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Polynomial Inequalities and Green's Functions

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The paper discusses some classical polynomial inequalities, their recent extensions to general sets, as well as the potential theory behind them. A unifying feature will be that in many cases the best constant is given in terms of the normal derivative of certain Green's functions.

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1. Classical Inequalities

More than 100 years ago S. N. Bernstein proved in [7] his famous inequality[1] for the derivatives of trigonometric polynomials

$$T_n(t) = \frac{a_0}{2} + (a_1 \cos t + b_1 \sin t) + \cdots + (a_n \cos nt + b_n \sin nt)$$

of degree at most n, namely for all θ

$$|T_n'(\theta)| \le n \sup_t |T_n(t)|. \tag{1}$$

This becomes an equality for example for $T_n(t) = \sin nt$ and $\theta = 0$. It will be convenient to rewrite it in the form

$$\|T_n'\| \le n\|T_n\|,$$

where $\|T_n\| = \sup_t |T_n(t)|$ is the supremum norm over the whole real line. In general, the supremum norm on a set E is defined as

$$\|f\|_E = \sup_{t\in E} |f(t)|.$$

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[1] Actually, the original [7] contained $2n$ instead of n in (1), see Section 6.

If

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

is an algebraic polynomial of degree at most $n = 1, 2, \ldots$[2], then $P_n(\cos t)$ is a trigonometric polynomial of degree at most n, and for it (1) takes the form

$$|P_n'(x)| \leq \frac{n}{\sqrt{1-x^2}}\|P_n\|_{[-1,1]}, \qquad x \in (-1,1), \tag{2}$$

which is again called "Bernstein's inequality". In it the right-hand side blows up as $x \to \pm 1$, and for x close to ± 1 a better estimate is due to A. A. Markov [18] from 1890:

$$\|P_n'\|_{[-1,1]} \leq n^2 \|P_n\|_{[-1,1]}. \tag{3}$$

2. Where Are These Polynomial Inequalities Used?

The polynomial inequalities from the previous section have various applications. For example, one of the major tasks of approximation theory is the characterization of the rate of approximation, say of $E_n(f) = \inf_{T_n} \|f - T_n\|$, where f is a 2π-periodic continuous function and the infimum is taken for all trigonometric polynomials of degree at most n. From practical point of view one needs a computable quantity for the characterization, and such a quantity is smoothness, for example the Lipshitz α class (Lip α) is defined by the property: $|f(x) - f(y)| \leq M|x-y|^\alpha$ for all x, y with some M. A typical result is the following: for $0 < \alpha < 1$ the equivalence

$$f \in \mathrm{Lip}\,\alpha \quad \Longleftrightarrow \quad E_n(f) \leq \frac{C}{n^\alpha} \quad \text{for some } C$$

is true. The direction $\Rightarrow$ can be proven by a direct construction like taking convolution with the Fejér kernel, but in the direction $\Leftarrow$ one needs Bernstein's inequality. Indeed, suppose there are trigonometric polynomials T_n of degree at most n such that $\|f - T_n\| \leq Cn^{-\alpha}$. Then, by Bernstein's inequality,

$$\begin{aligned}|(T_{2^{k+1}} - T_{2^k})'| &\leq 2^{k+1}\|T_{2^{k+1}} - T_{2^k}\| \\ &\leq 2^{k+1}(\|f - T_{2^{k+1}}\| + \|f - T_{2^k}\|) \leq 4C2^{k-\alpha}.\end{aligned} \tag{4}$$

Now for $2^{-n} \leq h \leq 2^{-n+1}$ we obtain, by the mean value theorem, with some ξ in between x and $x+h$

$$\begin{aligned}|f(x) - f(x+h)| &\leq |f(x) - T_{2^n}(x)| + |f(x+h) - T_{2^n}(x+h)| \\ &\quad + |T_{2^n}(x) - T_{2^n}(x+h)| \\ &\leq 2C(2^n)^{-\alpha} + h|T_{2^n}'(\xi)|,\end{aligned}$$

[2]In what follows the degree of P_n is always assumed to be at most n.

and, according to (4), here the last term is at most

$$h|T'_{2^n}(\xi)| = h\sum_{k=-1}^{n-1}|T'_{2^{k+1}}(\xi) - T'_{2^k}(\xi)| \le h\sum_{k=-1}^{n-1} 4C2^{k-\alpha} \le C_1 h 2^{n-\alpha} \le C_2 h^\alpha,$$

(since T_{-1} is constant). These show that $|f(x+h) - f(x)| \le C_3 h^\alpha$, which is the Lip α property.

In fact, a large part of approximation theory deals with direct (smoothness $\Rightarrow$ a given rate of approximation) and inverse (given rate of approximation $\Rightarrow$ smoothness) theorems, and the latter ones almost always involve a certain variant of the Bernstein or Markov inequalities.

