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Weighted Simultaneous Approximation by Kantorovich Sampling Operators

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We establish a direct estimate of the rate of the weighted simultaneous approximation by generalized Kantorovich sampling operators in L_p by a modulus of smoothness. The weights are power-type with nonpositive exponents at infinity. The unweighted case is also covered. We include an example of a sampling operator of that type whose kernel is supported on an arbitrarily fixed finite interval and which provides a rate of simultaneous approximation of any power-type order given in advance.

Keywords and Phrases: Sampling operator, sampling series, Kantorovich sampling operator, weighted approximation, Jackson-type estimate, direct inequality, direct estimate, modulus of smoothness, K-functional.

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1. Main Results

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lebesgue integrable function and $\chi : \mathbb{R} \to \mathbb{R}$. Let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of reals such that $t_k < t_{k+1}$ and $\lim_{k\to\pm\infty} t_k = \pm\infty$, as, moreover, $\theta \le \theta_k := t_{k+1} - t_k \le \Theta$ for all $k \in \mathbb{Z}$ with some constants $\theta, \Theta > 0$. Bardaro, Butzer, Stens and Vinti [6] introduced the Kantorovich-type sampling operators

$$\big(S_w^\chi f\big)(x) := \sum_{k\in\mathbb{Z}} \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du\, \chi(wx - t_k), \qquad x \in \mathbb{R}, \quad w > 0.$$

Under certain general assumptions on f and χ the series above is convergent in $\mathbb{R}$ and defines a function that belongs to the same space as f. For example, as it was shown in [6, Remark 3.2 and Corollary 5.1], if $f \in L_p(\mathbb{R})$, $1 \le p \le \infty$, and $\chi \in L_1(\mathbb{R})$ is such that $M_0(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u - t_k)| < \infty$, then

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$S_w^\chi f \in L_p(\mathbb{R})$ as well. Actually, if χ is Lebesgue measurable and $M_0(\chi) < \infty$, then $\chi \in L_1(\mathbb{R})$ (see the proof of [11, Lemma 1]). An important form of S_w^χ is the case when $t_k = k$.

The function χ is called a kernel. We will make use of the following standard characteristics of the kernel χ in order to investigate the approximation properties of S_w^χ:

(a) The discrete algebraic moment of χ of order $j \in \mathbb{N}_0$ w.r.t. $\{t_k\}_{k\in\mathbb{Z}}$, defined by
$$m_j(\chi, u) := \sum_{k\in\mathbb{Z}} (t_k - u)^j \chi(u - t_k), \qquad u \in \mathbb{R},$$
provided the series is convergent for all $u \in \mathbb{R}$,

(b) The discrete absolute moment of χ of order $\nu \geq 0$ w.r.t. $\{t_k\}_{k\in\mathbb{Z}}$, defined by
$$M_\nu(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |t_k - u|^\nu |\chi(u - t_k)|.$$

A slight modification of $m_j(\chi, u)$ will be useful. We introduce the *normalized* discrete algebraic moment of χ of order $j \in \mathbb{N}_0$ w.r.t. $\{t_k\}_{k\in\mathbb{Z}}$, defined by
$$\overline{m}_j(\chi, u) := \sum_{k\in\mathbb{Z}} \chi(u - t_k) \Big(\frac{t_k - u}{\theta_k}\Big)^j, \qquad u \in \mathbb{R},$$
provided the series is convergent for all $u \in \mathbb{R}$.

Clearly, in the case $t_k = k$ we have $\overline{m}_j(\chi, u) = m_j(\chi, u)$ for all $j \in \mathbb{N}_+$; $\overline{m}_0(\chi, u) = m_0(\chi, u)$ regardless of the values of t_k.

It is known and quite straightforward to see that if χ is bounded in a neighbourhood of 0 and $M_\nu(\chi)$ is finite for some $\nu > 0$, then $M_\delta(\chi)$ is finite for all $\delta \in [0, \nu)$.

Let χ be bounded in a neighbourhood of the origin, $m_0(\chi, u) \equiv 1$, and $M_\nu(\chi) < \infty$ with some $\nu > 0$. In [6, Theorem 4.1] it was shown that if f is a bounded Lebesgue measurable function on $\mathbb{R}$, then
$$\lim_{w\to\infty} S_w^\chi f(x) = f(x) \tag{1.1}$$
at every point x, at which f is continuous, and if, in addition, f is uniformly continuous and bounded on $\mathbb{R}$, then the convergence in (1.1) is uniform on $\mathbb{R}$ (see also [6, Remark 3.2]). The convergence of this approximation process in $L_p(\mathbb{R})$, $1 \leq p < \infty$ was established in [6, Corollary 5.2].

Our aim will be to extend this result into two directions. On the one hand, to generalize it to weighted approximation with the weight
$$\rho_{\alpha,\beta}(x) := \begin{cases} |x|^{-\alpha}, & x < -1, \\ 1, & -1 \leq x \leq 1, \\ x^{-\beta}, & x > 1, \end{cases}$$

where $\alpha, \beta \ge 0$; and, on the other hand, to consider simultaneous approximation, that is, approximation of the derivatives of f by the corresponding derivatives of $S_w^\chi f$ still in L_p-spaces with the above weight.

The phenomenon of simultaneous approximation in $L_p(\mathbb{R})$ of S_w^χ with $t_k = k$ has recently been established in [8], as the derivatives are considered in the weak sense. There results about fractional derivatives were given too. Also recently, the rate of simultaneous approximation of these operators in $L_p(\mathbb{R})$ has been characterized in [4, Theorems 3.6 and 3.7 and Corollaries 3.8 and 3.9] (see also [3]).

We will estimate the rate of approximation by the modulus of smoothness of order $s \in \mathbb{N}_+$ of f, defined for $t > 0$ by

$$\omega_s(f,t)_{p,\alpha,\beta} := \sup_{0<h\le t} \left\| \rho_{\alpha,\beta} \Delta_h^s f \right\|_p,$$

where $\Delta_h f(x) := f(x+h/2) - f(x-h/2)$, $x \in \mathbb{R}$, $h > 0$, and $\Delta_h^s := \Delta_h(\Delta_h^{s-1})$. Clearly, $\rho_{\alpha,\beta}\Delta_h^s f \in L_p(\mathbb{R})$ for every $h > 0$ if $\rho_{\alpha,\beta} f \in L_p(\mathbb{R})$.

We will make use of the following function spaces: $L_{1,loc}(\mathbb{R})$ is the space of the functions on $\mathbb{R}$, which are Lebesgue integrable on every closed finite interval, $AC_{loc}^r(\mathbb{R})$, $r \in \mathbb{N}_0$, is the space of the functions $f : \mathbb{R} \to \mathbb{R}$, which are r times differentiable on $\mathbb{R}$ and $f^{(j)}$, $j = 0, \dots, r$ are absolutely continuous on every closed finite interval on the real line, $C^r(\mathbb{R})$, $r \in \mathbb{N}_0$, is the space of the functions that are r-times continuously differentiable on $\mathbb{R}$ (as the derivatives are not necessarily bounded), and $L_p(\mathbb{R})$, where $1 \le p \le \infty$, is the standard Lebesgue L_p-spaces, equipped with the standard L_p-norm on the real line, which we will denote by $\| \circ \|_p$. When the L_p-norm is on an interval I, we will write $\| \circ \|_{p(I)}$.

We will establish the following direct estimate of the rate of the weighted simultaneous approximation by S_w^χ in L_p. Below and throughout c stands for a positive constant, generally with a different value at each occurrence, but independent of the approximated function and w.

Theorem 1.1. *Let $r \in \mathbb{N}_0$, $s \in \mathbb{N}_+$ and $1 \le p \le \infty$. Let $\alpha, \beta \ge 0$. Let $\chi \in C^r(\mathbb{R})$ be such that:*

(i) $\chi(u) = O(|u|^{-\gamma})$ *and* $\chi^{(r)}(u) = O(|u|^{-\gamma})$, *as* $u \to \pm\infty$, *where we assume* $\gamma > r + s + 1 + \max\{\alpha, \beta\}$,

(ii) $$\sum_{\ell=0}^{j} \binom{j+1}{\ell} \sum_{k\in\mathbb{Z}} \theta_k^{j-\ell} (t_k - u)^\ell \chi^{(r)}(u - t_k) \equiv \begin{cases} 0, & j = 0, \dots, r+s-1, \quad j \ne r, \\ (r+1)!, & j = r. \end{cases}$$

Then for all $f \in L_{1,loc}(\mathbb{R})$ such that $f \in AC_{loc}^{r-1}(\mathbb{R})$ and $\rho_{\alpha,\beta} f^{(r)} \in L_p(\mathbb{R})$, and all $w \ge 1$ there hold $S_w^\chi f \in C^r(\mathbb{R})$, $\rho_{\alpha,\beta}(S_w^\chi f)^{(r)} \in L_p(\mathbb{R})$ and

$$\left\| \rho_{\alpha,\beta} \big((S_w^\chi f)^{(r)} - f^{(r)} \big) \right\|_p \le c\, \omega_s(f^{(r)}, 1/w)_{p,\alpha,\beta}.$$

The assumption $f \in AC_{loc}^{r-1}(\mathbb{R})$ *is to be ignored in the case* $r = 0$.

Remark 1.1. In the case of equidistant nodes $\{t_k\}$, that is, $\theta_k = \theta > 0$, $k \in \mathbb{Z}$, assumption (ii) is equivalent to

$$\overline{m}_j(\chi^{(r)}, u) \equiv 0, \qquad j = 0, \dots, r-1, \quad r \geq 1, \qquad \theta^r \overline{m}_r(\chi^{(r)}, u) \equiv r!$$

and

$$1 + \theta^r \sum_{\ell=r+1}^{j-1} \frac{(j-r)\cdots(j-\ell+1)}{\ell!} \overline{m}_\ell(\chi^{(r)}, u) \equiv 0, \quad j = r+2, \dots, r+s, \ s \geq 2.$$

In particular, if $\theta_k = 1$ for all k, we have conditions (ii)-(iv) in [4, Theorem 3.6]. In this case, they can be reduced to relations about the Fourier transform of the kernel—see [4, Lemma 5.3]. As for conditions (i) and (v) in [4, Theorem 3.6] compared to condition (i) in Theorem 1.1, we note that, by virtue of [4, Lemma 4.1], the former imply the latter in the case $\rho_{\alpha,\beta}(x) \equiv 1$.

