Dynamic crack problems in functionally graded magnetoelectroelastic solids

Tsviatko Rangelov, Yonko Stoynov, Petia Dineva

An exponentially inhomogeneous transversely isotropic magnetoelectroelastic (MME) medium with a finite crack is studied. The crack is impermeable and subjected to anti-plane mechanical and in-plane electric and magnetic dynamic loads. The problem is solved by a non-hypersingular traction boundary integral equation method (BIEM) based on the usage of the analytically derived fundamental solution. A numerical scheme based on the collocation method and on the parabolic type of approximation of the field variables is proposed.
Program codes in Mathematica and Fortran are developed and validated by comparison tests for anisotropic elastic and piezoelectric materials. Illustrative examples reveal the dependence of the stress, electric and magnetic concentration fields near the crack-tips on the frequency and direction of the external load and on the magnitude and direction of the material gradient.

1 Introduction

The MEE composites are brittle and highly sensitive to the presence of defects like cracks, holes, impurities, etc. that can reach a critical size during service and thus compromise the structure safety, see Chue and Hsu [1].
The concept for functionally graded materials (FGM) was proposed in the last years, see Ma and Lee [2]. To enhance the promising applications, it is necessary to better understanding this new class of multifunctional intelligent composite materials in the context of their fracture state evaluation.
The solution of general boundary value problems for continuously inhomogeneous magneto-electric-elastic solids requires advanced numerical tool due to the high mathematical complexity arising from the electro-magneto-elastic coupling plus smooth variation of material characteristics.
The aim of this note is to propose nonhypersingular traction BIEM for the solution of the problem for wave propagation in a smooth exponentially inhomogeneous MEE plane with a finite crack subjected to an incident SH-wave. The BIEM technique is based on a frequency dependent fundamental solution derived analytically by the usage of an appropriate algebraic transformation for the displacement vector and the Radon transform.

2 Statement of the problem

In a Cartesian coordinate system consider a linear MEE medium poled in Ox_3 direction and subjected to a time-harmonic anti-plane mechanical load on Ox_3 axis and in-plane electrical and magnetic loads in the plane Ox_1x_2. The only non-vanishing fields are the anti-plane mechanical displacement u_3, the in-plane electrical displacement D_i, the in-plane magnetic induction B_i, the electric field $E_i = -\varphi_{,i}$ and the magnetic field $H_i = -\psi_{,i}$, where φ, ψ are electric and magnetic potentials correspondingly. The constitutive relations in the plane Ox_1x_2 are, see Soh and Liu [3]

$$\sigma_{iK} = C_{iKJl}u_{J,l}, \quad x \in R^2 \setminus \Gamma, \tag{1}$$

where $x = (x_1, x_2)$, $\Gamma = \Gamma^+ \cup \Gamma^-$ is a finite crack - an open arc. Here coma denotes partial differentiation, small indexes $i, l = 1, 2$, capital indexes $K, J = 3, 4, 5$ and it is assumed summation in repeating indexes. The generalized displacement is $u_J = (u_3, \phi, \varphi)$, and the generalized stress tensor is $\sigma_{iJ} = (\sigma_{i3}, D_i, B_i)$, where σ_{i3} is the stress. Generalized elasticity tensor C_{iJKl} is defined as: $C_{iJKl} = 0$ for $i \neq l$ and $C_{i33i} = c_{44}$; $C_{i34i} = C_{i43i} = e_{15}$; $C_{i35l} = C_{i53l} = q_{15}$; $C_{i44l} = -\varepsilon_{11}$; $C_{i45l} = C_{i54l} = -d_{11}$; $C_{i55l} = -\mu_{11}$.
Functions $c_{44}(x)$, $e_{15}(x)$, $\varepsilon_{11}(x)$ are: elastic stiffness, piezoelectric coupled coefficient and dielectric permittivity, while $q_{15}(x)$, $d_{11}(x)$, $\mu_{11}(x)$ are piezomagnetic, magnetoelectric coefficients and magnetic permeability correspondingly. It is assumed that $c_{44}(x)$, $\varepsilon_{11}(x)$ and $\mu_{11}(x)$ are positive that corresponds to a stable material, see [3].
Suppose that the material parameters C_{iJKl} and density ρ depend in the same manner exponentially on x