In [14] Bernstein's inequality played a decisive role in settling a conjecture in number theory on the uniform distribution of the argument of so called ultraflat polynomials (polynomials $P_n(z) = \sum_k a_k z^k$ with $|a_k| = 1$, which satisfy $|P_n(z)| \approx \sqrt{n}$ on the unit circle). It is also often used to estimate the values of (trigonometric) polynomials at points in close proximity, see [6] or [14] for such a use in number theory. Some other recent applications are related to heat diffusion [15], universality in random matrix theory [17], Hardy spaces [5], numerical analysis [24], Hilbert spaces [25], dynamical systems [16], partial differential equations [13], Fourier transforms [28], to name a few.

In the last 100 years many extensions and generalizations of the aforementioned classical polynomial inequalities have been given. A particularly intensive period has been the last 20 years, during which very general forms have been found. To discuss them we need a few notions from potential theory. For a general reference to logarithmic potential theory see [27].

3. Equilibrium Measures and Green's Functions

Let $E \subset \mathbb{C}$ be a compact subset of the plane. Think of E as a conductor, and put a unit charge on E, which can freely move in E. After a while the charge settles, it reaches an equilibrium. The mathematical formulation is the following (on the plane Coulomb's law takes the form that the repelling force between charged particles is proportional with the reciprocal of the distance): except for pathological cases, there is a unique probability measure μ_E on E, called the equilibrium measure of E, that minimizes the energy integral

$$\iint \log\frac{1}{|z-t|}\, d\mu(z)d\mu(t)$$

among all unit (Borel) measures supported on E. This μ_E certainly exists in all the cases we are considering in this paper.

When $E \subset \mathbb{R}$, then we shall denote by $\omega_E(t)$ the density of μ_E with respect to Lebesgue measure wherever it exists. It certainly exists in the (one dimensional) interior of E. For example,

$$\omega_{[-1,1]}(t) = \frac{1}{\pi\sqrt{1-t^2}}, \qquad t \in [-1,1],$$

is just the well known Chebyshev distribution. More generally, if T_N is an algebraic polynomial of degree N for which the complete inverse image

$$E = T_N^{-1}[-1,1] = \{x : T_N(x) \in [-1,1]\}$$

is part of the real line, then

$$\omega_E(t) = \frac{|T_N'(t)|}{\pi N\sqrt{1-T_N(t)^2}}, \qquad t \in E.$$

In a similar fashion, if E consists of disjoint smooth Jordan curves and arcs with arc measure s_E, then $d\mu_E = \omega_E ds_E$, and this ω_E is called equilibrium density. For example, if E is a circle of radius r then $\omega_E(z) \equiv 1/2\pi r$ on E. As another example, consider a lemniscate

$$\sigma = \{z : |T_N(z)| = 1\},$$

where T_N is an algebraic polynomial of degree N. Except for the points where σ crosses itself, we have

$$\omega_\sigma(z) = |T_N'(z)|/2\pi N.$$

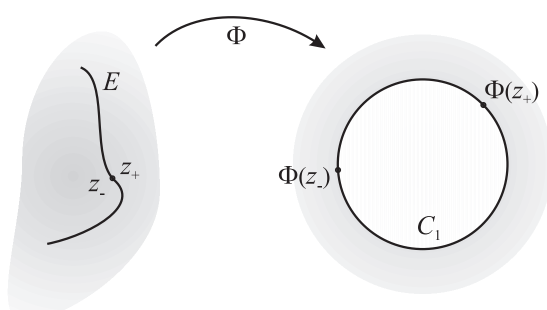

Figure 1. The conformal map from the exterior of E onto the exterior of the unit circle

For a further illustration, let E be a smooth (say C^2-smooth) Jordan curve (homeomorphic image of a circle) or arc (homeomorphic image of a segment), and Φ a conformal map from the exterior of E onto the exterior of the unit circle C_1 (that maps the point infinity to itself). This Φ can be extended to E as a continuously differentiable function (with the exception of the endpoints of E when E is a Jordan arc). Now if E is a Jordan curve, then simply $\omega_E(z) = |\Phi'(z)|/2\pi$. If E is a Jordan arc, then it has two sides, say positive and negative sides, and every point $z \in E$ different from the endpoints of E is considered to belong to both sides, where they represent different points $z_\pm$ (with different Φ-images), see Figure 1. In this case $\omega_E(z) = (|\Phi'(z_+)| + |\Phi'(z_-)|)/2\pi$. For example, if E is the arc of the unit circle that runs from $e^{-i\beta}$ to $e^{i\beta}$ counterclockwise, then

$$\omega_E(e^{it}) = \frac{1}{2\pi} \frac{\cos t/2}{\sqrt{\sin^2 \beta/2 - \sin^2 t/2}}, \qquad t \in (-\beta, \beta).$$

Let now $E \subset \mathbb{C}$ be a compact set. Under mild conditions (which always hold in the cases we are discussing) there is a unique function g_E on the unbounded component Ω of $\overline{\mathbb{C}} \setminus E$ such that

- $g_E \geq 0$ and g_E is harmonic on Ω,
- $g_E(z) \to 0$ as $z \to z_0 \in E$ (at least for "most" $z_0 \in E$),
- $g_E(z) \sim \log |z| + \text{const}$ as $z \to \infty$.