A direct estimate of a different type than the one in Theorem 1.1 was established for a very general class of sampling operators in the non-simultaneous case $r = 0$ for $2 \leq p \leq \infty$ with weights which include $\rho_{\alpha,\alpha}$ under certain additional assumptions on f in [10, Theorem 31 and Remark 34]. There the rate of approximation of a general class of multivariate quasi-projection operators in weighted L_p-spaces, $1 \leq p \leq \infty$, was considered. Estimates of the rate of approximation of the classical general non-integral sampling operators in spaces of continuous functions associated with the weight $\rho_{2,2}$ have been recently obtained in [1], and similar results for integral forms were established in [2, 5].

The contents of the paper are organized as follows. In Section 2, we collect several technical auxiliary results. In Section 3, we will establish the properties of S_w^χ, on which the proof of Theorem 1.1 is based. Then, in the next section, we prove this theorem. In the last section, we demonstrate the main result on a family of operators S_w^χ with time-limited kernels.

2. Auxiliary Results

We will often take into account the following simple fact, which relates any system of nodes $\{t_k\}_{k\in\mathbb{Z}}$ of the type we consider to its simplest form—$t_k = k$. To recall, we have set $\theta_k := t_{k+1} - t_k$.

Lemma 2.1. *Let* $\{t_k\}_{k\in\mathbb{Z}}$ *be a sequence of reals such that* $\theta \leq \theta_k \leq \Theta$ *for all* $k \in \mathbb{Z}$ *with some constants* $\theta, \Theta > 0$. *Then there exist positive constants* c_1 *and* c_2 *such that for all* $k \in \mathbb{Z}$

$$|t_k| \leq c_1(1 + |k|) \tag{2.1}$$

and

$$1 + |t_k| \geq c_2(1 + |k|). \tag{2.2}$$

Proof. The representations

$$t_k = \begin{cases} t_0 + \sum_{j=0}^{k-1} \theta_j, & k \geq 1, \\ t_0 - \sum_{j=1}^{-k} \theta_{-j}, & k \leq -1, \end{cases}$$

show that

$$\theta|k| - |t_0| \leq |t_k| \leq |t_0| + \Theta|k|, \qquad k \in \mathbb{Z}.$$

The right hand-side inequality above readily implies (2.1). The left hand-side inequality easily implies (2.2) with $c_2 = \theta/2$ if $|k| \geq 1 + 2|t_0|/\theta$. It is trivial to extend it to $|k| < 1 + 2|t_0|/\theta$, modifying c_2 if necessary. □

Next, we will establish auxiliary results, which we will use to show the boundedness of the family of operators $\{S_w^\chi\}$ in certain function spaces.

Lemma 2.2. *Let $\alpha, \beta \geq 0$. Let $\eta \in C(\mathbb{R})$ be such that $\eta(u) = O(|u|^{-\lambda})$, as $u \to \pm\infty$, where $\lambda > 1 + \max\{\alpha, \beta\}$. Then for all $x \in \mathbb{R}$ and $w \geq 1$ there holds*

$$\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} |\eta(wx - t_k)| \leq c\rho_{\alpha,\beta}(x)^{-1}.$$

Above c is a positive constant whose value is independent of x and w.

Proof. Let $x \geq 0$. In order to estimate the sum, we will split it into several parts. First, we consider the part on the negative t_k. We have

$$\begin{aligned} \sum_{\substack{k\in\mathbb{Z} \\ k:t_k<0}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} |\eta(wx - t_k)| &\leq c \sum_{\substack{k\in\mathbb{Z} \\ k:t_k<0}} \left(1 + \left|\frac{t_k}{w}\right|^\alpha\right) (wx - t_k)^{-\lambda} \\ &\leq c \sum_{\substack{k\in\mathbb{Z} \\ k:t_k<0}} (1 + |t_k|)^{\alpha-\lambda} \\ &\leq c \sum_{\substack{k\in\mathbb{Z} \\ k:t_k<0}} (1 + |k|)^{\alpha-\lambda} < \infty, \end{aligned}$$

where we have also taken into account (2.2).

Similarly, we have

$$\begin{aligned}\sum_{\substack{k\in\mathbb{Z}\\ k:t_k\ge 0}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}|\eta(wx-t_k)| &\le c\sum_{\substack{k\in\mathbb{Z}\\ k:t_k\ge 0}}\bigg(1+\Big(\frac{t_k}{w}\Big)^{\beta}\bigg)(1+|wx-t_k|)^{-\lambda}\\ &\le c\sum_{\substack{k\in\mathbb{Z}\\ k:t_k\ge 0}}\big(1+|wx-t_k|^{\beta}+x^{\beta}\big)(1+|wx-t_k|)^{-\lambda}\\ &\le c(1+x^{\beta})\sum_{\substack{k\in\mathbb{Z}\\ k:t_k\ge 0}}(1+|wx-t_k|)^{\beta-\lambda}.\end{aligned}$$

Thus, to complete the proof of the lemma for $x\ge 0$ it remains to show that the series

$$\sum_{\substack{k\in\mathbb{Z}\\ k:t_k\ge 0}}\frac{1}{(1+|u-t_k|)^{\delta}}, \tag{2.3}$$

where $\delta := \lambda-\beta > 1$, is convergent for all $u\ge 0$ and its sum is bounded on $[0,+\infty)$.

If $[0,u]$ contains at least one node t_k, we set $k'_u := \min\{k\in\mathbb{Z} : t_k\in[0,u]\}$ and $k''_u := \max\{k\in\mathbb{Z} : t_k\in[0,u]\}$. For $u\ge 0$ we have

$$\begin{aligned}\sum_{k=k'_u}^{k''_u}\frac{1}{(1+|u-t_k|)^{\delta}} &\le \frac{1}{\theta}\sum_{k=k'_u}^{k''_u-1}\frac{\theta_k}{(1+u-t_k)^{\delta}}+\frac{1}{(1+u-t_{k''_u})^{\delta}}\\ &\le \frac{1}{\theta}\int_0^u\frac{dv}{(1+u-v)^{\delta}}+1\\ &\le \frac{(u+1)^{1-\delta}}{\theta(\delta-1)}+1.\end{aligned}\tag{2.4}$$

As usually, if the upper index bound in a sum is smaller than the lower one, we consider the sum to be equal to 0.

Let us consider the remaining part of the sum on k with $t_k\ge 0$. We set $k'''_u := \min\{k\in\mathbb{Z} : t_k > u\}$. For $u\ge 0$ we have

$$\begin{aligned}\sum_{k=k'''_u}^{\infty}\frac{1}{(1+t_k-u)^{\delta}} &\le \frac{1}{(1+t_{k'''_u}-u)^{\delta}}+\frac{1}{\theta}\sum_{k=k'''_u+1}^{\infty}\frac{\theta_{k-1}}{(1+t_k-u)^{\delta}}\\ &\le 1+\frac{1}{\theta}\int_u^{\infty}\frac{dv}{(1+v-u)^{\delta}}\\ &= 1+\frac{1}{\theta(\delta-1)}.\end{aligned}\tag{2.5}$$

Estimates (2.4) and (2.5) show that the series (2.3) is convergent and its sum is bounded on $[0,+\infty)$.

That completes the proof of the lemma for $x\ge 0$.

The case $x \le 0$ can be reduced to the one we have already considered by means of the relation $\rho_{\alpha,\beta}(x) = \rho_{\beta,\alpha}(-x)$, $x \in \mathbb{R}$.

We set $\tau_k := -t_{-k}$ and $\tilde{\eta}(u) := \eta(-u)$. Clearly, we have $\tilde{\eta} \in C(\mathbb{R})$ and $\tilde{\eta}(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$.

Let $x \le 0$. Then, by what we have already shown but applied to the weight $\rho_{\beta,\alpha}$ and the system of nodes $\{\tau_k\}_{k\in\mathbb{Z}}$, we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} |\eta(wx - t_k)| &= \sum_{k\in\mathbb{Z}} \rho_{\beta,\alpha}\Big(\frac{\tau_k}{w}\Big)^{-1} |\tilde{\eta}(w(-x) - \tau_k)| \\ &\le c\rho_{\beta,\alpha}(-x)^{-1} = c\rho_{\alpha,\beta}(x)^{-1}.\end{aligned}$$

Thus the assertion of the lemma is established for $x \le 0$ as well. □

In particular, the following corollary of the last lemma will be useful.

Lemma 2.3. *Let $\nu \ge 0$. Let $\chi \in C(\mathbb{R})$ be such that $\chi(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$, where $\gamma > 1 + \nu$, then $M_\nu(\chi) < \infty$.*

Proof. We apply Lemma 2.2 with $\alpha = \beta = 0$, $\eta(u) = |u|^\nu \chi(u)$ and $\lambda = \gamma - \nu$. □

Lemma 2.4. *Let $\alpha, \beta \ge 0$ and $\eta \in C(\mathbb{R})$ be such that $\eta(u) = O(|u|^{-\lambda})$, as $u \to \pm\infty$, where $\lambda > 1 + \max\{\alpha, \beta\}$. Then for all $k \in \mathbb{Z}$ and $w \ge 1$ there holds*

$$\int_{\mathbb{R}} \rho_{\alpha,\beta}\Big(\frac{u}{w}\Big) |\eta(u - t_k)|\, du \le c\, \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big).$$

Above c is a positive constant, whose value is independent of k and w.

