4x^6 \\
& + 36106817040821373526169599672215142 4x^7 \\
& + 7187724785725739685794814686658560 0x^8 \\
& - 3500079718004869464115227551334400 0x^9 \\
& + 607469395667690921949992128000000 0x^{10} \\
& - 28944391153333462630400000000000x^{11} \\
& + 2251799813685248000000000000000x^{12}
\end{aligned}
$$

$$
\begin{aligned}
Q_{12,5}(x) = & \, 923643126562500000000000000000 \\
& - 60538952136363082623000000000000x \\
& - 1379505947701312305847898142033 6x^2 \\
& - 2453965031525954390671981172362 5x^3 \\
& + 238083513858480098725657246374 4x^4 \\
& + 5851413504632349434070447603916 8x^5 \\
& + 4655005511232816003203242822860 8x^6 \\
& - 3046411303055518012100058179174 4x^7 \\
& + 232611249783096528058764492800 0x^8 \\
& - 4426152935674048378631618560000 0x^9 \\
& + 880512890564496274227200000000 0x^{10} \\
& - 63703042444230656000000000000000x^{11} \\
& + 17179869184000000000000000000000x^{12}
\end{aligned}
$$

$$
\begin{aligned}
Q_{12,6}(x) = {} & 6250000000000000000000000\\
& + 15068760725475000000000000000x\\
& + 2106586116068039889928157750 40x^2\\
& + 677866128009890791245695953 7581x^3\\
& + 175521907641215976953779913997568x^4\\
& + 146633795017882331931298153993 8304x^5\\
& + 324553329039325491543121470239 5392x^6\\
& + 116850053291175440154535673158696 96x^7\\
& + 168461958706606046734435957604352000x^8\\
& + 532001946593574154881192060518400000x^9\\
& + 505317458896918566991298560000000000x^{10}\\
& + 51297286844061424025600000000000000x^{11}\\
& + 2814749767106560000000000000000000x^{12}
\end{aligned}
$$

The polynomials $R_{18,j}$ $(j = 0, \ldots, 6)$ of degree 18, utilized in Proposition 6:

$$
\begin{aligned}
R_{18,0}(x) = {} & 37421127923513015199026263635210173814759355000000 0\\
& + 1208946687260198804130148065800391637398690160000000x\\
& - 384390519498798639501099051016384977405477262149 8635x^2\\
& - 913165100253652212330371720522052961957439556961 4848x^3\\
& + 187910090494236191794366598353140011415036416654 33600x^4\\
& + 179036220280063949434145631049905282985474999771 46368x^5\\
& - 365235900003593682136425507841757395537557067986 37056x^6\\
& - 184877027780447730173330570741389652757702709535 04768x^7\\
& + 406674184749342389632685107983333160408160995775 61088x^8\\
& + 943389864584361901323511807256323126730072729950 6176x^9\\
& - 297246812290399647612550680253035125358539179619 77856x^{10}\\
& + 370374578262570004839592559191553634580123920577 3312x^{11}\\
& + 959416050363301243024289463753065652330872751679 0784x^{12}\\
& - 459656795540139673774918075740850465571360276375 1424x^{13}\\
& + 663309661581789490107409518099965621351912585560 064x^{14}\\
& - 420552546320259235439315303502722232422509880279 04x^{15}\\
& + 130207310849666406387134036398727432305744281600 0x^{16}\\
& - 191330148722493293543488792190436180492288000 00x^{17}\\
& + 1065270181901792577547754538637721600000000000x^{18}
\end{aligned}
$$

$$
\begin{aligned}
R_{18,1}(x) = & -1387721131302829154510845254858424239960427504\!5\\
&- 2114931517105784226152213281200173023003201206x\\
&+ 6861567753003158547123928590779461604616453929\!1x^2\\
&- 142637455507171354906843466169461183961549088128x^3\\
&- 70002847090838257792679362158842851955810110464x^4\\
&+ 364129186605362554833225972695346774959235170304x^5\\
&- 585261193006338216370710167928469995070873796608x^6\\
&+ 250016193125507468012141048302312307008282296320x^7\\
&- 35358815160560823889445906600113829673709862912x^8\\
&- 124066988890405937338691511082026508690045009920x^9\\
&+ 21374673536795225281522596088753851259485683712x^{10}\\
&+ 9858351440775228238449816532442125326985723904x^{11}\\
&- 22890889244003566948788470026302023423408209 92x^{12}\\
&- 242467500783452684111259876193655176076197888x^{13}\\
&+ 49202324151903720081568946390596211355680768x^{14}\\
&+ 6257663527320062890164060849609033975332864x^{15}\\
&+ 142857449654771321267572655391701467136000x^{16}\\
&+ 1135291757913750810712043370106060800000x^{17}\\
&+ 40636832500526145078573400064000000000000x^{18}
\end{aligned}
$$