This g_E is called the Green's function of the (unbounded component of the) complement of E.

For example,

$$g_{[-1,1]}(z) = \log \left|z + \sqrt{z^2 - 1}\right|,$$

if C_R is the circle $|z| = R$, then

$$g_{C_R}(z) = \log |z| - \log R,$$

and more generally, if $E = \{z : |T_N(z)| = 1\}$, $\deg(T_N) = N$, is a lemniscate, then

$$g_E(z) = \frac{1}{N} \log |T_N(z)|.$$

If E is connected and Φ is a conformal map of the complement of E onto the exterior of the unit disk (leaving the point ∞ invariant), then

$$g_E(z) = \log |\Phi(z)|.$$

Finally, for arbitrary E

$$g_E(z) = \sup_{\|P_n\|_E \leq 1} \frac{1}{n} \log |P_n(z)|,$$

where the supremum is taken for all polynomials P_n of degree $n = 1, 2, \ldots$.

We shall mostly use normal derivatives of Green's functions. Let E consist of disjoint smooth arcs (e.g. let $E \subset \mathbb{R}$ consist of intervals). Orient E somehow. Then it has two sides, and let $\mathbf{n}_\pm = \mathbf{n}_\pm(z)$ denote the two normals at $z \in E$.

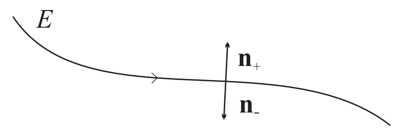

Figure 2. The two normal directions

The normal derivatives

$$g'_{\pm,E}(z) := g'_{\pm}(z) := \frac{\partial g_E(z)}{\partial \mathbf{n}_{\pm}}$$

exist if $z \in E$ is not an endpoint. In general, $g'_+(z) \neq g'_-(z)$, but if $E \subset \mathbb{R}$, then $g'_+ = g'_- =: g'_E$ by symmetry.

If E is a smooth Jordan arc, and Φ is a standard conformal map from $C \setminus E$ onto the exterior of the unit disk, then $z_\pm \in E$ are different points on the boundary of $\mathbb{C} \setminus E$, and $\Phi(z_\pm)$ are two different points on the unit circle, with which

$$g'_{\pm}(z) := |\Phi'(z_\pm)|. \tag{5}$$

In a similar vein, if E is a Jordan curve (homeomorphic to a circle) and Φ is a conformal map from the unbounded component of $\mathbb{C} \setminus E$ onto the exterior of the unit circle (leaving ∞ invariant), then

$$g'_+(z) = |\Phi'(z)|,$$

where n_+ is the outward normal.

If $E = \{z : |T_N(z)| = 1\}$, $\deg(T_N) = N$, is a lemniscate, then, as we have just mentioned, $g_E(z) = \frac{1}{N}\log|T_N(z)|$ and then

$$g'_+(z) = \frac{|T'_N(z)|}{N}, \tag{6}$$

where n_+ is the outward normal.

The normal derivatives are known if E consists of finitely many intervals on the real line:

$$E = \bigcup_{j=1}^{m} [a_{2j-1}, a_{2j}].$$

In this case

$$g'_E(x) = \frac{\prod_{j=1}^{m-1} |x - \xi_j|}{\sqrt{\prod_{j=1}^{2m} |x - a_j|}},$$

where the $\xi_j \in (a_{2j}, a_{2j+1})$, $j = 1, \ldots, m-1$, are the unique points with

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{j=1}^{m-1}(u - \xi_j)}{\sqrt{\prod_{j=1}^{2m} |u - a_j|}}\, du = 0, \qquad j = 1, \ldots, m-1.$$

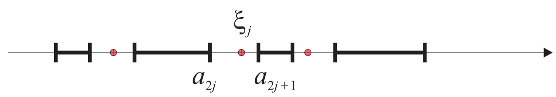

Figure 3. The position of the points ξ_j

Next, let us suppose that $z \in E$ lies on the outer boundary of E (i.e. it lies on the boundary of the unbounded component Ω of $\mathbb{C} \setminus E$), and that outer boundary is a C^2-smooth arc Γ in a neighborhood of z. In that case the equilibrium density ω_E (with respect to arc measure on Γ) of the equilibrium measure is given in terms of the normal derivatives of the Green's function:

$$\omega_E(z) = \frac{1}{2\pi}(g'_-(z) + g'_+(z)). \tag{7}$$

This formula should be understood in the sense that if one side of Γ does not belong to Ω (i.e. it belongs to E or to a bounded component of $\mathbb{C}\setminus E$), then the corresponding normal derivative is considered to be 0 (as the Green's function is considered to be 0 outside the unbounded component Ω of $\mathbb{C} \setminus E$). For example, if E is a Jordan curve (homeomorphic image of a circle), then

$$\omega_E(z) = \frac{1}{2\pi} g'_+(z),$$

where g'_+ is the normal derivative with respect to the outer normal to E. On the other hand, if E is a Jordan arc (homeomorphic image of a segment), then both derivatives appear in (7). In particular, if $E \subset \mathbb{R}$, then

$$\omega_E(z) = \frac{1}{\pi}\, g'_E(z) \tag{8}$$

because the two normal derivatives are the same.