Proof. For $t_k \le 0$, we have

$$\begin{aligned}\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \int_0^{+\infty} \rho_{\alpha,\beta}\Big(\frac{u}{w}\Big) |\eta(u - t_k)|\, du &\le \Big(1 + \Big|\frac{t_k}{w}\Big|\Big)^{\alpha} \int_0^{+\infty} |\eta(u - t_k)|\, du \\ &\le c(1 + |t_k|)^\alpha \int_0^{+\infty} \frac{du}{(1 + u - t_k)^\lambda} \\ &= c(1 + |t_k|)^\alpha \frac{1}{(\lambda - 1)(1 + |t_k|)^{\lambda - 1}} \\ &\le c,\end{aligned}$$

where at the last step we use that $\alpha \le \lambda - 1$.

Let us consider the case $t_k > 0$. Then

$$\begin{aligned}&\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \int_0^{+\infty} \rho_{\alpha,\beta}\Big(\frac{u}{w}\Big) |\eta(u - t_k)|\, du \\ &\qquad \le c\Big(\frac{t_k}{w}\Big)^{\beta} \int_0^{+\infty} \frac{1}{\big(1 + \frac{u}{w}\big)^{\beta} (1 + |u - t_k|)^\lambda}\, du.\end{aligned}$$

In order to estimate the integral on $[0,+\infty)$, we split it into three by means of the intermediate points $t_k/2$ and t_k. For each one of them, we have:

$$\int_0^{t_k/2} \frac{1}{\left(1+\frac{u}{w}\right)^\beta (1+t_k-u)^\lambda}\,du \le \int_0^{t_k/2} \frac{du}{(1+t_k-u)^\lambda} \le c\,t_k^{1-\lambda},$$

$$\int_{t_k/2}^{t_k} \frac{1}{\left(1+\frac{u}{w}\right)^\beta (1+t_k-u)^\lambda}\,du \le \int_{t_k/2}^{t_k} \frac{du}{\left(1+\frac{u}{w}\right)^\beta} \le c\left(\frac{w}{t_k}\right)^\beta$$

and

$$\int_{t_k}^{+\infty} \frac{1}{\left(1+\frac{u}{w}\right)^\beta (1+u-t_k)^\lambda}\,du \le \frac{1}{\left(1+\frac{t_k}{w}\right)^\beta}\int_0^{+\infty} \frac{du}{(1+u)^\lambda} \le c\left(\frac{w}{t_k}\right)^\beta,$$

as, for the last estimate, we have taken into account that $\lambda > 1$.

We combine all the above estimates and use that $\beta+1-\lambda \le 0$ to arrive at

$$\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \int_0^{+\infty} \rho_{\alpha,\beta}\Big(\frac{u}{w}\Big)|\eta(u-t_k)|\,du \le c, \qquad k\in\mathbb{Z},\quad w\ge 1. \tag{2.6}$$

We reduce the integral on $(-\infty,0]$ to the one estimated above in the same way as we dealt with the case $x \le 0$ in the proof of the previous lemma. We set $\tau_k := -t_{-k}$ and $\tilde\eta(u) := \eta(-u)$. Then, by virtue of (2.6) applied to the weight $\rho_{\beta,\alpha}$ and the system of nodes $\{\tau_k\}_{k\in\mathbb{Z}}$, we arrive at

$$\begin{aligned}
&\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \int_{-\infty}^{0} \rho_{\alpha,\beta}\Big(\frac{u}{w}\Big)|\eta(u-t_k)|\,du \\
&\quad = \rho_{\beta,\alpha}\Big(\frac{\tau_{-k}}{w}\Big)^{-1} \int_0^{+\infty} \rho_{\beta,\alpha}\Big(\frac{u}{w}\Big)|\tilde\eta(u-\tau_{-k})|\,du \le c, \qquad k\in\mathbb{Z},\quad w\ge 1.
\end{aligned}$$

Thus the assertion of the lemma is established. □

We will make use of an assertion about differentiation of the terms of convergent function sequences. It must be known, but I am not aware of any reference to it, so I include a proof for the sake of completeness.

Lemma 2.5. *Let $r \in \mathbb{N}_+$, $s_n \in C^r[a,b]$, $n = 1,2,\dots$, and $\{s_n(x)\}_{n=1}^\infty$ converge on $[a,b]$. We set $s(x) := \lim_{n\to\infty} s_n(x)$, $x\in[a,b]$. In addition, let $\{s_n^{(r)}(x)\}_{n=1}^\infty$ converge uniformly on $[a,b]$.*

Then $s \in C^r[a,b]$ and $\{s_n^{(j)}(x)\}_{n=1}^\infty$, $j = 0,\dots,r$, converge uniformly on $[a,b]$ to $s^{(j)}(x)$, respectively.

Proof. The case $r = 1$ is the classical theorem for term-by-term differentiation of function sequences. We assume that $r \ge 2$.

We expand $s_n(x)$ by Taylor's formula at a:

$$s_n(x) = T_{r-1,n}(x) + \frac{1}{(r-1)!}\int_a^x (x-u)^{r-1} s_n^{(r)}(u)\,du,$$

where $T_{r-1,n}(x)$ is the corresponding Taylor polynomial

$$T_{r-1,n}(x) := \sum_{j=0}^{r-1} \frac{s_n^{(j)}(a)}{j!}\,(x-a)^j.$$

Let $x_1, \ldots, x_{r-1}$ be distinct points in $(a, b]$. We consider the linear system

$$\sum_{j=1}^{r-1} \frac{(x_\ell - a)^{j-1}}{j!}\, s_n^{(j)}(a) = c_\ell, \qquad \ell = 1, \ldots, r-1,$$

where

$$c_\ell := \frac{1}{(x_\ell - a)} \left(s_n(x_\ell) - s_n(a) - \frac{1}{(r-1)!} \int_a^{x_\ell} (x_\ell - u)^{r-1} s_n^{(r)}(u)\, du \right),$$

with unknowns $s_n^{(j)}(a)$, $j = 1, \ldots, r-1$.

The matrix of the linear system is a non-zero constant multiple of the Vandermonde matrix. Therefore, $s_n^{(j)}(a)$, $j = 1, \ldots, r-1$, are determined by Cramer's rule, which expresses them as linear combinations of $s_n(a)$, $s_n(x_\ell)$ and

$$\int_a^{x_\ell} (x_\ell - u)^{r-1} s_n^{(r)}(u)\, du, \qquad \ell = 1, \ldots, r-1.$$

Since $\{s_n(x)\}_{n=1}^\infty$ converges on $[a, b]$, and $\{s_n^{(r)}(x)\}_{n=1}^\infty$ converges uniformly on $[a, b]$, it implies that the sequences $\{s_n^{(j)}(a)\}_{n=1}^\infty$, $j = 1, \ldots, r-1$, are convergent.

Set $\sigma_j := \lim_{n\to\infty} s_n^{(j)}(a)$, $j = 0, \ldots, r-1$, $r \geq 1$. Then the function sequence $\{T_{r-1,n}(x)\}_{n=1}^\infty$ converges (uniformly) on $[a, b]$ to

$$T_{r-1}(x) := \sum_{j=0}^{r-1} \frac{\sigma_j}{j!}\,(x-a)^j.$$

Let $\sigma_r(x) := \lim_{n\to\infty} s_n^{(r)}(x)$, $x \in [a, b]$. Then

$$s(x) = T_{r-1}(x) + \frac{1}{(r-1)!} \int_a^x (x-u)^{r-1} \sigma_r(u)\, du, \qquad x \in [a, b].$$

Consequently, $s \in C^r[a, b]$ and $s^{(r)}(x) = \sigma_r(x)$.

To complete the proof, we can use the inequality

$$|s_n^{(j)}(x) - s^{(j)}(x)| \leq \sum_{\ell=j}^{r-1} \frac{(b-a)^{\ell-j}}{(\ell-j)!}\, |s_n^{(\ell)}(a) - \sigma_\ell| + \frac{(b-a)^{r-j}}{(r-j)!} \sup_{u\in[a,b]} |s_n^{(r)}(u) - s^{(r)}(u)|, \qquad x \in [a, b],$$

where $j = 0, \ldots, r-1$. □

Remark 2.1. As the proof shows, it is sufficient to assume that the function sequence $\{s_n(x)\}_{n=1}^\infty$ converges at r distinct points in $[a, b]$ rather than on the whole interval.

3. Basic Properties of S_w^χ

First, we will show that, under certain assumptions on the kernel, the series defining S_w^χ is convergent and can be differentiated term-by-term.

Proposition 3.1. *Let $1 \le p \le \infty$ and $w \ge 1$. Let $\alpha, \beta \ge 0$. Let $\chi \in C(\mathbb{R})$ be such that $\chi(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$, where $\gamma > 1 + \max\{\alpha, \beta\}$. Let $f \in L_{1,loc}(\mathbb{R})$ be such that $\rho_{\alpha,\beta} f \in L_p(\mathbb{R})$. Then the series, defining $S_w^\chi f(x)$, is uniformly convergent on the compact intervals of $\mathbb{R}$ and $S_w^\chi f \in C(\mathbb{R})$.*

Certainly, the proposition holds for all $w > 0$. We assume $w \ge 1$ to avoid technical details. Actually, the case $w > 0$ can be reduced to $w = 1$, but, in view of further considerations, we will establish certain stronger results for all $w \ge 1$ than we actually need to prove it. The same holds for its extension to the differentiability of $S_w^\chi f$ stated further below.

Proof. It is enough to show that the series, defining $S_w^\chi f(x)$, is uniformly convergent on the compact intervals. To this end, we will apply the Weierstrass M-test.

