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Modeling of Natural Disasters via Cellular Neural Networks Approach

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ABSTRACT: In this paper we present several models of tsunami waves and tornado. We study shallow water waves. Two-component Camassa-Holm type system which admits peaked traveling waves is considered. Then we study two-dimensional Kuramoto-Tsuzuki equation as a model of tornado. Cellular Neural Network (CNN) approach is applied in order to study the structure of the traveling waves. Numerical simulations of the CNN models in both -tsunami and tornado models are presented.

1 INTRODUCTION

The study of propagation of tsunami from their small disturbance at the sea level to the size they reach approaching the coast has involved the interest of several scientists. It is clear that in order to predict accurately the appearance of a tsunami it is fundamental to built up a good model. From this point of view the most important tool in the context of water waves is soliton theory [8]. Frequently in the literature it is stated that a tsunami is produced by a large enough soliton. Solitons arise as special solutions of a widespread class weakly nonlinear dispersive PDEs modeling water waves, such as the KdV or Camassa-Holm equation [3], representing to various degrees of accuracy approximations to the governing equations for water waves in the shallow water regime. How the tsunami is initiated? The thrust of a mathematical approach is to examine how a wave, once initiated, moves, evolves and eventually becomes such a destructive force of nature. We aim to describe how an initial disturbance gives rise to a tsunami wave. Let h is the average depth of the water, λ is the typical wavelength of the wave and a is a typical amplitude. There are two important parameters: $\varepsilon = \frac{a}{h}$, called amplitude parameter, and the shallowness parameter $\delta = \frac{h}{\lambda}$. According to these parameters a rigorous validity ranges are obtained [1] for the main physical regimes encountered in modelling of two-dimensional water waves:

1. shallow-water, large amplitude ($\delta \ll 1$, $\varepsilon \sim 1$), leading at first order to the shallow-water equations [11] and at second order to the Green-Naghdi model [10];

2. shallow-water, medium amplitude regime ($\delta \ll 1$, $\varepsilon \sim \delta$) leading to the Serre equations [16] and to the Camassa-Holm equation [3];

3. shallow-water, small amplitude or long-wave regime ($\delta \ll$, $\varepsilon \sim \delta^2$) leading at first order to the linear wave equation

$$\varphi_{tt} - \varphi_{xx} = 0 \tag{1}$$

with general solution

$$\varphi(x,t) = \varphi_+(x-t) + \varphi_-(x+t), \tag{2}$$

where the sign $\pm$ refers to a wave profile $\varphi_\pm$ moving with unchanged shape to the right/left at constant unit speed. The small effects that were ignored at first order (small amplitude, long wave) build up on longer time/spatial scales to have a significant cumulative nonlinear effect so that on a longer time scale each of the waves that make up the solution (2) to (1) satisfies the KdV equation [8].

In this paper we shall present one more model - tornado dynamics. Observations of the tornado have a rich history, provided by many papers only for the 20th century. Brooks was the first observer, who put forward generally accepted assumption, that the funnel is a part of the parent cloud, the structure and dynamics of which represent a small tropical storm and having a helical structure[2]. Numerous observations of the parent cloud indicated the presence of long vortices in the horizontal plane in them; the vertical poles (funnels) are the continuation of which. This fact has no explanation. In 1951 in Texas during a tornado the funnel, passing over the observer, rose, and its edge was at the height of 6 meters with the inner cavity diameter of 130 meters. The wall thickness was the size of 3 meters. Vacuum in the cavity was absent, because it was easy to breathe during its passage. The walls was extremely fast spinning (Justice, 1930). Observations of the actual tornado, therefore, indicate a strong non- linearity and non-equilibrium of processes in atmosphere during the formation and existence of a tornado, that does not allow to create the perfect model of this exotic phenomenon. In the framework of the study of this unusual natural phenomenon the following questions are need to be answered:

1. Under what conditions in the atmosphere the appearance of a tornado happens?

2. What causes the existence of distinct lateral boundaries of the tornado? Why don't these boundaries spread in time?

3. What are the conditions for the existence of dissipative structures observed in the tornado a set of organized vortices? And what are their conditions of decay?

4. What causes the boundedness of tornadoes in height?

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5. What are the conditions for the existence and stability of stationary and others possible modes.
6. What determines the appearance of the tornado core (trunk) with significantly higher velocities?

For the mathematical modeling of highly non-equilibrium and nonlinear processes in a tornado authors propose the approach based on the nonlinear equations of momentum transfer with the model sources and sinks function. This approach can be assigned to the problems with peaking considered by Academician A. A. Samarskii[15]. For the first time the thermo- dynamic description was used to identify new principles of self-organization in the atmosphere in the model specification not entered before.

To gain some insight into the involved processes we have set up a numerical approach via Cellular Neural Networks (CNN) that treats a vortex as a fluid dynamical system.

2 STUDY OF THE SHALLOW WATER WAVES

An interesting phenomena in water channels is the appearance of waves with length much greater than the depth of the water. Korteweg and de Vries started the mathematical theory of this phenomenon and derived a model describing unidirectional propagation of waves of the free surface of a shallow layer of water. This is the well known KdV equation:

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0, & t > 0, \quad x \in \mathbf{R}, \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathbf{R} \end{cases}$$

where u describes the free surface of the water; for a presentation of the physical derivation of the equation. The beautiful structure behind the KdV equation initiated a lot of mathematical investigations.