4. The General Bernstein Inequality

The general form of Bernstein's inequality for sets on the real line were given in [4] and [32]: let $E \subset \mathbb{R}$ be a compact set. Then, for algebraic polynomials P_n of degree at most $n = 1, 2, \ldots$, we have

$$|P'_n(x)| \le n\pi\omega_E(x)\|P_n\|_E, \qquad x \in \mathrm{Int}(E). \tag{9}$$

This is sharp: if $x_0 \in \mathrm{Int}(E)$ is arbitrary, then there are polynomials P_n of degree at most $n = 1, 2, \ldots$ such that

$$|P'_n(x_0)| > (1 - o(1))n\pi\omega_E(x_0)\|P_n\|_E.$$

Using (8) the inequality (9) can be written in the alternative form:

$$|P_n'(x)| \le n g_E'(x) \|P_n\|_E, \qquad x \in \mathrm{Int}(E). \tag{10}$$

Note that in the special case $E = [-1, 1]$ this gives back the original Bernstein inequality (2) because $g_{[-1,1]}'(x) = 1/\sqrt{1-x^2}$.

Actually, for real polynomials more than (10) is true (see [36]):

$$\left(\frac{P_n'(x)}{g_E'(x)}\right)^2 + n^2 P_n(x)^2 \le n^2 \|P_n\|_E^2, \qquad x \in \mathrm{Int}(E),$$

which is the analogue of the beautiful inequality

$$\left(P_n'(x)\sqrt{1-x^2}\right)^2 + n^2 P_n(x)^2 \le n^2 \|P_n\|_{[-1,1]}^2, \qquad x \in (-1, 1),$$

of Szegő [31] and Schaake and van der Corput [30].

5. Markov's Inequality

The classical Markov inequality (3) complements Bernstein's inequality (2) when we have to estimate the derivative of a polynomial on $[-1, 1]$ close to the endpoints. What happens if we consider more than one intervals?

Let $E = \cup_{j=1}^m [a_{2j-1}, a_{2j}]$, $a_1 < a_2 < \cdots < a_{2m}$, consist of m real intervals. When we consider the analogue of the Markov inequality for E, actually we have to talk about one-one Markov inequality around every endpoint of E. Indeed, away from the endpoints (9) is true, therefore there the derivative can be only of order Cn, so an n^2 rate for the derivative can occur only close to the endpoints, and it is clear that different endpoints play different roles. Let a_j be an endpoint of E, and let E^j be the part of E that lies closer to a_j than to any other endpoint. Let M_j be the smallest constant for which

$$\|P_n'\|_{E^j} \le (1 + o(1)) M_j n^2 \|P_n\|_E \tag{11}$$

holds, where $o(1)$ tends to 0 (uniformly in the polynomials P_n) as n tends to infinity. This M_j depends on what endpoint a_j we are considering, and it is the asymptotically best constant in the corresponding local Markov inequality. Its value can be expressed in terms of the normal derivative g_E' of the Green's function g_E. Indeed, around a_j this normal derivative behaves like $\sim 1/\sqrt{|t - a_j|}$, and the limit

$$\Omega_j := \lim_{t \to a_j,\ t \in E} \sqrt{|t - a_j|}\, g_E'(t)$$

exists. With it the asymptotic Markov factor can be expressed (see [32]) as

$$M_j = 2\Omega_j^2, \qquad j = 1, \ldots, 2m.$$

As an example, consider $E = [-b,-a] \cup [a,b]$. In this case $m = 2$, $a_1 = -b$, $a_2 = -a$, $a_3 = a$, $a_4 = b$, and,

$$g_E'(t) = \frac{|t|}{\sqrt{(b^2-t^2)(t^2-a^2)}}.$$

Hence,

$$M_1 = M_4 = \frac{b}{b^2-a^2}, \qquad M_2 = M_3 = \frac{a}{b^2-a^2}.$$

Since $M_1 = M_4 > M_2 = M_3$, we obtain that

$$\|P_n'\|_{[-b,-a]\cup[a,b]} \le (1+o(1))n^2 \frac{b}{b^2-a^2} \|P_n\|_{[-b,-a]\cup[a,b]},$$

which is a result of Borwein from [11].

As an immediate consequence of the theorem we get the following asymptotically best possible global Markov inequality:

$$\|P_n'\|_E \le (1+o(1))n^2 \left(\max_{1\le j\le 2m} 2\Omega_j^2 \right) \|P_n\|_E.$$

Here the $o(1)$ tends to 0 uniformly in the polynomials P_n as $n \to \infty$, and this term cannot be dropped. It seems to be a difficult problem to find, on several intervals, for each n the best Markov constant for polynomials of degree at most n.