$$C_{iKJl}(x) = C^0_{iKJl}e^{2<a,x>}, \quad \rho(x) = \rho^0 e^{2<a,x>}, \tag{2}$$

where $<,>$ means the scalar product in R^2, $a = (a_1, a_2)$ and we use the notations $a_1 = r\cos\alpha$, $a_2 = r\sin\alpha$, $r = |a|$ is the magnitude and α is the direction of the material inhomogeneity.
Assuming the quasistatic approximation of MEE material in the absence of body forces, electric charges and magnetic current densities, the balance equation is

$$\sigma_{iK,i} + \rho_{KJ}\omega^2 u_J = 0. \tag{3}$$

where $\rho_{QJ} = \begin{cases} \rho, Q = J = 3 \\ 0, Q, J = 4 \text{ or } 5 \end{cases}$ and ω is the frequency of the applied time-harmonic load.
The boundary condition on the crack is

$$t_J|_\Gamma = 0. \tag{4}$$

where $t_J = \sigma_{iJ}n_i$ is the generalized traction and $n = (n_1, n_2)$ is the normal vector to Γ. That means the crack is impermeable, i.e. the crack line is free of mechanical traction, electric charge and magnetic current. In the following we will study the case $\omega > \omega_0$ when the dynamic behavior of the MEE material is characterized with a wave propagation phenomena. The total generalized displacement u_J and traction t_J field is a sum of an incident SH-wave and scattered by the crack wave, i.e. $u_J = u_J^{in} + u_J^{sc}$ and $t_J = t_J^{in} + t_J^{sc}$. Here $\omega_0 = \sqrt{\frac{\det M}{(\varepsilon^0_{11}\mu^0_{11} - d^{02}_{11})\rho^0}}|a|$, where $M = \begin{pmatrix} c^0_{44} & e^0_{15} & q^0_{15} \\ e^0_{15} & -\varepsilon^0_{11} & -d^0_{11} \\ q^0_{15} & -d^0_{11} & -\mu^0_{11} \end{pmatrix}$.

Suppose that $U_J(x,\omega) = e^{<a,x>} u_J(x,\omega)$ satisfies Sommerfeld–type condition at infinity, more specifically $U_3 = o(|x|^{-1}), U_4 = o(e^{-|a||x|}), U_5 = o(e^{-|a||x|}) \quad \text{for} \quad |x| \to \infty$. This condition ensures uniqueness of the scattering field u_J^{sc} for a given incident field u_J^{in} and it can be proved that the boundary value problem (BVP) (3), (4) admits continuous differentiable solutions.
The non-hypersingular traction BIE is derived following Wang and Zhang [4] for the homogeneous, Rangelov et al. [5] for the inhomogeneous piezoelectric case and Stoynov and Rangelov [6, 7] for the MEE case. The following system of BIE, that is equivalent to the BVP (3), (4) is obtained

$$\begin{aligned} -t_J^{in}(x,\omega) &= C_{iJKl}(x)n_i(x)\int_{\Gamma^+}[(\sigma^*_{\eta PK}(x,y,\omega)\triangle u_{P,\eta}(y,\omega) \\ &- \rho_{QP}(y)\omega^2 u^*_{QK}(x,y,\omega)\triangle u_P(y,\omega))\delta_{\lambda l} \\ &- \sigma^*_{\lambda PK}(x,y,\omega)\triangle u_{P,l}(y,\omega)]n_\lambda(y)d\Gamma, \quad x\in\Gamma^+. \end{aligned} \tag{5}$$

where u^*_{JQ} is the fundamental solution of (3), obtained with Radon transform in Stoynov and Rangelov [7], $\sigma^*_{iJQ} = C_{iJMl}u^*_{MQ,l}$ is its stress, $\Delta u_J = u_J|_{\Gamma^+} - u_J|_{\Gamma^-}$ is the generalized crack opening displacement, x, y denote the field and the source point respectively. Equation (5) is traction non-hypersingular BIE on the crack line Γ for the unknown Δu_J. Once having a solution or the generalized crack opening displacement, the generalized displacement u_J can be obtained at every point in $R^2 \setminus \Gamma$ by using the corresponding representation formulae, see Stoynov and Rangelov [7].