Recently, Camassa and Holm proposed a new model for the same phenomenon:

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbf{R}, \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathbf{R}. \end{cases}$$

The variable $u(t,x)$ in the Camassa-Holm (CH) equation represents the fluid velocity at time t in the x direction in appropriate nondimensional units (or, equivalently, the height of the water's free surface above a flat bottom). Unlike KdV, which is derived by asymptotic expansions in the equation of motion, CH is obtained by using an asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime. The novelty of the Camassa and Holm's work was the physical derivation of CH equation and the discovery that the equation has solitary waves (solitons) that retain their individuality under interaction and eventually emerge with their original shapes and speeds.

As an alternative model to KdV, Benjamin, Bona and Mahoney [19] proposed the so-called BBM-equation:

$$u_t + u_x + uu_x - u_{xxt} = 0, t > 0, x \in \mathbf{R}.$$

Numerical work of Bona, Pritchard and Scott [19] shows that the solitary waves of the BBM-equation are not solitons.

As noted by Whitham [19], it is intriguing to find mathematical equations including the phenomena of breaking and peaking, as well as criteria for the occurrence of each. He observed that solutions of the KdV-equation do not break as physical water waves do. Whitham suggested to replace the KdV-model by the nonlocal equation

$$u_t + uu_x + \mathbf{K} = 0, t > 0, x \in \mathbf{R},$$

for which he conjectured that breaking solutions exist. Here $\mathbf{K}$ is a Fourier operator with symbol $k(\xi) = \sqrt{(tanh\xi)/\xi}$. Whitham's conjecture was proved [19]. The numerical calculations carried out for the Whitham equation do not support any strong claim that soliton interaction can be expected.

On the other hand, Camassa, Holm and Hyman [3] show that the solitary waves have a discontinuity in the first derivative at their peak and that soliton interactions occur in CH equation. The advantage of the new equation in comparison with the well-established models KdV, BBM and the Whitham equation is clear: The Camassa-Holm equation has peaked solitons, breaking waves, and permanent waves.

In order to derive the model equation of tsunami wave we assume an initial disturbance of the form of a two-dimensional wave and we are interested in understanding the dynamics of the wave as it propagates across the ocean. Choose Cartesian coordinates (X, Y) with the Y-axis pointing vertically upwards, the X-axis being the direction of wave propagation, and with the origin located on the mean water level $Y = 0$. Let $(U(X,Y,T), V(X,Y,T))$ be the velocity field of the two-dimensional flow propagating in the X-direction over the flat bed $Y = -h$, and let $Y = H(X,T)$ be the water's free surface with mean water level $Y = 0$. The equation of mass conservation

$$U_X + V_Y = 0$$

is a consequence of assuming constant density, a physically reasonable assumption for gravity water waves.Under the assumption of inviscid flow (which is realistic since experimental evidence confirms that the length scales associated with an adjustment of the velocity distribution due to laminar viscosity or turbulent mixing are long compared to typical wave-lengths) the equation of motion is Euler's equation:

$$\begin{cases} U_T + UU_X + VU_Y = -\frac{1}{\rho}P_X, \\ V_T + UV_X + VV_Y = -\frac{1}{\rho}P_Y - g, \end{cases}$$

where P is the pressure, g is the constant acceleration of gravity and ρ is the constant density of water. We also have the boundary conditions $P = P_{atm}$ on $Y = H(X,T)$, where P_{atm} is the (constant) atmospheric pressure at the water's free surface, $V = H_T + UH_X$ on $Y = H(X,T)$, and $V = 0$ on $Y = -h$. This conditions express the fact that water particles can not cross the free surface, respectively, the impermeable rigid bed, while $P = P_{atm}$ decouples the motion of the water from that of the air above it in the absence of surface tension; for wavelength larger than a few mm (and in our case we deal with hundreds of km) the effects of surface tension are known to be negligible. We will consider irrotational flows with zero vorticity

$$U_Y - V_X = 0,$$

a hypothesis that allows for uniform currents but neglects the effects of non-uniform currents in the fluid.

Finding exact solutions to the nonlinear governing equations for water waves is not possible even with the aid of the most advanced computers. In order to derive approximations to the governing equations it is useful to write them in non-dimensional form. We assume that the two-dimensional waves under investigation have acquired a certain pattern. We assume that the wave pattern under investigation represents a weakly irregular perturbation of a wave train in the sense that averages over suitable times/distances resemble a wave train.Since h is the average depth of the water, the non-dimensionalisation Y_0 of Y should be $Y = hy$, which is to be understood as replacing the dimensional, physical variable Y by hy, where y is now a non-dimensional version of the original Y. The non-dimensionalisation of the horizontal spatial variable is also obvious; if λ is some average of typical wavelength of the wave, we set $X = \lambda x$. The corresponding non-dimensionalsation of time is $T = \frac{\lambda}{\sqrt{gh}}t$.

Then the governing equation for irrotational water waves equations in nondimensionalized form is:

$$\begin{cases} \delta^2\Phi_{xx} + \Phi_{yy} = 0 & \text{in} \quad \Gamma(t), \\ \Phi_y = 0, & \text{on} \quad y = -1, \\ \xi_t + \varepsilon\xi_x\Phi_x + \frac{\varepsilon}{\delta^2}\phi_y = 0 & \text{on} \quad y = \varepsilon\xi, \\ \Phi_t + \frac{\varepsilon}{2}\Phi_x^2 + \frac{\varepsilon}{2\delta^2}\Phi_y^2 + \xi = 0 & \text{on} \quad y = \varepsilon\xi, \end{cases}$$

where $x \mapsto \varepsilon\xi(x,t)$ is a parametrization on the free surface at time t, $\Gamma(t) = \{(x,y), -1 < y < \varepsilon\xi(x,t)\}$ is the fluid domain delimited above by the free surface and below by the flat bed $\{y = -1\}$, and where $\Phi(.,.,t) : \Gamma(t) \to \mathbf{R}$ is the velocity potential associated to the flow, so that the two-dimensional velocity field is given by (Φ_x, Φ_y).