6. M. Riesz and Hilbert's Lemniscate Theorem

Let $C_1 = \{z : |z| = 1\}$ be the unit circle and P_n an algebraic polynomial of degree at most n. Then $P_n(e^{it})$ is a trigonometric polynomial of degree at most n, so by Bernstein's inequality (1) we have

$$\left| \frac{dP_n(e^{it})}{dt} \right| \le n \max |P_n|.$$

The left hand side is $|P_n'(e^{it}) i e^{it}| = |P_n'(e^{it})|$, so the previous inequality can be rewritten as

$$|P_n'(z)| \le n\|P_n\|_{C_1}, \qquad z \in C_1. \tag{12}$$

This inequality is due to M. Riesz, and was proved in the paper [29] which contained the first proof of Bernstein's inequality (2) (Bernstein had $2n$ instead of n in (2)).

Riesz' inequality has been extended to Jordan curves and families of Jordan curves in [23]: if E is a finite union of disjoint C^2 Jordan curves (homeomorphic images of circles), then for polynomials P_n of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(z)| \le (1+o(1))n g_+'(z)\|P_n\|_E, \qquad z \in E, \tag{13}$$

where $g'_+(z)$ is the normal derivative of the Green's function g_E taken with respect to the outer normal. Here $o(1)$, which tends to 0 uniformly in P_n as $n \to \infty$, cannot be dropped. Furthermore, (13) is best possible: if $z_0 \in E$, then there are polynomials $P_n \not\equiv 0$ of degree at most $n = 1, 2, \ldots$ for which

$$|P'_n(z_0)| \geq (1 - o(1))ng'_+(z_0)\|P_n\|_E.$$

For the unit circle we have $g'_+(z) = 1$, so, modulo the factor $(1 + o(1))$, (13) gives back the original inequality (12) of M. Riesz (which, in general, cannot be dropped in the Jordan curve case).

So far we have not said anything about how to prove the general versions of the classical polynomial inequalities, so let us indicate the proof for (13). The key is to consider lemniscates, i.e. level sets of polynomials. A typical lemniscate is of the form $\sigma = T_N^{-1}C_1 = \{z : |T_N(z)| = 1\}$, where T_N is a polynomial of some degree N. We have already mentioned (see (6)) that $g'_+(z) = |T'_N(z)|/N$, so if $E = \sigma$ and the P_n in (13) is of the special form $P_n(z) = R_m(T_N(z))$ with some polynomial R_m, then $n = mN$, and from Riesz' inequality applied to R_m we have for $z \in \sigma$

$$\begin{aligned} |P'_n(z)| = |R'_m(T_N(z))T'_n(z)| &= |R'_m(T_N(z))|Ng'_{+,\sigma}(z) \\ &\leq m\|R_m\|_{C_1}Ng'_{+,\sigma}(z) = ng'_{+,E}(z)\|P_n\|_E, \end{aligned}$$

which is (13) without the factor $(1 + o(1))$.

Note that even though this is only a very special case of (13) (the set is a lemniscate and the polynomial is of the special form $R_m(T_N)$), a crucial thing has happened in this simple step: although we have started from the unit circle, we got a result on a set that may consist of several components.

The next step is to get rid of the special form $R_m(T_N)$ of P_n to get the full (13), but still on a lemniscate $E = \sigma$. This is quite subtle, see [23] or [34] how to do that. The final step is to approximate a union of smooth Jordan curves by lemniscates, which is done by a sharp form of Hilbert's lemniscate theorem. Hilbert's lemniscate theorem claims that if K is a compact set on the plane with connected complement and U is a neighborhood of K, then there is a lemniscate σ that separates K and $\mathbb{C} \setminus U$, i.e. it lies within U but encloses K. We can reformulate this as follows. Let γ_j, Γ_j, $j = 1, \ldots, m$, be Jordan curves (i.e. homeomorphic images of the unit circle), γ_j lying interior to Γ_j and the Γ_j's lying exterior to one another, and set $\gamma^* = \cup_j \gamma_j$, $\Gamma^* = \cup_j \Gamma_j$. Then there is a lemniscate σ that is contained in the interior of Γ^* which also contains γ^* in its interior, i.e. σ separates γ^* and Γ^* in the sense that it separates each γ_j from the corresponding Γ_j. This is not enough for our approximation, what we need is the following sharpened form (see [23]). Let γ^* and Γ^* be twice continuously differentiable in a neighborhood of a point P where now we assume that they touch each other. We also assume that their curvature at P are different. Then there is a lemniscate σ that separates γ^* and Γ^* and touches both γ^* and Γ^* at P, see Figure 4. Furthermore, σ lies strictly in between γ^* and Γ^* except for the point P, it has precisely one connected component in between each γ_j and

Γ_j, $j = 1, \ldots, m$, and these m components are Jordan curves. (Actually, the same statement is true if γ^* and Γ^* touch each other in finitely many points.)