First, we will show that

$$\left| \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \right| \le c\, w^{1/p} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \|\rho_{\alpha,\beta} f\|_{p[t_k/w, t_{k+1}/w]}, \quad k \in \mathbb{Z}. \tag{3.1}$$

In the case $p = 1$ we just have

$$\left| \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \right| \le \max_{u \in [t_k/w, t_{k+1}/w]} \rho_{\alpha,\beta}(u)^{-1} \|\rho_{\alpha,\beta} f\|_{1[t_k/w, t_{k+1}/w]}. \tag{3.2}$$

We have

$$\rho_{\alpha,\beta}(x)^{-1} \le \rho_{\alpha,\beta}(a)^{-1} + \rho_{\alpha,\beta}(b)^{-1}, \qquad x \in [a, b], \tag{3.3}$$

for any interval $[a, b]$.

In addition, since $t_{k+1}/w \le t_k/w + \Theta$, $k \in \mathbb{Z}$, $w \ge 1$, and $\rho_{\alpha,\beta}(x)^{-1} \ge 1$, $x \in \mathbb{R}$, we have

$$\rho_{\alpha,\beta}\Big(\frac{t_{k+1}}{w}\Big)^{-1} \le c\, \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}, \qquad k \in \mathbb{Z}, \quad w \ge 1. \tag{3.4}$$

Therefore,

$$\max_{u \in [t_k/w, t_{k+1}/w]} \rho_{\alpha,\beta}(u)^{-1} \le c\, \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}, \qquad k \in \mathbb{Z}, \quad w \ge 1. \tag{3.5}$$

Inequality (3.1) for $p = 1$ follows from (3.2) and (3.5).

To complete the proof for $p > 1$, we need only to apply Hölder's inequality to arrive at

$$\|\rho_{\alpha,\beta} f\|_{1[t_k/w, t_{k+1}/w]} \le c\, w^{-1/q} \|\rho_{\alpha,\beta} f\|_{p[t_k/w, t_{k+1}/w]}, \tag{3.6}$$

where q is the conjugate exponent to p.

Now, (3.1) for $p > 1$ follows from (3.2), (3.5) and (3.6).

Further, we set $\delta := \max\{\alpha, \beta\}$. Clearly,

$$\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \le (1 + |t_k|)^{\delta}, \qquad k \in \mathbb{Z}, \quad w \ge 1. \tag{3.7}$$

Next, we will estimate $|\chi(wx - t_k)|$. By virtue of the assumptions on χ, we have $|\chi(u)| \le c(1 + |u|)^{-\gamma}$, $u \in \mathbb{R}$. Let $[a, b]$ be an arbitrary closed finite interval on $\mathbb{R}$ and $c' := 1 + w \max\{|a|, |b|\}$. Then

$$|\chi(wx - t_k)| \le c(c' + |t_k| - |wx|)^{-\gamma} \le c''(1 + |t_k|)^{-\gamma}, \quad x \in [a, b], \ \ k \in \mathbb{Z}. \tag{3.8}$$

Here c'' is a positive constant, whose value is independent of x (but may depend on w).

Estimates (3.1), (3.7), (3.8) and (2.2) yield

$$\left|\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\right| |\chi(wx - t_k)| \le c'''(1 + |k|)^{\delta-\gamma}, \qquad x \in [a, b], \ \ k \in \mathbb{Z},$$

where c''' is a positive constant, whose value is independent of x and k (but may depend on f or w).

Since $\delta - \gamma < -1$, the series

$$\sum_{k\in\mathbb{Z}} (1 + |k|)^{\delta-\gamma}$$

is convergent and then the Weierstrass M-test implies that the series, defining $S_w^{\chi} f(x)$, is uniformly convergent on $[a, b]$. □

Proposition 3.2. *Let $r \in \mathbb{N}_+$, $1 \le p \le \infty$ and $w \ge 1$. Let $\alpha, \beta \ge 0$. Let $\chi \in C^r(\mathbb{R})$ be such that $\chi(u) = O(|u|^{-\gamma})$ and $\chi^{(r)}(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$, where $\gamma > r+1-1/p+\max\{\alpha, \beta\}$. Let $f \in L_{1,loc}(\mathbb{R})$ be such that $f \in AC_{loc}^{r-1}(\mathbb{R})$ and $\rho_{\alpha,\beta} f^{(r)} \in L_p(\mathbb{R})$. Then $S_w^{\chi} f(x)$ is well defined on $\mathbb{R}$, $S_w^{\chi} f \in C^r(\mathbb{R})$ and*

$$\begin{aligned}&\big(S_w^{\chi} f\big)^{(j)}(x)\\ &\quad = w^j \sum_{k\in\mathbb{Z}} \frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\, \chi^{(j)}(wx - t_k), \quad x \in \mathbb{R}, \quad j = 1, \dots, r. \end{aligned}\tag{3.9}$$

Proof. By virtue of Lemma 2.5, it is enough to show that the series for $j = 0, r$ on the right above are uniformly convergent on the compact intervals. Again, we will do so by means of the Weierstrass M-test.

We will establish an analogue of (3.1) under the given assumptions on the r-th derivative of f. We expand $f(u)$ by Taylor's formula at 0 and integrate on $[t_k/w, t_{k+1}/w]$ to get

$$\left|\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\right| \le c\sum_{\ell=0}^{r-1}\frac{|f^{(\ell)}(0)|}{\ell!}\left(\left|\frac{t_k}{w}\right|^{\ell}+1\right) + \frac{1}{(r-1)!}\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w}\left|\int_0^u (u-v)^{r-1}f^{(r)}(v)\,dv\right|du. \quad (3.10)$$

To estimate the inner integral on the right above, we use (3.3) and (3.5) to get for v between 0 and u, and $u\in[t_k/w, t_{k+1}/w]$

$$\rho_{\alpha,\beta}(v)^{-1} \le 1+\rho_{\alpha,\beta}(u)^{-1} \le 1+\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}$$

and apply Hölder's inequality to arrive at

$$\begin{aligned}\left|\int_0^u (u-v)^{r-1}f^{(r)}(v)\,dv\right| &\le \big(1+\rho_{\alpha,\beta}(u)^{-1}\big)\,|u|^{r-1/p}\big\|\rho_{\alpha,\beta}f^{(r)}\big\|_p\\ &\le c\left(1+\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}\right)\left(\left|\frac{t_k}{w}\right|^{r-1/p}+1\right)\big\|\rho_{\alpha,\beta}f^{(r)}\big\|_p\\ &\le c\left(1+\left|\frac{t_k}{w}\right|^{r-1/p}\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}\right)\big\|\rho_{\alpha,\beta}f^{(r)}\big\|_p, \qquad u\in\Big[\frac{t_k}{w},\frac{t_{k+1}}{w}\Big].\end{aligned}$$

Now, (3.10) yields

$$\left|\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\right| \le c\sum_{\ell=0}^{r-1}\frac{|f^{(\ell)}(0)|}{\ell!}\left(\left|\frac{t_k}{w}\right|^{\ell}+1\right) + c\left(1+\left|\frac{t_k}{w}\right|^{r-1/p}\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}\right)\big\|\rho_{\alpha,\beta}f^{(r)}\big\|_p. \quad (3.11)$$

Let $[a,b]$ be an arbitrary closed finite interval and $c' := 1 + w\max\{|a|,|b|\}$. We have $|\chi^{(j)}(u)| \le c(c'+|u|)^{-\gamma}$ for $u\in\mathbb{R}$ and $j=0,r$, hence

$$\begin{aligned}\big|\chi^{(j)}(wx-t_k)\big| &\le c(c'+|t_k|-|wx|)^{-\gamma}\\ &\le c''(1+|t_k|)^{-\gamma}, \qquad x\in[a,b],\quad k\in\mathbb{Z},\end{aligned} \quad (3.12)$$

where c'' is a positive constant, whose value is independent of x (but may depend on w).

Now, similarly to the proof of the previous proposition, in view of (3.11), (3.7), (3.12) and (2.2), the uniform convergence of the series in (3.9) with $j=0,r$ on $[a,b]$ follows by the Weierstrass M-test from the convergence of the series

$$\sum_{k\in\mathbb{Z}}(1+|k|)^{r-1-\gamma} \quad \text{and} \quad \sum_{k\in\mathbb{Z}}(1+|k|)^{r+\max\{\alpha,\beta\}-1/p-\gamma}.$$

Now, the assertion of the proposition follows from Lemma 2.5. □

The next basic property of the family of operators $\{S_w^\chi\}_w$ we will prove shows that it is uniformly bounded in the weighted L_p-norm associated with $\rho_{\alpha,\beta}$.

Proposition 3.3. *Let $1 \le p \le \infty$ and $\alpha, \beta \ge 0$. Let $\chi \in C(\mathbb{R})$ be such that $\chi(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$, where $\gamma > 1 + \max\{\alpha, \beta\}$.*

Then for all $f \in L_{1,loc}(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in L_p(\mathbb{R})$, and all $w \ge 1$ there hold $\rho_{\alpha,\beta} S_w^\chi f \in L_p(\mathbb{R})$ and

$$\|\rho_{\alpha,\beta} S_w^\chi f\|_p \le c\, \|\rho_{\alpha,\beta} f\|_p. \tag{3.13}$$

Proof. In view of Proposition 3.1, $S_w^\chi f$ is well-defined and is Lebesgue measurable on $\mathbb{R}$.

We will prove (3.13) for $p = 1$ and $p = \infty$. Then the Riesz-Thorin interpolation theorem will imply it for $1 < p < \infty$.

(a) $p = \infty$. By virtue of (3.1) with $p = \infty$, we have

$$|\rho_{\alpha,\beta}(x) S_w^\chi f(x)| \le c\, \rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} |\chi(wx - t_k)|\, \|\rho_{\alpha,\beta} f\|_\infty, \quad x \in \mathbb{R},$$

and (3.13) with $p = \infty$ follows from Lemma 2.2 with $\eta = \chi$ and $\lambda = \gamma$.