3 TRAVELING WAVE SOLUTIONS FOR TOW-COMPONENT CAMASSA-HOLM TYPE SYSTEM

Our aim in this section is to find model equations which admit peaked travelling waves: waves that are smooth except at their crest and which capture therefore the main features of the waves of greatest height encountered as solutions to governing equations for water waves [8].

The motion of inviscid fluid with a constant density is described by the well-known Euler's equations. On the other hand, the motion of a shallow water over a flat bottom is described by a 3×3 semilinear system of first order partial differential equations. Recently model of the shallow water waves was derived, Camassa-Holm (CH) equation:

$$u_t - u_{txx} + ku_x + 3uu_x = 2u_xu_{xx} + uu_{xxx},$$

attracted a lot of attention. CH is also an integrable equation and its solitary waves are smooth if $k > 0$ and become peaked in the limit $k \to 0$. Moreover, the shape of these peaked waves remains stable under small perturbation- these waves are orbitally stable - so that these wave patterns are recognizable. We point our also that CH models waves in the presence of vorticity, arising thus as a model of wave-current interactions. This leads to the idea that CH equation might be relevant to the modeling of tsunamis.

A recent development related to CH consists in its extension to an integrable two-component system of the form:

$$\left| \begin{array}{l} m_t + 2u_xm + um_x + \rho\rho_x = 0 \\ \rho_t + (u\rho)_x = 0, \text{where} \quad m = u - u_{xx}. \end{array} \right. \tag{3}$$

We are looking for travelling wave solutions of (3), i. e. $u(t,x) = u(\xi)$, $m(t,x) = m(\xi)$, $\xi = x - ct$, $c = const$. Substituting $u(\xi)$, $m(\xi)$ in (2.19) we get from the second equation that $-c\rho' + (u\rho)' = 0 \Rightarrow c\rho(\xi) = u(\xi)\rho(\xi) - \alpha$, $\alpha = const \Rightarrow \rho = \frac{\alpha}{u-c}$; $m = u - u''$. Therefore, the first equation in (3) implies:

$$-cm' + (um)' + u'm + \rho\rho' = 0 \Rightarrow$$

$$-c(u' - u''') + (um)' + u'(u - u'') + \rho\rho'' = 0$$

and after an integration with respect to ξ we get that

$$-cu + cu'' + u^2 - uu'' + \frac{1}{2}u^2 - \frac{1}{2}(u')^2 + \frac{1}{2}\rho^2 = \frac{\beta}{2} = const,$$

i. e.

$$-cu + cu'' + \frac{3}{2}u^2 - uu'' - \frac{1}{2}(u')^2 + \frac{\alpha^2}{(u-c)^2} = \frac{\beta}{2}. \tag{4}$$

Assuming $c > 0$ we make in (4) the change $u = c(1+z)$. In the case $c < 0$ we shall make the change $u = c(1-z)$. We shall confine ourselves to the case when $c > 0$ and $z = \frac{u}{c} - 1$ as the other case is treated similarly. Consequently, $\rho = \frac{\alpha}{cz}$, $z \neq 0$ and then (4) can be written as:

$$-c^2(1+z) + c^2 z^{''} + \frac{3}{2}c^2(1+z)^2 - c^2(1+z)z^{''} - \frac{1}{2}c^2(z^{'})^2 + \frac{1}{2}\frac{\alpha^2}{c^2z^2} = \frac{\beta}{2} \Rightarrow$$

$$zz^{''} + \frac{1}{2}(z^{'})^2 - \frac{3}{2}z^2 - 2z - \frac{\alpha^2}{2c^4z^2} + \frac{\beta - c^2}{2c^2} = 0. \qquad (5)$$

We multiply (5) by $z^{'}$ then integrate in ξ and having in mind that $zz^{'}z^{''} + \frac{1}{2}(z^{'})^3 = \frac{1}{2}(z(z^{'})^2)'$ we obtain:

$$\frac{1}{2}z(z^{'})^2 - \frac{z^3}{2} - z^2 + \frac{\alpha^2}{2c^4z} + \frac{\beta - c^2}{2c^2}z = \frac{\gamma}{2} = const,$$

i. e.

$$z^2(z^{'})^2 = z^4 + 2z^3 + \frac{c^2 - \beta}{c^2}z^2 + \gamma z - \frac{\alpha^2}{c^4}. \qquad (6)$$

Put $\frac{c^2-\beta}{c^2} = \beta_1 \Leftrightarrow \beta = c^2(1-\beta_1)$, $-\frac{\alpha^2}{c^4} = \alpha_1 < 0 \Leftrightarrow \alpha = \pm c^2\sqrt{-\alpha_1}$, i. e. $\alpha \to 0 \Leftrightarrow \alpha_1 \to 0$.

Let us fix $c > 0$. Then $\beta_1 \in \mathbf{R}$, $\gamma \in \mathbf{R}$ and $\alpha_1 < 0$ are arbitrary real constants.