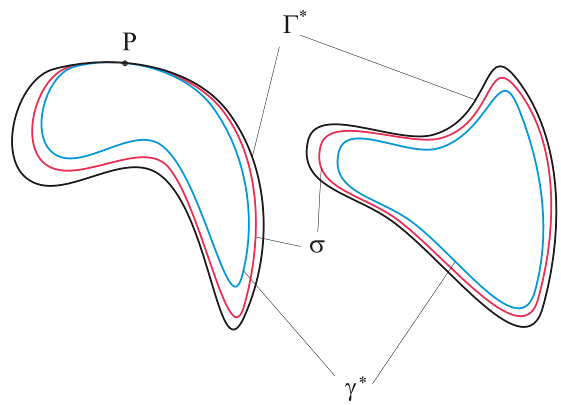

Figure 4. The lemniscate σ separating γ^* and Γ^*

Now having settled (13) for lemniscates, the final step in proving (13) for a system of (C^2) Jordan curves E at a specific point $z_0 \in E$ is to use the sharp form of the Hilbert lemniscate theorem to $\Gamma^* = E$, and to a γ^* which touches E as in the theorem, and which otherwise lies very close to E. This γ^* can be chosen so close to E that at z_0 we have $\omega_{\gamma^*}(z_0) \le (1+\varepsilon)\omega_E(z_0)$, provided $\varepsilon > 0$ is given. Then automatically for the σ lying in between γ^* and E we have

$$g'_{+,\sigma}(z_0) \le g'_{+,\gamma^*}(z_0) \le (1+\varepsilon) g'_{+,E}(z_0),$$

and by the σ-version of (13) we obtain

$$|P_n'(z_0)| \le (1+o(1)) n g'_{+,\sigma}(z_0) \|P_n\|_\sigma \le (1+o(1)) n (1+\varepsilon) g'_{+,E}(z_0) \|P_n\|_E,$$

because $\|P_n\|_\sigma \le \|P_n\|_E$ by the maximum principle. Since $\varepsilon > 0$ is arbitrary, (13) follows.

The general Bernstein and Markov inequalities ((9) and (11)) can be proven along similar lines using "real lemniscates", i.e. polynomial inverse images $T_N^{-1}[-1,1]$, see [32, 34].

7. Jordan Arcs

So far, in the general inequalities we have considered, always one of the normal derivatives of the Green's function gave the (asymptotically) best Bernstein-factors, and the Markov-factors have also been expressed in terms of them. In some sense this was accidental, it is due to either a symmetry (when $E \subset \mathbb{R}$)

or to an absolute lack of symmetry (when E was a Jordan curve for which the two sides of E, the exterior and interior of E, play absolutely different roles). The case of Jordan arcs (which have not been considered so far), show the true nature of these inequalities.

So let E be a Jordan arc, i.e. a homeomorphic image of a segment. We assume C^{2+} (a little more than C^2) smoothness of E. As has already been discussed in Section 2, E has two sides, and every point $z \in E$ different from the endpoints of E gives rise to two different points $z_\pm$ on the two sides and formula (5) holds.

Now the Bernstein inequality on E takes the form (for $z \in E$ being different from the two endpoints of E)

$$|P_n'(z)| \le (1+o(1))n \max\left(g_-'(z), g_+'(z)\right)\|P_n\|_E, \tag{14}$$

where $o(1)$ tends to 0 uniformly in P_n as $n \to \infty$, see [22]. This is best possible, one cannot write anything smaller than $\max(g_-'(z), g_+'(z))$ on the right. The first result in this direction was in the paper [21] by Nagy and Kalmykov which contains (14) for analytic arcs.

To appreciate the strength of (14) (or that of (13)) let us mention that the smooth Jordan arc (curve) in it can be arbitrary, and a general smooth Jordan arc (curve) can be pretty complicated, see for example, Figure 5.

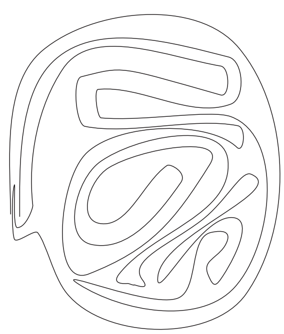

Figure 5. A "wild" Jordan arc

As for Markov's inequality, let now w be one of the endpoints of E, and let $\tilde{E}$ be the part of E that is closer to z than to the other endpoint of E. As $z \to w$, the common limits

$$\Omega_w = \lim_{z\to w,\ z\in E} \sqrt{|z-w|}\, g_-'(z) = \lim_{z\to w,\ z\in E} \sqrt{|z-w|}\, g_+'(z)$$

exist, and with it we have the Markov-inequality around w (see [22]):

$$\|P_n'\|_{\tilde{E}} \le (1+o(1))n^2 2\Omega_w^2 \|P_n\|_E, \tag{15}$$

and this is best possible in the sense that one cannot write a smaller number than $2\Omega_w^2$ on the right.