(b) $p = 1$. Similarly, by (3.1) with $p = 1$ and Lemma 2.4 with $\eta = \chi$ and $\lambda = \gamma$, we arrive at

$$\begin{aligned}
\|\rho_{\alpha,\beta} S_w^\chi f\|_1 &\le cw \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \int_{\mathbb{R}} \rho_{\alpha,\beta}(x) |\chi(wx - t_k)|\, dx\, \|\rho_{\alpha,\beta} f\|_{1[t_k/w, t_{k+1}/w]} \\
&\le c \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \int_{\mathbb{R}} \rho_{\alpha,\beta}\Big(\frac{u}{w}\Big) |\chi(u - t_k)|\, du\, \|\rho_{\alpha,\beta} f\|_{1[t_k/w, t_{k+1}/w]} \\
&\le c \sum_{k\in\mathbb{Z}} \|\rho_{\alpha,\beta} f\|_{1[t_k/w, t_{k+1}/w]} = c\, \|\rho_{\alpha,\beta} f\|_1. \qquad \square
\end{aligned}$$

Next, we will extend the above assertion to the derivatives.

Proposition 3.4. *Let $r \in \mathbb{N}_+$, $1 \le p \le \infty$ and $\alpha, \beta \ge 0$. Let $\chi \in C^r(\mathbb{R})$ be such that:*

(i) $\chi(u) = O(|u|^{-\gamma})$ *and* $\chi^{(r)}(u) = O(|u|^{-\gamma})$, *as $u \to \pm\infty$, where we assume $\gamma > r + 1 + \max\{\alpha, \beta\}$,*

(ii) $\displaystyle\sum_{\ell=0}^{j} \binom{j+1}{\ell} \sum_{k\in\mathbb{Z}} \theta_k^{j-\ell} (t_k - u)^\ell \chi^{(r)}(u - t_k) \equiv 0, \quad j = 0, \dots, r-1.$

Then for all $f \in L_{1,loc}(\mathbb{R})$ such that $f \in AC_{loc}^{r-1}(\mathbb{R})$ and $\rho_{\alpha,\beta} f^{(r)} \in L_p(\mathbb{R})$, and all $w \ge 1$ there hold $S_w^\chi f \in C^r(\mathbb{R})$, $\rho_{\alpha,\beta}(S_w^\chi f)^{(r)} \in L_p(\mathbb{R})$ and

$$\big\|\rho_{\alpha,\beta}(S_w^\chi f)^{(r)}\big\|_p \le c\, \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_p.$$

Proof. We have $S_w^\chi f \in C^r(\mathbb{R})$ and

$$(S_w^\chi f)^{(r)}(x) = w^r \sum_{k\in\mathbb{Z}} \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du\chi^{(r)}(wx - t_k), \qquad x \in \mathbb{R}, \tag{3.14}$$

by virtue of Proposition 3.2.

We expand $f(u)$ at $x \in \mathbb{R}$ by Taylor's formula:

$$f(u) = \sum_{j=0}^{r-1} \frac{f^{(j)}(x)}{j!}(u-x)^j + \frac{1}{(r-1)!}\int_x^u (u-v)^{r-1} f^{(r)}(v)\,dv.$$

Then

$$\begin{aligned}&\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du \\ &\quad = \frac{w}{\theta_k}\sum_{j=0}^{r-1}\frac{f^{(j)}(x)}{(j+1)!}\left[\Big(\frac{t_{k+1}}{w}-x\Big)^{j+1} - \Big(\frac{t_k}{w}-x\Big)^{j+1}\right] + R_{r,k,w}f(x),\end{aligned} \tag{3.15}$$

where we have set

$$R_{r,k,w}f(x) := \frac{1}{(r-1)!}\,\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w}\left(\int_x^u (u-v)^{r-1} f^{(r)}(v)\,dv\right)du.$$

We represent the first term in the square brackets on the right hand-side of (3.15) by means of the binomial formula:

$$\Big(\frac{t_{k+1}}{w}-x\Big)^{j+1} = \left[\frac{\theta_k}{w} + \Big(\frac{t_k}{w}-x\Big)\right]^{j+1} = \frac{1}{w^{j+1}}\sum_{\ell=0}^{j+1}\binom{j+1}{\ell}\theta_k^{j+1-\ell}(t_k - wx)^\ell.$$

Then we substitute this expression in (3.15) and get

$$\begin{aligned}&\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du \\ &\quad = \sum_{j=0}^{r-1}\frac{f^{(j)}(x)}{(j+1)!}w^{-j}\sum_{\ell=0}^{j}\binom{j+1}{\ell}\theta_k^{j-\ell}(t_k-wx)^\ell + R_{r,k,w}f(x).\end{aligned} \tag{3.16}$$

We use the last relation to derive from (3.14) the representation

$$\begin{aligned}(S_w^\chi f)^{(r)}(x) &= \sum_{j=0}^{r-1}\frac{f^{(j)}(x)}{(j+1)!}w^{r-j}\sum_{\ell=0}^{j}\binom{j+1}{\ell}\sum_{k\in\mathbb{Z}}\theta_k^{j-\ell}(t_k-wx)^\ell\chi^{(r)}(wx-t_k) \\ &\quad + w^r\sum_{k\in\mathbb{Z}} R_{r,k,w}f(x)\chi^{(r)}(wx-t_k), \qquad x\in\mathbb{R}.\end{aligned} \tag{3.17}$$

By virtue of assumption (ii), relation (3.17) takes the form

$$(S_w^\chi f)^{(r)}(x) = w^r \sum_{k\in\mathbb{Z}} R_{r,k,w} f(x) \chi^{(r)}(wx - t_k), \qquad x \in \mathbb{R}. \tag{3.18}$$

It remains to estimate the weighted L_p-norm of the right hand-side of (3.18). We will do that for $p = 1$ and $p = \infty$, and then complete the proof of the proposition by means of the Riesz-Thorin interpolation theorem.

(a) $p = \infty$. We use (3.3) to deduce the estimate

$$\begin{aligned}
|R_{r,k,w} f(x)| &\le \frac{w}{(r-1)!\,\theta_k} \int_{t_k/w}^{t_{k+1}/w} \left| \int_x^u (u-v)^{r-1} |f^{(r)}(v)|\, dv \right| du \\
&\le \frac{w}{(r-1)!\,\theta_k} \int_{t_k/w}^{t_{k+1}/w} \left| \int_x^u (u-v)^{r-1} \rho_{\alpha,\beta}(v)^{-1} dv \right| du \, \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty \\
&\le \frac{w}{r!\,\theta_k} \int_{t_k/w}^{t_{k+1}/w} |u-x|^r \rho_{\alpha,\beta}(u)^{-1} du \, \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty \\
&\quad + \frac{w}{r!\,\theta_k} \int_{t_k/w}^{t_{k+1}/w} |u-x|^r du \, \rho_{\alpha,\beta}(x)^{-1} \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty \\
&\le \frac{w}{r!\,\theta_k} \int_{t_k/w}^{t_{k+1}/w} |u-x|^r du \\
&\quad \times \left(\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} + \rho_{\alpha,\beta}\Big(\frac{t_{k+1}}{w}\Big)^{-1} + \rho_{\alpha,\beta}(x)^{-1} \right) \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty \\
&\le \frac{1}{r!\,w^r} \big(|wx - t_k|^r + |wx - t_{k+1}|^r \big) \\
&\quad \times \left(\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} + \rho_{\alpha,\beta}\Big(\frac{t_{k+1}}{w}\Big)^{-1} + \rho_{\alpha,\beta}(x)^{-1} \right) \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty .
\end{aligned}$$

Next, we use that $|wx - t_{k+1}| \le |wx - t_k| + \Theta$ and (3.4) to derive

$$\begin{aligned}
&|R_{r,k,w} f(x)| \\
&\quad \le \frac{c}{w^r} \left(\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} + \rho_{\alpha,\beta}(x)^{-1} \right) \big(1 + |wx - t_k|^r\big) \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty .
\end{aligned} \tag{3.19}$$

We apply this estimate to (3.18) and get

$$\begin{aligned}
&|\rho_{\alpha,\beta}(x) (S_w^\chi f)^{(r)}(x)| \\
&\quad \le c\, \rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \big(1 + |wx - t_k|^r\big) |\chi^{(r)}(wx - t_k)| \, \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty \\
&\qquad + c \sum_{k\in\mathbb{Z}} \big(1 + |wx - t_k|^r\big) |\chi^{(r)}(wx - t_k)| \, \big\|\rho_{\alpha,\beta} f^{(r)}\big\|_\infty .
\end{aligned} \tag{3.20}$$

By virtue of Lemma 2.2 with $\eta(u) = \chi^{(r)}(u)$, $\lambda = \gamma$ and $\eta(u) = u^r \chi^{(r)}(u)$, $\lambda = \gamma - r$ we have

$$\rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1} \big(1 + |wx - t_k|^r\big) \big|\chi^{(r)}(wx - t_k)\big| \le c, \qquad x \in \mathbb{R}, \quad w \ge 1,$$

and, by the same lemma with the same ηs and λs, respectively, but $\alpha = \beta = 0$ (or Lemma 2.3), we have

$$\sum_{k\in\mathbb{Z}} \big(1 + |wx - t_k|^r\big) \big|\chi^{(r)}(wx - t_k)\big| \le c, \qquad x \in \mathbb{R}, \quad w \ge 1.$$

Now, the assertion of the proposition for $p = \infty$ follows from (3.20).