Put $P_4(z) = z^4 + 2z^3 + \beta_1 z^2 + \gamma z + \alpha_1$.

If $\gamma = 0$ we write $\tilde{P}_4(z) = z^4 + 2z^3 + \beta_1 z^2 + \alpha_1$. Define now $w = \tilde{\tilde{P}}_4(z) = z^2(z^2 + 2z + \beta_1)$. Evidently, $\tilde{\tilde{P}}_4(z) = 0 \Leftrightarrow z_{1,2} = 0$, $z_{3,4} = -1 \pm \sqrt{1-\beta_1}$. We shall assume further on that

$$0 < \beta_1 < 1 \Leftrightarrow 0 < 1 - \beta_1 < 1 \Leftrightarrow -1 < z_3 < 0,\ -2 < z_4 < -1. \qquad (7)$$

Consider now the cross points of the biquadratic parabola $w = \tilde{\tilde{P}}_4(z)$ and the straight line $w = -\alpha_1 > 0$, $0 < -\alpha_1 \ll 1 \Leftrightarrow 0 < |\alpha| \ll 1$. Geometrically we have the Fig.1.

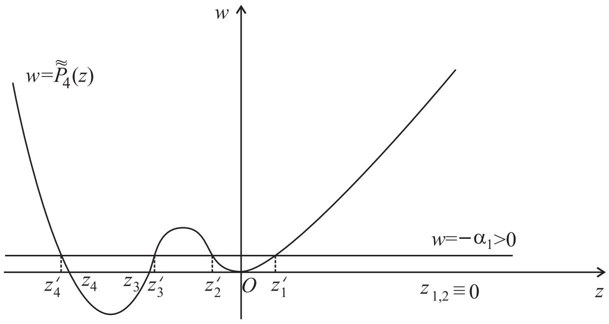

Figure 1:

Evidently, the curve $w = \tilde{\tilde{P}}_4(z)$ crosses the line $w = -\alpha_1$, $|\alpha_1| \ll 1$ at the points $z_4^{'} < z_4 < z_3 < z_3^{'} < z_2^{'} < 0 < z_1^{'}$. Therefore $\tilde{P}_4(z){=}0 \Leftrightarrow \tilde{P}_4(z_j^{'}) = 0$, $z_{j,\,1\le j\le 4}$ being 4 simple roots of the algebraic equation $\tilde{P}_4(z) = 0$. As it is known from Analysis, the simple roots of the algebraic equations depend continuously on the coefficients of the corresponding polynomials. This way we come to the Proposition 1.

Proposition 1. *Consider the 4^{th} order polynomial $P_4(z)$, fix the constant $c > 0$ and suppose that $0 < \beta_1 < 1$. Then one can find some $0 < \varepsilon_0$ such that if $|\gamma| \le \varepsilon_0$, $|\alpha| \le \varepsilon_0$, then $P_4(z) = 0$ has 4 simple roots $z_4^{'} < z_3^{'} < z_2^{'} < 0 < z_1^{'}$.*

According to [13] the equation

$$\left| \begin{array}{l} (z^{'})^2 = \frac{P_4(z)}{z^2} \ge 0 \\ z(0) = z_0 \in [z_3^{'},\, z_2^{'}] \end{array} \right. \qquad (8)$$

possesses a smooth periodic solution $z(\xi)$, $z(0) = z_0$, $z_3^{'} \le z(\xi) \le z_2^{'}$, i. e. $z \neq 0$. In fact, $P_4(z) \ge 0$ for $z \in [z_3^{'},\, z_2^{'}]$ (see Fig.1), while $P_4(z) < 0$ for $z \in (z_4^{'},\, z_3^{'})$ or $z \in (z_2^{'},\, z_1^{'})$. At first we construct the solution of $z^{'} = \frac{\sqrt{P_4(z)}}{-z} > 0$, $z(0) = z_0$ as $z \in [z_3^{'},\, z_2^{'}] \Rightarrow -z > 0$, i. e.

$$\xi = \int_{z_3^{'}}^{z} \frac{-\lambda d\lambda}{\sqrt{P_4(\lambda)}} = H(z) \quad \text{and} \quad \text{for} \quad z_0 = z_3^{'}. \qquad (9)$$

Evidently, $H^{'}(z) > 0$ for $z \in (z_3^{'},\, z_2^{'})$, $H'(z_3') = \infty$, $H'(z_2') = \infty$, i. e. $\left(H^{-1}(\xi)\right)' > 0$, $H(z_3') = 0$. Put $0 < \dfrac{T}{2} = \displaystyle\int_{z_3'}^{z_2'} \frac{-\lambda d\lambda}{\sqrt{P_4(\lambda)}} \Rightarrow H(z_2') = \dfrac{T}{2}$, $z = z(\xi)$, $0 \le \xi \le \dfrac{T}{2}$, $z'(0) = z'\left(\dfrac{T}{2}\right) = 0$. Then we continue $z(\xi)$

as an even function on the interval $-\frac{T}{2} \le \xi \le 0$, i. e. $z(-\xi) = z(\xi)$. One can see easily that $z(\xi)$ satisfies the ODE $z'^2 = \frac{P_4(z)}{z^2}$ on $\left[-\frac{T}{2}, \frac{T}{2}\right]$. Our last step is to continue $z(\xi)$ as a smooth periodic function with period T on the real line $\mathbf{R}^1_\xi$. According to [13], for $z'_3 \le z \le z'_2$