3 Numerical realization

The numerical procedure for the solution of the BVP follows the numerical algorithm developed and validated in Rangelov et al. [5] for the inhomogeneous piezoelectric material and in Stoynov and Rangelov [7] for the homogeneous MEE case. The crack Γ is discretized by quadratic boundary elements (BE) away from the crack-tips and special crack-tip quarter-point BE near the crack-tips to model the asymptotic behavior of the displacement and the traction. Applying the shifted point scheme, the singular integrals converge in Cauchy principal value (CPV) sense, since the smoothness requirements $\Delta u_J \in C^{1+\alpha}(\Gamma)$ of the approximation are fulfilled.
In the numerical examples the crack Γ with a half-length $c = 5mm$, occupying an interval $(-c, c)$ on Ox_1 axis is considered. The crack is divided into 7 BE with lengths correspondingly: $l_1 = l_7 = 0.15c$, $l_2 = \cdots = l_6 = 0.34c$, 1^{st} BE is a left quarter point BE, 7^{th} BE is a right quarter point BE and the rest BE are ordinary BEs.
The material is magnetoelectroelastic composite $BaTiO_3/CoFe_2O_4$ with reference material constants C^0_{iJKl} given in Song and Sih [8].
The described numerical scheme is validated by benchmark examples describing fracture behaviour of a line finite crack in an infinite plane subjected to a normal incident time–harmonic SH–wave in three different kinds of material, more specifically: (a) graded elastic anisotropic, see Daros [9]; (b) graded piezoelectric, see Rangelov et al. [5]; (c) homogeneous MEE composite, see Stoynov and Rangelov [7].
The dynamic fracture state of MEE is characterized by the leading term of the asymptotic of the generalized displacement and the generalized traction near the crack–tips

presented by the generalized intensity factor (GIF). For the considered MEE media GIFs are stress intensity factor K_{III}, electric field intensity factor K_E and magnetic field intensity factor K_H. For the straight crack on Ox_1, $\Gamma = (-c, c)$ they are defined as

$$\begin{array}{l} K_{III} = \lim_{x_1 \to \pm c} t_3 \sqrt{2\pi(x_1 \mp c)}, \\ K_E = \lim_{x_1 \to \pm c} E_2 \sqrt{2\pi(x_1 \mp c)}, \\ K_H = \lim_{x_1 \to \pm c} H_2 \sqrt{2\pi(x_1 \mp c)} \end{array} \tag{6}$$

where t_3 and E_2, H_2 are calculated at the point $(x_1, 0)$ close to the crack-tip.
In the figures the normalized frequency is $\Omega = ck^0, k^0 = \sqrt{\rho^0/c^0_{44}}\omega$ and normalized GIFs mechanical stress intensity factor $K^*_{III} = \frac{K_{III}}{t_3^{in}\sqrt{\pi c}}$, electric field intensity factor $K^*_E = \frac{10K_E}{t_3^{in}\sqrt{\pi c}}$ and magnetic field intensity factor $K^*_H = \frac{10^4 K_H}{t_3^{in}\sqrt{\pi c}}$, are plotted.
Fig. 1 shows the frequency dependence of the GIF K^*_{III}, K^*_E and K^*_H for the left crack tip, at different magnitudes of the material gradient $\beta = 2rc$ for $\beta = 0.0; 0.2; 0.4; 0.6$, at direction of material inhomogeneity along the crack, i.e. $\alpha = 0$ and in the case of a normal incident wave, i.e. $\theta = \pi/2$. Analysis of these results leads to the following observations: (a) there is a frequency $\Omega = 1.1$ where dynamic overshoot occurs and this frequency is not shifted when the material inhomogeneity is involved; (b) the magnitude of the material gradient has influence on all stress, electric field and magnetic induction concentration near the crack. A comparison between the results for the homogeneous material and for the inhomogeneous one with magnitude $rc = 0.3$ shows K^*_{III}, K^*_E and K^*_H increase with about 19%, 24% and 22% respectively when the observer point is near the left crack-tip.
The sensitivity of the generalized stress concentration with respect to the direction of the material gradient $\alpha = k\pi/2, k = 0.0, 0.1 \cdots 1$ is demonstrated on Fig. 2, where case (a) is for the right crack tip and case (b) is for the left crack tip correspondingly. The fixed parameters are: $\Omega = 1.0$, $\theta = \pi/2$ and $\beta = 0.2, 0.4, 0.6$. The obtained results show that stress concentration fields are different at both crack-tips and even they have quite different behaviour: (a) the right crack-tip shows the maximal values for GIF in the case when the direction of material gradient is $\alpha = \pi/2$, while in contrast, the left crack-tip has its maximal values of GIF at $\alpha = 0.0$. These presented results show that in functional graded MEE material the local stress fields depend on the magnitude and direction of material gradient r, α.