(b) $p = 1$. By virtue of (3.18), we have

$$\big\|\rho_{\alpha,\beta}(S_w^\chi f)^{(r)}\big\|_1 \le w^r \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \rho_{\alpha,\beta}(x) \big|R_{r,k,w} f(x) \chi^{(r)}(wx - t_k)\big| dx. \tag{3.21}$$

We have for the terms on the right hand-side

$$\begin{aligned} &\int_{\mathbb{R}} \rho_{\alpha,\beta}(x) \big|R_{r,k,w} f(x) \chi^{(r)}(wx - t_k)\big| dx \\ &\quad \le c\, w \int_{t_k/w}^{t_{k+1}/w} \left(\int_{\mathbb{R}} \rho_{\alpha,\beta}(x) \left| \int_x^u |u - v|^{r-1} |f^{(r)}(v)|\, dv \right| \big|\chi^{(r)}(wx - t_k)\big|\, dx \right) du \\ &\quad \le c \sup_{u\in[t_k/w, t_{k+1}/w]} \int_{\mathbb{R}} \rho_{\alpha,\beta}(x) \left| \int_x^u |u - v|^{r-1} |f^{(r)}(v)|\, dv \right| \big|\chi^{(r)}(wx - t_k)\big|\, dx. \end{aligned} \tag{3.22}$$

For any fixed $u \in [t_k/w, t_{k+1}/w]$ we split the integral on x above into two parts—on $(-\infty, u]$ and $[u, +\infty)$, and interchange the order of integration on x and v. We have for the integral on $x \le u$

$$\begin{aligned} &\int_{-\infty}^u \rho_{\alpha,\beta}(x) \left(\int_x^u (u - v)^{r-1} |f^{(r)}(v)|\, dv \right) \big|\chi^{(r)}(wx - t_k)\big|\, dx \\ &\quad = \int_{-\infty}^{t_k/w} (u - v)^{r-1} |f^{(r)}(v)| \left(\int_{-\infty}^v \rho_{\alpha,\beta}(x) \big|\chi^{(r)}(wx - t_k)\big|\, dx \right) dv \\ &\qquad + \int_{t_k/w}^u (u - v)^{r-1} |f^{(r)}(v)| \left(\int_{-\infty}^v \rho_{\alpha,\beta}(x) \big|\chi^{(r)}(wx - t_k)\big|\, dx \right) dv. \end{aligned} \tag{3.23}$$

For the first iterated integral on the right above, we use (i) to get for $t_k/w < 1$

$$\begin{aligned}
&\int_{-\infty}^{t_k/w} (u-v)^{r-1}|f^{(r)}(v)| \left(\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)\big|\chi^{(r)}(wx-t_k)\big|\,dx\right) dv \\
&\quad \le c \int_{-\infty}^{t_k/w} (u-v)^{r-1}\rho_{\alpha,\beta}(v)|f^{(r)}(v)| \left(\int_{-\infty}^{v} (1+t_k-wx)^{-\gamma} dx\right) dv \\
&\quad \le \frac{c}{w} \int_{-\infty}^{t_k/w} (u-v)^{r-1}(1+t_k-wv)^{1-\gamma}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv \qquad (3.24)\\
&\quad \le \frac{c}{w^r} \int_{-\infty}^{t_k/w} (t_{k+1}-wv)^{r-1}(1+t_k-wv)^{1-\gamma}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv \\
&\quad \le \frac{c}{w^r} \int_{-\infty}^{t_k/w} (1+t_k-wv)^{r-\gamma}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv.
\end{aligned}$$

Let us consider now the case $t_k/w \ge 1$. Just as above we establish that

$$\begin{aligned}
&\int_{-\infty}^{1} (u-v)^{r-1}|f^{(r)}(v)| \left(\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)\big|\chi^{(r)}(wx-t_k)\big|\,dx\right) dv \\
&\qquad\qquad \le \frac{c}{w^r} \int_{-\infty}^{1} (1+t_k-wv)^{r-\gamma}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv. \qquad (3.25)
\end{aligned}$$

Further, for the remaining part of the integral on v, we use Lemma 2.4 with $\eta(y) = (1+|y|)^{r+\varepsilon}\chi^{(r)}(y)$ and $\lambda = \gamma - r - \varepsilon$, where $\varepsilon > 0$ is such that $\gamma > r + 1 + \varepsilon + \max\{\alpha, \beta\}$, to get

$$\begin{aligned}
&\int_{1}^{t_k/w} (u-v)^{r-1}|f^{(r)}(v)| \left(\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)\big|\chi^{(r)}(wx-t_k)\big|\,dx\right) dv \\
&\quad \le \frac{c}{w^{r-1}} \int_{1}^{t_k/w} (1+t_k-wv)^{r-1}|f^{(r)}(v)| \left(\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)\big|\chi^{(r)}(wx-t_k)\big|\,dx\right) dv \\
&\quad \le \frac{c}{w^{r-1}} \int_{1}^{t_k/w} (1+t_k-wv)^{-1-\varepsilon}|f^{(r)}(v)| \left(\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)|\eta(wx-t_k)|\,dx\right) dv \\
&\quad \le \frac{c}{w^r} \int_{1}^{t_k/w} (1+t_k-wv)^{-1-\varepsilon}|f^{(r)}(v)| \left(\int_{\mathbb{R}} \rho_{\alpha,\beta}\Big(\frac{y}{w}\Big)|\eta(y-t_k)|\,dy\right) dv \\
&\quad \le \frac{c}{w^r} \int_{1}^{t_k/w} (1+t_k-wv)^{-1-\varepsilon}\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)|f^{(r)}(v)|\,dv \\
&\quad \le \frac{c}{w^r} \int_{1}^{t_k/w} (1+t_k-wv)^{-1-\varepsilon}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv, \qquad \frac{t_k}{w} \ge 1. \qquad (3.26)
\end{aligned}$$

Now, (3.24)–(3.26) yield

$$\int_{-\infty}^{t_k/w} (u-v)^{r-1}|f^{(r)}(v)| \left(\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)\big|\chi^{(r)}(wx-t_k)\big|\,dx\right) dv$$
$$\le \frac{c}{w^r}\int_{-\infty}^{t_k/w} (1+t_k-wv)^{-1-\varepsilon}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv, \qquad k\in\mathbb{Z}. \tag{3.27}$$

In order to estimate the second iterated integral on the right of (3.23), we first apply Lemma 2.4 with $\eta=\chi^{(r)}$ and $\lambda=\gamma$ to get

$$\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)\big|\chi^{(r)}(wx-t_k)\big|\,dx \le \frac{1}{w}\int_{\mathbb{R}} \rho_{\alpha,\beta}\left(\frac{y}{w}\right)\big|\chi^{(r)}(y-t_k)\big|\,dy$$
$$\le \frac{c}{w}\,\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big), \qquad v\in\mathbb{R}.$$

In addition, we take into account (3.1) with $p=1$ and $f^{(r)}$ in place of f, to deduce the estimate

$$\begin{aligned}\int_{t_k/w}^{u} (u-v)^{r-1}|f^{(r)}(v)| &\left(\int_{-\infty}^{v} \rho_{\alpha,\beta}(x)\big|\chi^{(r)}(wx-t_k)\big|\,dx\right) dv\\
&\le \frac{c}{w}\,\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)\int_{t_k/w}^{u} (u-v)^{r-1}|f^{(r)}(v)|\,dv\\
&\le \frac{c}{w^r}\,\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)\int_{t_k/w}^{t_{k+1}/w} |f^{(r)}(v)|\,dv\\
&\le \frac{c}{w^r}\,\big\|\rho_{\alpha,\beta}f^{(r)}\big\|_{1[t_k/w,t_{k+1}/w]}, \qquad u\in\Big[\frac{t_k}{w},\frac{t_{k+1}}{w}\Big],\quad k\in\mathbb{Z}.\end{aligned} \tag{3.28}$$

Estimates (3.23), (3.27) and (3.28) imply

$$\begin{aligned}\int_{-\infty}^{u} \rho_{\alpha,\beta}(x)&\left(\int_{x}^{u} (u-v)^{r-1}|f^{(r)}(v)|\,dv\right)\big|\chi^{(r)}(wx-t_k)\big|\,dx\\
&\le \frac{c}{w^r}\int_{-\infty}^{t_k/w} (1+t_k-wv)^{-1-\varepsilon}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv\\
&\quad+\frac{c}{w^r}\,\big\|\rho_{\alpha,\beta}f^{(r)}\big\|_{1[t_k/w,t_{k+1}/w]}, \qquad u\in\Big[\frac{t_k}{w},\frac{t_{k+1}}{w}\Big],\quad k\in\mathbb{Z}.\end{aligned} \tag{3.29}$$

Next, we reduce the part of integral on x on the last line of (3.22) which is on $[u,+\infty)$ to the part on $(-\infty,-u]$, which we have just estimated above. We set $\tau_k := -t_{-k}$ and $\tilde g(y) := g(-y)$. Let $u\in[t_k/w,t_{k+1}/w]$. Then we have $-u\in[\tau_{-k-1}/w,\tau_{-k}/w]$. By what we have already established in (3.29) with $-u$ in place of u and w.r.t. the system of nodes $\{\tau_k\}_{k\in\mathbb{Z}}$, the weight $\rho_{\beta,\alpha}$ and the function $\tilde{f}$, we get the following estimate

$$\begin{aligned}
&\int_u^\infty \rho_{\alpha,\beta}(x)\left(\int_u^x (v-u)^{r-1}|f^{(r)}(v)|\,dv\right)\left|\chi^{(r)}(wx-t_k)\right|dx\\
&\quad = \int_{-\infty}^{-u} \rho_{\alpha,\beta}(-y)\left(\int_u^{-y} (v-u)^{r-1}|f^{(r)}(v)|\,dv\right)\left|\chi^{(r)}(-wy-t_k)\right|dx\\
&\quad = \int_{-\infty}^{-u} \rho_{\beta,\alpha}(y)\left(\int_y^{-u} (-u-v)^{r-1}|\tilde{f}^{(r)}(v)|\,dv\right)\left|\tilde{\chi}^{(r)}(wy-\tau_{-k})\right|dx\\
&\quad \le \frac{c}{w^r}\int_{-\infty}^{\tau_{-k-1}/w} (1+\tau_{-k-1}-wv)^{-1-\varepsilon}\rho_{\beta,\alpha}(v)|\tilde{f}^{(r)}(v)|\,dv\\
&\qquad + \frac{c}{w^r}\left\|\rho_{\beta,\alpha}\tilde{f}^{(r)}\right\|_{1[\tau_{-k-1}/w,\tau_{-k}/w]} \qquad (3.30)\\
&\quad \le \frac{c}{w^r}\int_{-\infty}^{\tau_{-k}/w} (1+\tau_{-k}-wv)^{-1-\varepsilon}\rho_{\beta,\alpha}(v)|\tilde{f}^{(r)}(v)|\,dv\\
&\qquad + \frac{c}{w^r}\left\|\rho_{\beta,\alpha}\tilde{f}^{(r)}\right\|_{1[\tau_{-k-1}/w,\tau_{-k}/w]}\\
&\quad = \frac{c}{w^r}\int_{t_k/w}^{\infty} (1+wv-t_k)^{-1-\varepsilon}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv\\
&\qquad + \frac{c}{w^r}\left\|\rho_{\alpha,\beta}f^{(r)}\right\|_{1[t_k/w,t_{k+1}/w]}.
\end{aligned}$$