$$-\xi = -H(z) = \int_{z'_3}^{z} \frac{\lambda d\lambda}{\sqrt{P_4(\lambda)}} = \frac{2}{\sqrt{(z'_1 - z'_3)(z'_2 - z'_4)}} \left\{ (z'_3 - z'_4) \times \Pi\left(\delta, \frac{z'_2 - z'_3}{z'_2 - z'_4}, q\right) + z'_4 F(\delta, q) \right\},$$

where $q = \sqrt{\frac{(z'_2 - z'_3)(z'_1 - z'_4)}{(z'_1 - z'_3)(z'_2 - z'_4)}}$, $\delta = \arcsin\sqrt{\frac{(z'_2 - z'_4)(z - z'_3)}{(z'_2 - z'_3)(z - z'_4)}}$ and F, Π are respectively (see [13]). Certainly, $H(z) = |\xi|$ for $|\xi| \le \frac{T}{2}$, $z'_3 \le z \le z'_2$.

This way we expressed a class of periodic solutions of (3) – travelling wave type – by the famous Legendre's elliptic functions. In fact $z \neq 0$ is periodic with period T and $u = c(1+z)$, $\rho = \frac{\alpha}{cz}$.

Remark 1. *Consider the ODE (8), where* $P_4(z) = z^4 + 2z^3 + \beta_1 z^2 + \gamma z + \alpha_1$, $\beta_1, \gamma, \alpha_1 < 0$ *being arbitrary constants. Certainly,* $P_4(z) = 0$ *possesses at least two real roots as* $P_4(0) = \alpha_1 < 0$. *We assume that* $k_1 = k_2 < 0$ *is a double root of* $P_4(z)$*, while* $k_1 < k_3 < 0 < k_4$ *are simple roots. We have supposed that* $k_3 < 0 < k_4$ *as* $k_1^2 k_3 k_4 = \alpha_1 < 0$. *Therefore,* $k_1 < z < k_3 \Rightarrow P_4(z) > 0$. *Then the Cauchy problem*

$$\left| \begin{array}{l} z^2 z'^2 = P_4(z) = (z - k_1)^2 (z - k_3)(z - k_4) \\ z(0) = k_3 \end{array} \right.$$

possesses a smooth solution $z(\xi)$, $\xi \in \mathbf{R}^1$*, such that* $z(-\xi) = z(\xi)$, $\forall \xi \in \mathbf{R}^1$, $z'(0) = 0$, $z'(\xi) < 0$ *for* $\xi > 0$. *This is a soliton, of course (see Fig.2). Moreover, we can give an explicit formula for the solution as the corresponding integral* $\xi = \int_{k_3}^{z} \frac{\lambda d\lambda}{(\lambda - k_1)\sqrt{(\lambda - k_3)(\lambda - k_4)}} \equiv H_1(z) > 0$, $k_1 < z < k_3$, $H'_1(z) < 0$, $H_1(k_3) = 0$, $\lim_{z \to k_1} H_1(z) = +\infty$, $H'_1(k_3) = -\infty$ *can be calculated by using the standard Euler's substitutions.*

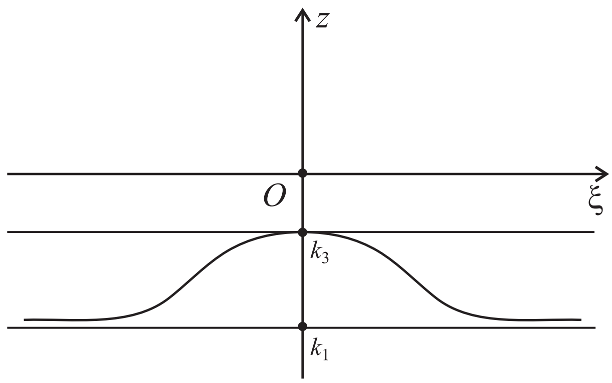

Figure 2:

The case when $P_4(z)$ has a triple root or a pair of complex roots $w, \overline{w} \in \mathbf{C}^1 \backslash \mathbf{R}^1$ can be studied in a similar way and we omit the details. They are left to the reader. We complete our study. We are ready now to discuss more general problems concerning larger classes of travelling wave solutions.

4 MODELING TORNADO DYNAMICS

In the non-equilibrium thermodynamics it is accepted to characterize the processes within the system under the influence of the external environment by the so-called entropy production σ^i per unit volume of the layer. There are also other local thermodynamic characteristics- the external ow of entropy σ^ε and the rate of change of entropy $\dot{S}$, equal to their sum. It is believed that in the self-organization systems full change of the entropy decreases with time: $\dot{S} < 0$. This approach allows one to record the system of equations for the velocity components in the case of an incompressible fluid in dimensionless form as two-dimensional Kuramoto-Tsuzuki equation[15] for the atmospheric layer:

$$\frac{\partial \Phi}{\partial t} = v_1(1 + ic_1)\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) + q\Phi - \alpha_1(1 + ic_2)|\Phi|^2 \Phi \tag{10}$$

where $\Phi = \nu_x + i\nu_y$, v_2/v_1 is related to viscosity, α_2/α_1 due to sinks.

Geometrically we have the following tornado pressure field oscillation. On the Figure 3 isobaric surfaces and the pressure gradient field (orthogonal to the surface) are displayed [15].