8. Higher Derivatives

The Markov inequality (3) can be iterated to get for any $k = 1, 2, \dots$ for the k-th derivative

$$\|P_n^{(k)}\|_{[-1,1]} \le n^2(n-1)^2 \cdots (n-k+1)^2 \|P_n\|_{[-1,1]}.$$

However, this is not sharp, the correct bound was proven as early as 1892 by V. A. Markov [19], the brother of A. A. Markov:

$$\|P_n^{(k)}\|_{[-1,1]} \le \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)}\|P_n\|_{[-1,1]}. \tag{16}$$

This turns into an equality for $P_n(x) = \cos(n \arccos x)$, i.e. the Chebyshev polynomials. The corresponding inequality for several intervals or for a Jordan arc is not known and it is pretty hopeless, but we do have the asymptotically sharp Markov-inequality which involves the factor $1/1\cdot 3\cdots(2k-1) = 1/(2k-1)!!$ from (16). For example, (11) for higher derivatives takes the form

$$\|P_n^{(k)}\|_{E^j} \le (1+o(1))n^{2k}\frac{(2\Omega_j^2)^k}{(2k-1)!!}\,\|P_n\|_E$$

while (15) has the extension

$$\|P_n^{(k)}\|_{\tilde{E}} \le (1+o(1))n^{2k}\frac{(2\Omega_w^2)^k}{(2k-1)!!}\,\|P_n\|_E,$$

and both of these are best possible (no smaller number can be written on the right).

9. Almost Everywhere Results

Let $E \subset \mathbb{R}$ be an arbitrary arbitrary compact set of positive linear measure. As early as 1916 Privalov proved [26] that for every $\varepsilon > 0$ there is a C_ε such that

$$|P_n'(x)| \le C_\varepsilon n \|P_n\|_E$$

for all $x \in E$ with the exception of a set of measure $< \varepsilon$. A sharper form is contained in [35, Corollary 2.3]: define for $x \in E$,

$$\widehat{\omega}_E(x) = \lim_{\delta \to 0} \omega_{E_\delta}(x),$$

where E_δ is the δ-neighborhood of E (the limit exists, for $\omega_{E_\delta}(x)$ decreases as δ increases). By Fatou's lemma

$$\int \widehat{\omega}_{E_\delta}(x)\,dx \le 1.$$

Now if $x \in E$ is any point (not just interior point), then for any algebraic polynomial P_n of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(x)| \le n\pi\widehat{\omega}_E(x)\|P_n\|_E.$$

Conversely, if $\gamma < \pi\widehat{\omega}_E(x)$, then there are algebraic polynomials $P_n \not\equiv 0$ of arbitrarily large degree n such that

$$|P_n'(x)| \ge n\gamma\|P_n\|_E.$$

These show that in Privalov's theorem one can choose, for example, $C_\varepsilon = \pi/\varepsilon$.

10. Local Results

Let again $E \subset \mathbb{R}$. We shall now address the problem when

$$|P_n'(x_0)| \le Cn\|P_n\|_E$$

is true at a given point x_0 with some constant C independent of P_n and n. Without loss of generality we may choose $x_0 = 0$, so the question is what structural properties of E guarantee

$$|P_n'(0)| \le Cn\|P_n\|_E. \tag{17}$$

Andrievskii proved in [3] that (17) is true if and only if

$$g_E(z) \le C|z|, \qquad z \in \mathbb{C}, \tag{18}$$

i.e. if and only if the Green's function g_E is Lip 1 at the point 0. Furthermore, if (18) is true, then the normal derivative

$$g_E'(0) = \lim_{t\to 0} \frac{g_E(it)}{t}$$

exists, and it is the asymptotically best C in (17).

It is quite remarkable, that (17) is equivalent to a similar inequality for higher derivatives (see [37]): If $k \ge 2$ fixed, then (17) is true if and only if

$$|P_n^{(k)}(0)| \le Cn^k\|P_n\|_E \tag{19}$$

holds (naturally, with a possibly different constant C than in (17)). Note that neither direction of the equivalence (17) $\Leftrightarrow$ (19) is trivial, even not $\Rightarrow$, for (17) cannot be iterated since the local Bernstein inequality in it is known only at the single point 0.

Another somewhat surprising fact is that for (17) to hold the set does not need to be thick in measure-theoretical sense, namely there is an E of Lebesgue-measure 0 for which (17) is true. (20)

A proof of this fact follows from the equivalence of (17) and (18), from [33, Corollary 5.2] and from [12, Corollary 1.12].