Estimates (3.29) and (3.30) yield

$$\begin{aligned}
&\int_{\mathbb{R}} \rho_{\alpha,\beta}(x)\left|\int_x^u (u-v)^{r-1}|f^{(r)}(v)|\,dv\right|\left|\chi^{(r)}(wx-t_k)\right|dx\\
&\qquad \le \frac{c}{w^r}\int_{\mathbb{R}} (1+|wv-t_k|)^{-1-\varepsilon}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv\\
&\qquad\quad + \frac{c}{w^r}\left\|\rho_{\alpha,\beta}f^{(r)}\right\|_{1[t_k/w,t_{k+1}/w]}, \qquad u\in\left[\frac{t_k}{w},\frac{t_{k+1}}{w}\right],\quad k\in\mathbb{Z},
\end{aligned}$$

hence (3.22) implies

$$\begin{aligned}
&\int_{\mathbb{R}} \rho_{\alpha,\beta}(x)\left|R_{r,k,w}f(x)\chi^{(r)}(wx-t_k)\right|dx\\
&\qquad \le \frac{c}{w^r}\int_{\mathbb{R}} (1+|wv-t_k|)^{-1-\varepsilon}\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv \qquad (3.31)\\
&\qquad\quad + \frac{c}{w^r}\|\rho_{\alpha,\beta}f^{(r)}\|_{1[t_k/w,t_{k+1}/w]}, \qquad k\in\mathbb{Z}.
\end{aligned}$$

Now, (3.21) and (3.31) yield

$$\begin{aligned}
\left\|\rho_{\alpha,\beta}(S_w^\chi f)^{(r)}\right\|_1 &\le c\int_{\mathbb{R}}\left(\sum_{k\in\mathbb{Z}}(1+|wv-t_k|)^{-1-\varepsilon}\right)\rho_{\alpha,\beta}(v)|f^{(r)}(v)|\,dv\\
&\quad + c\left\|\rho_{\alpha,\beta}f^{(r)}\right\|_1\\
&\le c\left\|\rho_{\alpha,\beta}f^{(r)}\right\|_1,
\end{aligned}$$

since the series $\sum_{k\in\mathbb{Z}}(1+|u-t_k|)^{-1-\varepsilon}$ is convergent on $\mathbb{R}$ and its sum is bounded (see (2.3) and henceforward to the end of the proof of Lemma 2.2). $\square$

Next, we will establish a Jackson-type estimate for the weighted (simultaneous) approximation by S_w^χ in L_p.

Proposition 3.5. *Let $r \in \mathbb{N}_0$, $s \in \mathbb{N}_+$ and $1 \le p \le \infty$. Let $\alpha, \beta \ge 0$. Let $\chi \in C^r(\mathbb{R})$ be such that:*

(i) $\chi(u) = O(|u|^{-\gamma})$ *and* $\chi^{(r)}(u) = O(|u|^{-\gamma})$, *as $u \to \pm\infty$, where we assume $\gamma > r+s+1+\max\{\alpha,\beta\}$,*

(ii)
$$\sum_{\ell=0}^{j}\binom{j+1}{\ell}\sum_{k\in\mathbb{Z}}\theta_k^{j-\ell}(t_k-u)^\ell\chi^{(r)}(u-t_k) \equiv \begin{cases}0, & j=0,\dots,r+s-1,\ j\ne r,\\ (r+1)!, & j=r.\end{cases}$$

Then for all $f \in L_{1,loc}(\mathbb{R})$ such that $f \in AC_{loc}^{r+s-1}(\mathbb{R})$ and $\rho_{\alpha,\beta}f^{(r)}, \rho_{\alpha,\beta}f^{(r+s)} \in L_p(\mathbb{R})$, and all $w \ge 1$ there hold $S_w^\chi f \in C^r(\mathbb{R})$ and

$$\big\|\rho_{\alpha,\beta}\big((S_w^\chi f)^{(r)} - f^{(r)}\big)\big\|_p \le \frac{c}{w^s}\big\|\rho_{\alpha,\beta}f^{(r+s)}\big\|_p. \tag{3.32}$$

Proof. If $r=0$, we have $S_w^\chi f \in C(\mathbb{R})$ by Proposition 3.1, and if $r \ge 1$, we have $S_w^\chi f \in C^r(\mathbb{R})$ and formula (3.14) by Proposition 3.2.

To prove (3.32), we proceed as in the proof of Proposition 3.4. We expand $f(u)$ at x by Taylor's formula with an integral remainder, in which the derivative is of order $r+s$. Just as in that proof, we use this expansion to get (3.16) with $r+s$ in place of r and then apply (ii) to arrive at

$$(S_w^\chi f)^{(r)}(x) - f^{(r)}(x) = w^r\sum_{k\in\mathbb{Z}}R_{r+s,k,w}f(x)\chi^{(r)}(wx-t_k), \quad x\in\mathbb{R}. \tag{3.33}$$

We complete the proof of (3.32) just as in the previous proposition—we consider the cases $p=1$ and $p=\infty$ and apply the Riesz-Thorin theorem. Basically, we follow the same arguments, as we keep $\chi^{(r)}$ but everywhere else we substitute r with $r+s$.

For $p=\infty$ we have by means of (3.33) and (3.19), the latter with $r+s$ in place of r (cf. (3.20)),

$$\begin{aligned}
&\big|\rho_{\alpha,\beta}(x)\big((S_w^\chi f)^{(r)}(x) - f^{(r)}(x)\big)\big|\\
&\quad\le \frac{c}{w^s}\rho_{\alpha,\beta}(x)\sum_{k\in\mathbb{Z}}\rho_{\alpha,\beta}\Big(\frac{t_k}{w}\Big)^{-1}\big(1+|wx-t_k|^{r+s}\big)\big|\chi^{(r)}(wx-t_k)\big|\big\|\rho_{\alpha,\beta}f^{(r+s)}\big\|_\infty\\
&\qquad\quad + \frac{c}{w^s}\sum_{k\in\mathbb{Z}}\big(1+|wx-t_k|^{r+s}\big)\big|\chi^{(r)}(wx-t_k)\big|\big\|\rho_{\alpha,\beta}f^{(r+s)}\big\|_\infty\\
&\quad\le \frac{c}{w^s}\big\|\rho_{\alpha,\beta}f^{(r+s)}\big\|_\infty,
\end{aligned}$$

where, at the last estimate, we have applied Lemma 2.2 with $\eta(u) = \chi^{(r)}(u)$, $\lambda = \gamma$ and $\eta(u) = u^{r+s}\chi^{(r)}(u)$, $\lambda = \gamma - r - s$.

For $p = 1$, similar small variations of the proof of Proposition 3.4 verify the assertion in this case.

We have, by (3.33),

$$\begin{aligned}\big\|\rho_{\alpha,\beta}\big((S_w^\chi f)^{(r)} - f^{(r)}\big)\big\|_1 \\ \le w^r \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \rho_{\alpha,\beta}(x)\big|R_{r+s,k,w}f(x)\chi^{(r)}(wx - t_k)\big|dx. \quad (3.34)\end{aligned}$$

Just similarly to (3.31), we establish that

$$\begin{aligned}&\int_{\mathbb{R}} \rho_{\alpha,\beta}(x)\left|R_{r+s,k,w}f(x)\chi^{(r)}(wx - t_k)\right| dx \\ &\le \frac{c}{w^{r+s}} \int_{\mathbb{R}} (1 + |wv - t_k|)^{-1-\varepsilon}\rho_{\alpha,\beta}(v)|f^{(r+s)}(v)|\, dv \qquad (3.35) \\ &\quad + \frac{c}{w^{r+s}} \big\|\rho_{\alpha,\beta}f^{(r+s)}\big\|_{1[t_k/w,t_{k+1}/w]}, \quad k \in \mathbb{Z}.\end{aligned}$$

We apply Lemma 2.4 once with $\eta(y) = (1+|y|)^{r+s+\varepsilon}\chi^{(r)}(y)$ and $\lambda = \gamma-r-s-\varepsilon$, where $\varepsilon > 0$ is such that $\gamma > r + s + 1 + \varepsilon + \max\{\alpha, \beta\}$, and again with $\eta(y) = \chi^{(r)}(y)$ and $\lambda = \gamma$.

Estimates (3.34) and (3.35) yield (3.32) for $p = 1$. □

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is quite standard. We will make use of the K-functional

$$\begin{aligned}K_s(f, t)_{p,\alpha,\beta} := \inf \big\{&\|\rho_{\alpha,\beta}(f - g)\|_p + t\,\|\rho_{\alpha,\beta}g^{(s)}\|_p \\ &: g \in AC_{loc}^{s-1}(\mathbb{R}),\ \rho_{\alpha,\beta}g, \rho_{\alpha,\beta}g^{(s)} \in L_p(\mathbb{R})\big\},\end{aligned}$$

where $s \in \mathbb{N}_+$, $f \in L_{1,loc}(\mathbb{R})$, $\rho_{\alpha,\beta}f \in L_p(\mathbb{R})$ and $t > 0$.