$$
\begin{aligned}
R_{18,2}(x) = & \ 72784323858813787410859921576663154582879023705500000000\\
&+ 2917315488818713340579833933234609030480934641800000000 0x\\
&+ 349119186963722179265424753186759910401970549890734764 65x^2\\
&- 400628322410740285241927513370026063117619394883890790 4x^3\\
&- 348642376066009835453555644303852751072270904244785192 96x^4\\
&- 148395651383240068759619595185087920892767105456039526 40x^5\\
&+ 915366933159482277830679298255561257655661717854198169 6x^6\\
&+ 617654432481202642512825509014524361714805073259580620 8x^7\\
&- 651774390953972290740514771906925041088191427273293824x^8\\
&- 631668577779859072125107514035164366518817010487394304x^9\\
&+ 66306056064455261685665274786126371505520699986935808x^{10}\\
&+ 18805999219427251760297976254484110618899799529750528x^{11}\\
&- 2028811035021755162393686387643059879579108283252736x^{12}\\
&- 236931603037870946794042761855532259073584567156736x^{13}\\
&+ 17852845278096792821385161195285448722839999873024x^{14}\\
&+ 1439208967540208885346041663079224644021774188544x^{15}\\
&+ 32367866297734314274783027277998696509630054400x^{16}\\
&+ 282453731781792242313777687646909537189888000x^{17}\\
&+ 852216145521434062038203630910177280000000x^{18}
\end{aligned}
$$

$$
\begin{aligned}
R_{18,3}(x) = & - 196146162968720464664576000000000000000000 \\
& + 835932436806084264208850080112000000000000000x \\
& - 1548411798571866064410058342484906703972855x^{2} \\
& + 1087886545626205239465477547986883175028416x^{3} \\
& + 43885458300525617611240315392261695559449 6x^{4} \\
& - 1470543781549689773040006809746045853118464x^{5} \\
& + 854708627752910380202189337605376590962688x^{6} \\
& + 658056845744400144134721497429483530485 76x^{7} \\
& - 401281123174423891741821780600678798852096x^{8} \\
& + 148448690947691116599114132562594641215488x^{9} \\
& - 731418513153846580370067679507073151 7952x^{10} \\
& - 243382274964353816569866983514859739 5456x^{11} \\
& + 407829250460124723475096330160664739 84x^{12} \\
& + 204004565810407242634060786082523381 76x^{13} \\
& - 235590847182898726094962529253982208x^{14} \\
& - 39089352211475381654103774583586816x^{15} \\
& + 5524880431431752457722318028800 00x^{16} \\
& - 82371602978128161799898398720 00x^{17} \\
& + 708650120335626134880256000000x^{18}
\end{aligned}
$$

$$
\begin{aligned}
R_{18,4}(x) = & - 854416871939511154904293048654080000000000000000000 \\
& - 1035387617970252420569467195146686732544000000000000000x \\
& - 614560421720704102555534816864562854283616298376881 5x^{2} \\
& - 2292023280317554704022721685761995684943293634135654 4x^{3} \\
& - 6472926788144890074061601158364885995610815904837222 4x^{4} \\
& - 1327460151061801650232392382518649408615929285197168 64x^{5} \\
& - 2485766538916582907782665019490942992401938763502059 52x^{6} \\
& - 3103333320349464976743155122607711739656798047609815 04x^{7} \\
& - 2107835874510268676684061212140297457783401620239810 56x^{8} \\
& - 748258797287498601653646871147550238224850135568875 52x^{9} \\
& - 1399205054270887172150417209080345594088265299997491 2x^{10} \\
& - 1183272092761666795542449986562022754997227241865216x^{11} \\
& + 550005515309705426512810589322055481984526466416 64x^{12} \\
& + 217130604139987517220218773874774125149935957442 56x^{13} \\
& - 293013949858912293664755301733741253683031572 48x^{14} \\
& - 197913900523626149864473316617238493598461198336x^{15} \\
& + 1092414393969218110671994670021684140168970240 0x^{16} \\
& - 16724040963546088664220182583365925129420800 0x^{17} \\
& + 852216145521434062038203630910177280000000x^{18}
\end{aligned}
$$