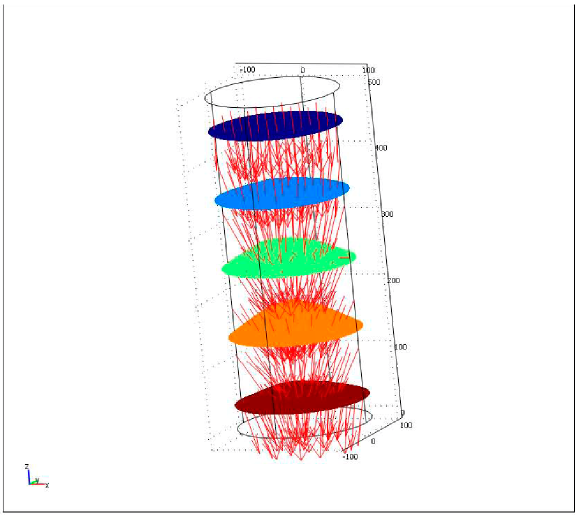

Figure 3: Tornado isobaric surfaces.

5 CELLULAR NEURAL NETWORK APPROACH

It is known [4,5,14,17] that some autonomous CNNs represent an excellent approximation to nonlinear partial differential equations (PDEs). The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate . An one-dimensional discretized Laplacian template will be in the following form:

$$A_1 = (1, -2, 1),$$

This is the two-dimensional discretized Laplacian A_2 template:

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let us use 2-D grid given on Fig.3:

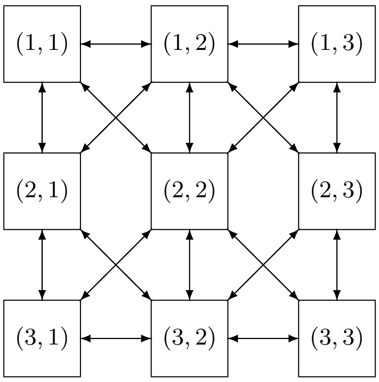

Figure 4: 3×3 CNN.

The squares are the circuit units - cells, and the links between the cells indicate that there are interactions between linked cells. One of the key features of a CNN is that the individual cells are nonlinear dynamical systems, but that the coupling between them is linear. Roughly speaking, one could say that these arrays are nonlinear but have a linear spatial structure, which makes the use of techniques for their investigation common in engineering or physics attractive.

We will give the general definition of a CNN which follows the original one:

Definition 1. *The CNN is a*

a). 2-, 3-, or n- dimensional array of

b). mainly identical dynamical systems, called cells, which satisfies two properties:

c). most interactions are local within a finite radius r, and

d). all state variables are continuous valued signals.

Definition 2. *An $M \times M$ cellular neural network is defined mathematically by four specifications:*

1). CNN cell dynamics;

2). CNN synaptic law which represents the interactions (spatial coupling) within the neighbor cells;
3). Boundary conditions;
4). Initial conditions.

Now in terms of definition 2 we can present the dynamical systems describing CNNs. For a general CNN whose cells are made of time-invariant circuit elements, each cell $C(ij)$ is characterized by its CNN cell dynamics :

$$\dot{x}_{ij} = -g(x_{ij}, u_{ij}, I^s_{ij}), \tag{11}$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usually a scalar. In most cases, the interactions (spatial coupling) with the neighbour cell $C(i+k, j+l)$ are specified by a CNN synaptic law:

$$\begin{aligned} I^s_{ij} &= A_{ij,kl}x_{i+k,j+l} + \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \\ &+ \tilde{B}_{ij,kl} * u_{i+k,j+l}(t). \end{aligned} \tag{12}$$

The first term $A_{ij,kl}x_{i+k,j+l}$ of (12) is simply a linear feedback of the states of the neighborhood nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood.

Definition 3. *For any cloning template A which defines the dynamic rule of the cell circuit, we define the convolution operator $*$ by the formula*

$$A * z_{ij} = \sum_{C(k,l)\in N_r(i,j)} A(k-i, l-j)z_{kl},$$

where $A(m, n)$ denotes the entry in the pth row and rth column of the cloning template, $p = -1, 0, 1$, and $r = -1, 0, 1$, respectively.

6 TRAVELING WAVE SOLUTIONS OF OUR CNN MODELS

Our next step is to construct CNN model of the two-component Camassa-Holm type system (3). In our case we have the following $N \times N$ CNN system:

$$\left| \begin{array}{l} \frac{du_{ij}}{dt} - \frac{d}{dt}(A_2 * u_{ij}) + 2A_1 * u_{ij}(u_{ij} - A_2 * u_{ij}) + \rho_{ij}A_1 * \rho_{ij} = 0 \\ \frac{d\rho_{ij}}{dt} + \rho_{ij}A_1 * u_{ij} + u_{ij}A_1 * \rho_{ij} = 0 \end{array} \right. , \tag{13}$$

where $1 \leq i, j \leq N$.