We have that there exists $c > 0$ such that for all $f \in L_{1,loc}(\mathbb{R})$ such that $\rho_{\alpha,\beta}f \in L_p(\mathbb{R})$, and all $t \in (0, 1]$ there holds

$$c^{-1}\omega_s(f, t)_{p,\alpha,\beta} \le K_s(f, t^s)_{p,\alpha,\beta} \le c\,\omega_s(f, t)_{p,\alpha,\beta}. \qquad (4.1)$$

These relations were established e.g. in [9, Chapter 6, Theorem 2.4] in the unweighted case: $\alpha = \beta = 0$; the proof for $\alpha, \beta \ge 0$ is similar. We will need only the right hand-side inequality above.

Proof of Theorem 1.1. Let $g \in AC_{loc}^{r+s-1}(\mathbb{R})$ be such that $\rho_{\alpha,\beta}g^{(r)}, \rho_{\alpha,\beta}g^{(r+s)} \in L_p(\mathbb{R})$. Then, by virtue of Proposition 3.3 (if $r = 0$), Proposition 3.4 (if $r \ge 1$)

and Proposition 3.5, we get

$$\begin{aligned}\|\rho_{\alpha,\beta}\big((S_w^\chi f)^{(r)} - f^{(r)}\big)\|_p &\le \big\|\rho_{\alpha,\beta}\big(S_w^\chi(f-g)\big)^{(r)}\big\|_p + \big\|\rho_{\alpha,\beta}\big((S_w^\chi g)^{(r)} - g^{(r)}\big)\big\|_p \\ &\quad + \big\|\rho_{\alpha,\beta}(g^{(r)} - f^{(r)})\big\|_p \\ &\le c\Big(\big\|\rho_{\alpha,\beta}(f^{(r)} - g^{(r)})\big\|_p + \frac{1}{w^s}\,\big\|\rho_{\alpha,\beta}g^{(r+s)}\big\|_p\Big).\end{aligned}$$

We take the infimum on g and arrive at the estimate

$$\big\|\rho_{\alpha,\beta}\big((S_w^\chi f)^{(r)} - f^{(r)}\big)\big\|_p \le c\,K_s(f^{(r)}, w^{-s})_{p,\alpha,\beta}.$$

Now, the assertion of the theorem follows from the right hand-side inequality in (4.1). □

5. An Example

We will demonstrate the efficiency of the main theorem on one example. The central B-spline of order $\ell \in \mathbb{N}_+$, $\ell \ge 2$, is defined by

$$B_\ell(x) := \frac{1}{(\ell-1)!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\Big(\frac{\ell}{2} + x - j\Big)_+^{\ell-1}, \qquad x \in \mathbb{R}, \tag{5.1}$$

where $(x)_+^k := \max\{x^k, 0\}$, $x \in \mathbb{R}$. The central B-spline of order 1 is the characteristic function of $[-1/2, 1/2]$. Clearly, $B_\ell \in C^{\ell-2}(\mathbb{R})$, $\ell \ge 2$, and is of finite support.

We will make use of the Fourier transform of $f \in L_1(\mathbb{R})$ given by

$$\hat{f}(v) := \int_{\mathbb{R}} f(u)\,e^{-ivu}\,du, \qquad v \in \mathbb{R},$$

and the convolution of $f, g \in L_1(\mathbb{R})$

$$f * g(u) := \int_{\mathbb{R}} f(u-v)g(v)\,dv, \qquad u \in \mathbb{R}.$$

Following Butzer and Stens [7, Section 5.2.2] we will show that given any positive integer $s \ge 2$ there exists a kernel, which is a linear combination of translates of central B-splines and thus of finite support, such that the corresponding Kantorovich sampling operator has a rate of simultaneous approximation estimated above by the modulus of smoothness of order s. Moreover, the support of the kernel can be pre-fixed arbitrarily; in particular, so that the samplings are taken only from the past.

Theorem 5.1. *Let $r, n, s \in \mathbb{N}_0$, $n \le r$, $s \ge 2$, $1 \le p \le \infty$, $\alpha, \beta \ge 0$, $b_\mu \in \mathbb{R}$, $\mu = 0, \dots, s-1$, be such that $b_0 < \cdots < b_{s-1}$, and let $a_\mu \in \mathbb{R}$, $\mu = 0, \dots, s-1$ be the unique solution of the linear system*

$$\sum_{\mu=0}^{s-1} \Big(\frac{1}{2} - b_\mu\Big)^j a_\mu = (-i)^j \bigg(\frac{1}{\widehat{B_{r+s+1}}}\bigg)^{(j)}(0), \qquad j = 0, \dots, s-1. \tag{5.2}$$

We set

$$\big(S_w^\psi f\big)(x) := \sum_{k\in\mathbb{Z}} w \int_{k/w}^{(k+1)/w} f(u)\, du\, \psi(wx-k) \ \text{ with } \psi(t) := \sum_{\mu=0}^{s-1} a_\mu B_{r+s}(t-b_\mu).$$

Then for all $f \in L_{1,loc}(\mathbb{R})$ with $f \in AC_{loc}^{n-1}(\mathbb{R})$ and $\rho_{\alpha,\beta} f^{(n)} \in L_p(\mathbb{R})$, and all $w \ge 1$ there hold $S_w^\psi f \in C^n(\mathbb{R})$, $\rho_{\alpha,\beta}(S_w^\psi f)^{(n)} \in L_p(\mathbb{R})$ and

$$\big\|\rho_{\alpha,\beta}\big((S_w^\psi f)^{(n)} - f^{(n)}\big)\big\|_p \le c\, \omega_s(f^{(n)}, 1/w)_{p,\alpha,\beta}.$$

The assumption $f \in AC_{loc}^{n-1}(\mathbb{R})$ is to be ignored for $n = 0$.

Proof. To recall, $B_{\ell+1} = B_1 * B_\ell$ and

$$\widehat{B_\ell}(v) = \Big(\frac{\sin(v/2)}{v/2}\Big)^\ell, \qquad v \in \mathbb{R}, \quad \ell \in \mathbb{N}_+,$$

(see e.g. [7, (5.2.10)] or [12, Theorem 4.33]).

First, let us note that the determinant of the linear system (5.2) is of Vandermonde type; hence it has a unique solution. Moreover, it is real because the Fourier transform of B_{r+s+1} is an even function, so is its reciprocal and hence the right hand side of (5.2) is equal to 0 if j is odd—see [7, p. 168].

We will show that the kernel ψ satisfies the assumptions in Theorem 1.1. Then this theorem will imply Theorem 5.1.

Clearly, ψ satisfies the assumptions of smoothness and decay at infinite for any $n = 0, \dots, r$ in place of r.

It remains to show that ψ satisfies assumption (ii) in Theorem 1.1 with any $n = 0, \dots, r$ in place of r. Since $t_k = k$, it takes the form given in Remark 1.1. As it was shown in the proof of [4, Lemma 5.3], the latter is equivalent to

$$\widehat{\overline{\psi}}^{(j)}(2\pi k) = \begin{cases} 1, & k = j = 0, \\ 0, & k = 0,\ j = 1, \dots, s-1, \\ 0, & k \in \mathbb{Z},\ \ k \ne 0,\ \ j = 0, \dots, r+s, \end{cases} \tag{5.3}$$

where $\overline{\psi}(u) := \kappa * \psi(u)$ with $\kappa(u) := B_1(u + 1/2)$, that is, $\kappa(u)$ is the characteristic function of the interval $[-1, 0]$. Consequently, if ψ satisfies (ii) with some r, then it satisfies it for any $n < r$ in place of r as well. Thus to prove the theorem it remains to show that $\overline{\psi}$ satisfies (5.3).

Just as in the proof of [7, Theorem 3] we show that the function (cf. [7, (5.2.18)])

$$\varphi(u) := \sum_{\mu=0}^{s-1} a_\mu B_{r+s+1}(u-\varepsilon_\mu), \qquad u \in \mathbb{R},$$

where $\varepsilon_\mu := b_\mu - 1/2$, satisfies the relations

$$\widehat{\varphi}^{(j)}(2\pi k) = \begin{cases} 1, & k=j=0, \\ 0, & k=0, \quad j=1,\dots,s-1, \\ 0, & k\in\mathbb{Z}, \quad k\neq 0, \quad j=0,\dots,r+s. \end{cases}$$

In this regard, let us note that (5.2) can be written in the form (see [7, (5.2.17)])

$$\sum_{\mu=0}^{s-1} a_\mu(-i\varepsilon_\mu)^j = \left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(j)}(0), \qquad j=0,\dots,s-1.$$

The Fourier transforms of $\overline{\psi}$ and φ are

$$\widehat{\overline{\psi}}(v) = \widehat{\kappa}(v)\widehat{\psi}(v) = e^{iv/2}\Big(\widehat{B_1}(v)\Big)^{r+s+1} \sum_{\mu=0}^{s-1} a_\mu e^{-ib_\mu v}$$

and

$$\widehat{\varphi}(v) = \Big(\widehat{B_1}(v)\Big)^{r+s+1} \sum_{\mu=0}^{s-1} a_\mu e^{-i\varepsilon_\mu v}.$$

Thus we have $\widehat{\overline{\psi}}(v) = \widehat{\varphi}(v)$ for all $v \in \mathbb{R}$ and hence, in view of the uniqueness of the Fourier transform and the continuity of $\overline{\psi}$ and φ, we arrive at $\overline{\psi}(u) = \varphi(u)$ for all $u \in \mathbb{R}$.

Thus we have shown that $\overline{\psi}$ satisfies (5.3) and, therefore, ψ satisfies assumptions (ii) in Theorem 1.1. □

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