However, the set E must be thick at 0 in potential-theoretical sense as is shown by the following characterization of (18). For that we need the notion of logarithmic capacity. Recall that

$$g_E(z) \sim \log|z| + \text{const} \qquad \text{as } z \to \infty,$$

and in fact the limit

$$\lim_{z\to\infty} (\log|z| - g_E(z))$$

exists, and we set

$$\log \text{cap}(E) := \lim_{z\to\infty} (\log|z| - g_E(z)).$$

For example, the capacity of a line segment of length ℓ is $\ell/4$, and the capacity of a circle/disk of radius r is r. If E is connected and Φ is a conformal map of the complement onto the exterior of the unit disk, then around ∞

$$\Phi(z) = \frac{z}{\text{cap}(E)} + c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots.$$

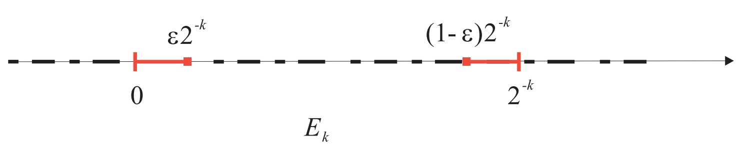

Figure 6. The depiction of the sets E_k

With some fixed $0 < \varepsilon < 1/3$ define now $I_k = [0, 2^{-k}]$,

$$E_k = (E \cap I_k) \cup [0, \varepsilon 2^{-k}] \cup [(1-\varepsilon)2^{-k}, 2^{-k}],$$

and similarly define I_k and E_k for negative k by using $|k|$ in the just given formulae.

Define also the capacitary defect

$$\theta_k = \big(\text{cap}(I_k) - \text{cap}(E_k)\big)/\text{cap}(I_k) = 2^{|k|+2}\big(\text{cap}(I_k) - \text{cap}(E_k)\big).$$

With this the characterization of the Lip 1 property reads as (see [12, Theorem 11])

$$g_E(z) \le C|z| \quad \Longleftrightarrow \quad \sum_k \theta_k < \infty.$$

11. Bernstein's Approximation Theorem

The aforementioned results are connected with the famous theorem of S. N. Bernstein on the rate of polynomial approximation of $|x|$. Let

$$\mathcal{E}_n(f(x), E) = \inf_{\deg(P_n)=n} \|f(x) - P_n(x)\|_E$$

be the rate of the best approximation of $f(x)$ on E by polynomials of degree n. Bernstein's result says (see [8]) that

$$\lim_{n\to\infty} n\mathcal{E}_n(|x|, [-1,1]) = \sigma$$

exists, finite and positive. The value of σ is still not known today. Later Bernstein extended his result ([9, 10]): if $p > 0$ is not an even integer, then

$$\lim_{n\to\infty} n^p \mathcal{E}_n^p(|x|^p, [-1,1]) = \sigma_p, \tag{21}$$

and he also considered the non-symmetric case: if $a < 0 < b$, then

$$\lim_{n\to\infty} n^p \mathcal{E}_n(|x|^p, [a,b]) = \sqrt{|a|b} \cdot \sigma_p,$$

with the same σ_p as in (21).

R. K. Vasiliev considered approximation of $|x|^p$ on an arbitrary compact $E \subseteq \mathbb{R}$. His main result is that if $0 \in \mathrm{Int}(E)$, then

$$\lim_{n\to\infty} n^p \mathcal{E}_n(|x|^p, E) = \frac{\sigma_p}{g_E'(0)}. \tag{22}$$

This result is from [38], where it is stated in a completely different form and it is one of the two theorems in that book. Unfortunately, the second theorem is not correct (contradicts (20)), and it is difficult to tell what went wrong in the close to 160 pages of reasonings. The form (22) of Vasiliev's theorem was given in [33, Theorem 10.5] along with a relatively short, about 5 pages proof.

There is also an analogue of (20): if p is not an even integer, then there is a set E of Lebesgue-measure zero for which

$$\liminf_{n\to\infty} n^p \mathcal{E}_n(|x|^p, E) > 0.$$

See [33, Corollary 10.4].

On the other hand, a recent result of Andrievskii (see [1, 2]) claims that (for p not an even integer)

$$\liminf_{n\to\infty} n^p \mathcal{E}_n(|x|^p, E) > 0$$

if and only if

$$g_E(z) \le C|z|,$$

so $\ge c/n^p$ rate of polynomial approximation of $|x|^p$ is equivalent to the local Bernstein-inequality (17) which in turn is equivalent to the Lip 1 property of the Green's function at the point 0.

12. Endpoint results

The preceding results have an analogue for endpoints that will be proven in [37]. In fact, suppose that $E \subset \mathbb{R}$ is compact, $0 \in E$, but $(-a, 0) \cap E = \emptyset$ for some $a > 0$ (0 is an "endpoint" of E). At such points we have the complete analogue of what were discussed above: for a fixed $k \geq 2$ and for $p > 0$ not an integer the following are equivalent.

- $|P_n'(0)| \leq Cn^2 \|P_n\|_E$,
- $|P_n^{(k)}(0)| \leq Cn^{2k} \|P_n\|_E$,
- $g_E(z) \leq C|z|^{1/2}$,
- $\sum_{k>0} \theta_k < \infty$,
- $\liminf_{n\to\infty} n^{2p} \mathcal{E}_n(|x|^p, E) > 0$.

As before, here the C may be different at different occurrences.

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