Our objective in this paper is to study the structure of the travelling wave solutions of the CNN model of two component Camassa-Holm type system (13). We shall study the travelling wave solutions of the CNN model (13) of the form:

$$\left| \begin{array}{l} u_{ij} = \Theta_1(icos\Omega + jsin\Omega - ct), \\ \rho_{ij} = \Theta_2(icos\Omega + jsin\Omega - ct), \end{array} \right. \tag{14}$$

for some continuous functions $\Theta_1, \Theta_2 : \mathbf{R}^1 \to \mathbf{R}^1$ and for some unknown real number c. As we mentioned above $s = icos\Omega + jsin\Omega - ct$. Let us substitute (14) in our CNN model (13). Therefore we consider solution $\Theta_1(s; c)$, $\Theta_2(s; c)$ of:

$$\left| \begin{array}{l} -c\Theta_1'(s; c) + G_1(\Theta_1(s; c), \Theta_2(s; c)) = 0, \\ -c\Theta_2'(s; c) + G_2(\Theta_1(s; c), \Theta_2(s; c)) = 0, \end{array} \right. \tag{15}$$

where $G_1(\Theta_1, \Theta_2)$, $G_2(\Theta_1, \Theta_2) \in \mathbf{R}^1$ are satisfying

$$\left| \begin{array}{l} lim_{s\to\pm\infty}\Theta_1(s; c) = 0, \\ lim_{s\to\pm\infty}\Theta_2(s; c) = 0, \end{array} \right. \tag{16}$$

for some $c > 0$. We shall investigate the basic properties of the solutions of (13).

Suppose that our CNN model (13) are finite circular arrays of $L = N.N$ cells. For this case we have finite set of frequences [3]:

$$\Omega = \frac{2\pi k}{L}, \;\; 0 \leq k \leq L - 1. \tag{17}$$

The following proposition then hold:

Proposition 2. *Suppose that $u_{ij}(t) = \Theta_1(icos\Omega + jsin\Omega - ct)$, $\rho_{ij} = \Theta_2(icos\Omega + jsin\Omega - ct)$ are the travelling wave solutions of the CNN model (13) of the two component Camassa-Holm type system (3) with $\Theta_1, \Theta_2 \in C^1(\mathbf{R^1}, \mathbf{R^1})$ and $\Omega = \frac{2\pi k}{L}$, $0 \leq k \leq L - 1$. Then there exist constants $c > 0$ and $s_0 > 0$ such that*
(i) for $s < s_0$ the solutions $\Theta_1(s; c)$, $\Theta_2(s; c)$ of (15) satisfying (16) is increasing;
(ii) for $s > s_0$ the solutions $\Theta_1(s; c)$, $\Theta_2(s; c)$ of (15) satisfying (16) is decreasing;
(iii) for $s = s_0$ the solutions $\Theta_1(s; c)$, $\Theta_2(s; c)$ of (15) have maximum of angle type with positive opening, peakon.

Moreover, the solutions $\Theta_1(s;c)$, $\Theta_2(s;c)$ *are either non vanishing everywhere or compactly supported, i.e.* $\Theta_1(s;c) = 0$, $\Theta_2(s;c) = 0$ *for* $|s - s_0| \geq d$, *d being an appropriate positive constant.*

Remark 2. There has been many studies on travelling wave solutions of spatially and time discrete systems [12]. However, as far as we know travelling wave solutions of peakon type have been hardly studied in such discrete systems. For this reason we apply CNN approach and the numerical simulations of our CNN model (13) confirm the proposed results.

The simulations of the CNN model (13) give us the following Figure 5:

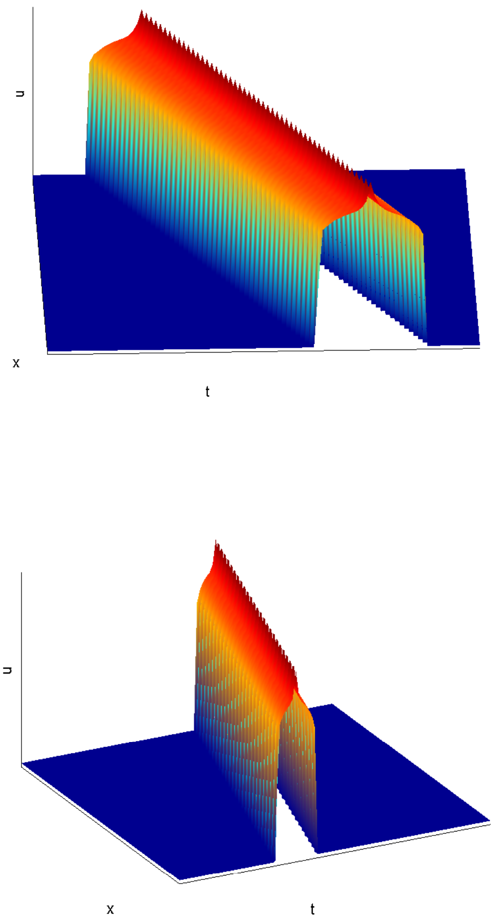

Figure 5: Peakon wave solution of (13).

CNN model of the two-dimensional Kuramoto-Tsuzuki equation (10) is:

$$\frac{d\Phi_{ij}}{dt} = v_1(1 + c_1)A_2 * \Phi_{ij} + q\Phi_{ij} - \alpha_1(1 + ic_2|\Phi_{ij}|^2\Phi_{ij}, \tag{18}$$

where A_2 is two-dimensional discretized Laplacian template, $*$ is a convolution operator [4]. After simulation of this model we get the following result:

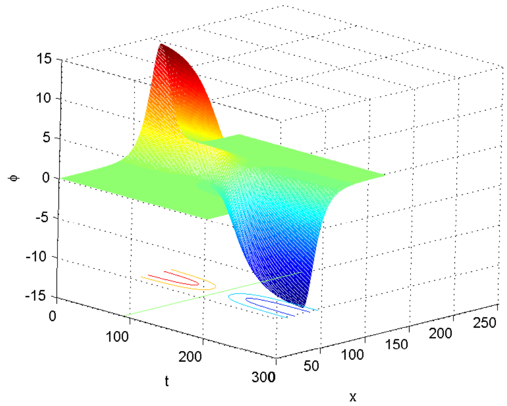

Figure 6: Simulation of CNN model of Kuramoto-Tsuzuki equation.