$$
\begin{aligned}
R_{18,5}(x) = {} & 17395813858131648000000000000000000000 \\
& - 1188837597980191038363157151520000000000000x \\
& + 2232179849443880115484899404068394796513 87x^{2} \\
& - 10526230463483087357995172363074624724279 04x^{3} \\
& + 19680147193883415392813138556968402969637888x^{4} \\
& + 15562946039077035961584183292982899781468 16x^{5} \\
& - 2661728114832387070254402219162536542474207232x^{6} \\
& - 692931399865743819939426116508476007518306304x^{7} \\
& + 12450240381984556412152768707866015375871180800 0x^{8} \\
& - 270664557911312936733877314255640068259611410432x^{9} \\
& + 175140368563567911169366704815172177352441462784x^{10} \\
& - 4413580166380572438790471161352159643041792000 0x^{11} \\
& + 36477581852458170327142518147324359951421276 16x^{12} \\
& + 91787283078759700066330865022589847753195520x^{13} \\
& - 22574613727719174860321245222785959529021440x^{14} \\
& + 2302242145674265066782685100652613924290 56x^{15} \\
& - 782599585494520214989954653999857664000x^{16} \\
& - 37264044068306756146018816054263808000 00x^{17} \\
& + 406368325005261450785734000640000000000x^{18}
\end{aligned}
$$

$$
\begin{aligned}
R_{18,6}(x) = {} & 358400000000000000000000000000 \\
& + 644779254034265600000000000000000000x \\
& + 39657740843854205719597705127973384 5x^{2} \\
& + 8782303711687340126832873553337804390 4x^{3} \\
& + 69346665336716697117465051049185409392 64x^{4} \\
& + 61892653825901675071887315630609325935 8208x^{5} \\
& + 886586192230126891174149293624385552273 89952x^{6} \\
& - 837810667318134286171629236107339434330 947584x^{7} \\
& + 341953356467185603832338310494231897306281541632x^{8} \\
& + 1157399942746759700554035731876857138354751694438 4x^{9} \\
& + 401940017749676510088362718325830455705273054003 20x^{10} \\
& + 341793611275365435832784811138520750652498714296 32x^{11} \\
& - 940286412798768495120251608038595918188798961254 4x^{12} \\
& - 540551448157519520743285645232504874656380380774 4x^{13} \\
& - 494372206073054551072746041614465679422620958 72x^{14} \\
& + 52670135321918390141590888436044006384149620326 4x^{15} \\
& + 861311374941237008993087463141980699266908160 00x^{16} \\
& + 47313496039579098954018964923873323122688000 00x^{17} \\
& + 106527018190179257754775453863772160000000000x^{18}
\end{aligned}
$$

The polynomials $S_{14,j}$ $(j = 0, \ldots, 6)$ of degree 14, utilized in Proposition 9:

$$\begin{aligned}
S_{14,0}(x) = {} & 4645518131862241933150220520577833380000000 \\
& - 52636183590440871268641329288252732270000000x \\
& - 3642796800928790180199632798236729946182764 60x^2 \\
& + 1801242177223678066238975653453642108984575 19x^3 \\
& + 8036389322455501635227676995604942859308569 60x^4 \\
& - 3407015301166466279528743148752558251568496 64x^5 \\
& - 6766536966108389203520187182703197507446374 40x^6 \\
& + 3613807849236938535347908680731100155635302 40x^7 \\
& + 1582888406172205223027309180271198362350387 20x^8 \\
& - 1489133464295722841420651562409528827922350 08x^9 \\
& + 35699493083114384415333479038288313660211200x^{10} \\
& - 2959552977670964814569803412436162019590144x^{11} \\
& + 315326968115003811365878686463914147840 00x^{12} \\
& - 658385327413470634751054013530112000 00x^{13} \\
& + 387293081176346778088243200000000000x^{14}
\end{aligned}$$

$$\begin{aligned}
S_{14,1}(x) = {} & - 857866456847674918273534106865638920150 05 \\
& + 5521528403367427279927070416862951984558 91x \\
& - 15182709808663332276245366898134018942947 35x^2 \\
& + 12002582924115404611262772833967766502029 05x^3 \\
& - 4242526222375574685320674984171333100086016x^4 \\
& - 2128573719509660950632736830220191303475 20x^5 \\
& + 2371778845565469947292202718972744147927 04x^6 \\
& - 821914093907536631805480086038326710108 16x^7 \\
& - 1274334702686834356670306207796465500160 0x^8 \\
& + 892793294928046432049751803076279061708 8x^9 \\
& - 275767572462673385187272371729007116288 0x^{10} \\
& + 376850751065624164135425891340841385984x^{11} \\
& + 29674360646192485569108403436912640 00x^{12} \\
& + 494804340516862970674513182720 0000x^{13} \\
& + 236384937241422594048000000000 0x^{14}
\end{aligned}$$

$$
\begin{aligned}
S_{14,2}(x) = {} & - 2890020866700088058304675568974783179400000000 \\
& - 8107072827850385785033884753989608346670000000x \\
& - 5791926185578022678172805747725738191755978020x^2 \\
& + 951531334157636461440110063301976252580455719x^3 \\
& + 1504656580805854707587942122159077392012213760x^4 \\
& + 508267586500110903201376252459844513676165 12x^5 \\
& - 1411578991241748665659928453109831178282598 40x^6 \\
& - 81763448543632196321243052530690565341184 00x^7 \\
& + 55560780725796979349288866287923254158950 40x^8 \\
& + 2819746074245096640372457067970872223989 76x^9 \\
& - 88970854620485359722634617948720185999360x^{10} \\
& - 3540581248596039193443866781372408397824x^{11} \\
& + 48159592796498925520328498409017180160 0x^{12} \\
& + 7737531062714744645228024058347520 00x^{13} \\
& + 30983446494107742247059456000000 0x^{14}
\end{aligned}
$$

$$
\begin{aligned}
S_{14,3}(x) = {} & - 332270369254627738609792000000000000000 \\
& + 1261118465473858252417421761696000000 0x \\
& - 194102361505890474591906830435883856 20x^2 \\
& + 168106589074546986487730409249618830 79x^3 \\
& - 89534471761863378662493688630273505 28x^4 \\
& + 18377854493616197653010218314563266 56x^5 \\
& + 1537972029876910729790751894510305 28x^6 \\
& - 451551508357159115873649589007155 20x^7 \\
& + 557363267084351691460337878853222 4x^8 \\
& - 16748500092946733109139537563484 16x^9 \\
& - 10895851733382865104173380560486 40x^{10} \\
& - 4342865474934062378017825908326 4x^{11} + 39013302794640820964708843520 0x^{12} \\
& - 3684701331725505339064320 00x^{13} + 9233786610993070080000 0x^{14}
\end{aligned}
$$

$$
\begin{aligned}
S_{14,4}(x) = {} & - 74162591550000000000000000000000000000000 \\
& - 5564709971350473820853250000000000000000000x \\
& - 14556758174784231129271196996256762468750000x^2 \\
& - 10388403613456883747498644642483271687109375x^3 \\
& - 11703450050993737928005331390824827960000000x^4 \\
& - 7908715893090945948158634082250556672000000x^5 \\
& - 1515309190645844769958494551965335552000000x^6 \\
& + 324941032945670620392259212487635763200 00x^7 \\
& - 11836573287465825834786174999545354649600 0x^8 \\
& - 5401670953287050324426915260176059596800 0x^9 \\
& - 16548707282464840591523612790982128435 20x^{10} \\
& + 8560228279695705451451842586490552975 36x^{11} \\
& - 3171714399892970015140936072927641600x^{12} \\
& - 56300556420215168339377548951552000x^{13} \\
& + 3098344649410774224705945600000000x^{14}
\end{aligned}
$$

$$
\begin{aligned}
S_{14,5}(x) = {} & 6903810902108160000000000000000000000 \\
& - 163525289285569953221711539200000000000x \\
& + 1841120010425531216671161409137877592976x^2 \\
& - 3122059864708808192178626527336380173613x^3 \\
& + 19419272913930339929456819182722800281344x^4 \\
& - 31081372751700012119691858683542470586368x^5 \\
& - 12398066182553714521742532511884451774464x^6 \\
& + 16608327046356091843417682711118499282944x^7 \\
& + 25212718044369653941994700922760969846784x^8 \\
& - 13673304254663428889586445393598494015488x^9 \\
& + 2245251052299390396568110380758234824704x^{10} \\
& - 1266067108428921542022069750084966943104x^{11} \\
& + 1967971339177920251836530954338304000x^{12} \\
& + 4451698112834864798092807372800000x^{13} \\
& + 2363849372414225940480000000000x^{14}
\end{aligned}
$$

$$
\begin{aligned}
S_{14,6}(x) = {} & 3276800000000000000000000000000 \\
& + 202435709495552000000000000000000x \\
& + 78368555558113415318706786901200960x^2 \\
& + 2646453688844485315429827964540380809x^3 \\
& + 152659965155082101218099272818480053760x^4 \\
& + 3982140239506167613485666515147065049088x^5 \\
& + 31236434546629690920205352532225629880320x^6 \\
& + 19779379368189663746146082140250716504064x^7 \\
& + 153722275605319209732374545551582497341440x^8 \\
& + 277217401260646401496924991417964736544768x^9 \\
& + 179212005343633666208826173870123484446720x^{10} \\
& + 44761940737683842686686142335757880131584x^{11} \\
& + 3532403907920740209280107036083748864000x^{12} \\
& - 23843035990060348547822705757388800000x^{13} \\
& + 38729308117634677808824320000000000x^{14}
\end{aligned}
$$

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