Remark 3. *We consider a CNN programmable realization allowing the calculation of all necessary processing steps in real time. The network parameter values of CNN models, are determined in a supervised optimization process.During the optimization process the mean square error is minimized using Powell method and Simulated Annealing [18]. The results are obtained by the CNN simulation system MATCNN applying 4th order Runge-Kutta integration.*

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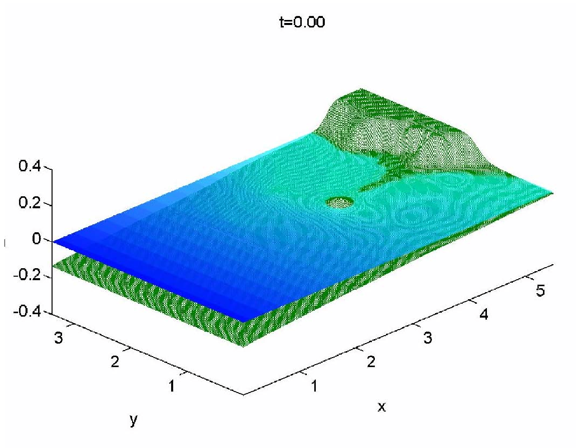
8 CONCLUSIONS AND DISCUSSION

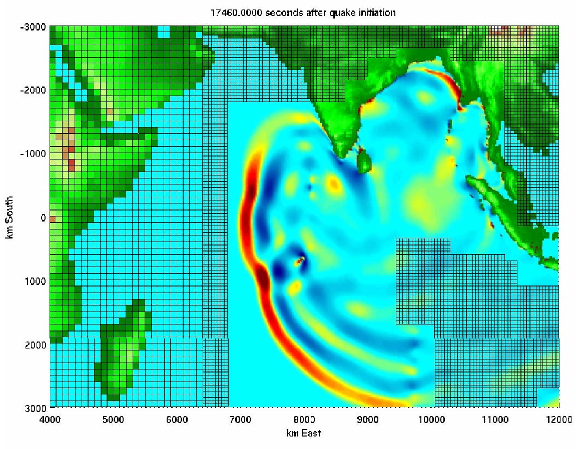
To quantify the dynamics of tsunami waves as they impact on costal areas is a challenging mathematical and physical problem of outmost importance. In this paper we discuss mathematical models of tsunamis and tornado- two-component CH type system and two-dimensional Kuramoto-Tsuzuki equation. In order to study the dynamics of our models we use CNN approach in order to discretized the governing equation over a suitable grid. We study the traveling wave solutions for the CNN model. For two-component CH system we obtain peakon solutions and present numerical simulations of the corresponding CNN model.

Let us conclude with a brief discussion of the wave dynamics as the tsunami propagate towards the coast. The previous considerations show that from initiation until reaching towards the costal region, a good approximation in non-dimensional variables of tsunami waves is provided by the solutions of the corresponding model equation. In the original physical variables this means that up until near-shore the wave profile remained unaltered propagating at constant speed $\sqrt{gh_0}$. The linear model breaks down when the tsunami waves enter the shallower water of the coastal regions and for understanding of the tsunamis close to the shore the appropriate equations are those modeling he propagation of long water over variable depth. Before the waves reach the breaking state, their front steepens and dispersion, no matter how weak, becomes relevant. In this region faster wave fronts can catch up slower ones(but they can never overtake them) as a manifestation of the ”confluence of shocks” and can result in large amplitude wave fronts building up behind smaller ones.

Let us take for example the tsunami of 2004 in the Indian Ocean. For modeling purposes, outside of the Bay of Bengal the two-dimensional character of the tsunami waves can not be taken any more more for granted since diffraction around islands and reflection from steep shores alter this feature considerably. The earthquake that generated the tsunami changed the shape of the ocean floor by raising it by a few m to the west of the epicenter and lowering it to the east (over 100 km in the east-west direction and about 900 km in the north-south direction).

The initial shape of the wave pattern that developed into the tsunami wave featured therefore to the west of the epicenter a wave of elevation followed by a wave of depression (that is, with water levels higher, respectively lower than normal), while to the east of the epicenter the initial wave profile consisted of a depression followed by an elevation.The fact that as the tsunami waves reached the shore in either direction, the shape of the initial disturbance (first wave of elevation, then wave of depression, respectively vice-versa) was not altered is of utmost importance in validating a theory for the wave dynamics on this occasion. This observation suggests that perhaps the shape of the tsunami waves remained approximately constant as they propagated across the Bay of Bengal. These clearly show a leading wave of elevation, followed by a wave of depression, a feature common both to the initial wave profile west of the epicenter and to the tsunami as it entered the coastal regions of India and Sri Lanka. These measurements also confirm another essential feature of tsunami waves: even though these waves reach large amplitudes due to the diminishing depth effect as they approach the shore (waves as high as 30m were observed near the city Banda Aceh on the west coast of the northern tip of Sumatra about 160 km away from the epicenter of the earthquake), tsunami waves are barely noticeable at sea due to





their small amplitude. Indeed, the satellite data shows that the maximum amplitude of the waves, whether positive or negative with respect to the usual sea level, was less than 0.8m over distances of more than 100 km.

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