4 Conclusion

A dynamic fracture analysis of an exponentially inhomogeneous MEE cracked plane subjected to time-harmonic anti-plane mechanical and in-plane electromagnetic loads is presented in this study. The results show the sensitivity of the GSIFs to the type of the material inhomogeneity characteristics, to the coupled nature of MEE continua and to the properties of the applied dynamic electro-magneto mechanical load. The presented method can be successfully used for the more complex problems of crack interactions, cracks with arbitrary shapes and composites with different combinations of piezoelectric and piezomagnetic constituents.

Acknowledgement. The authors acknowledge the support of the BNSF under the Grant No. DID 02/15 and the support of TU Sofia under the Grant No. 102 NI 218-11.

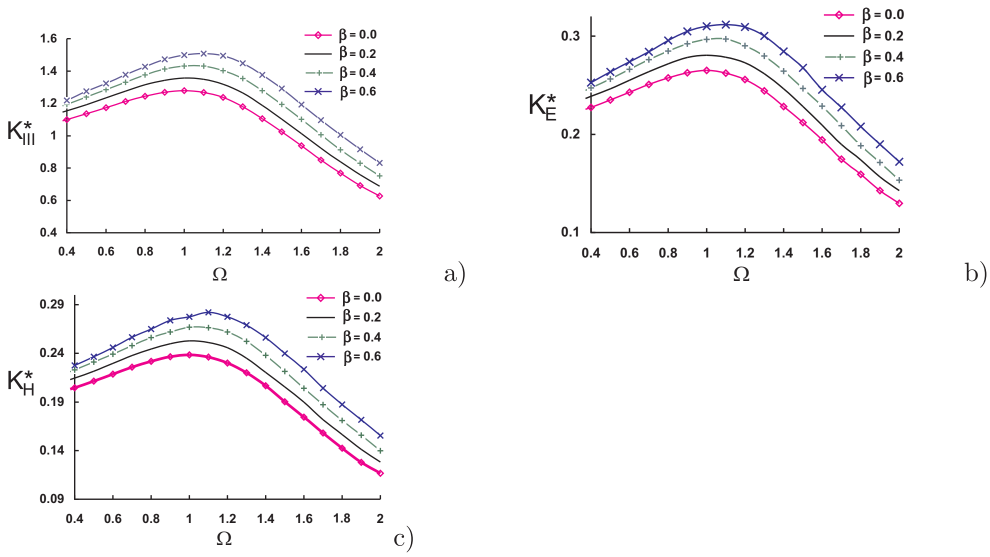

Figure 1: GIF versus normalized frequency Ω at the left crack–tip for different values of the magnitude β at a direction of material inhomogeneity $\alpha = 0.0$ and a wave incident angle $\theta = \pi/2$: (a)K^*_{III} ; (b) K^*_E; (c) K^*_H.

Acknowledgement. The authors acknowledge the support of the BNSF under the Grant No. DID 02/15 and the support of TU Sofia under the Grant No. 102 NI 218-11.

References

[1] C. H. Chue and W. H. Hsu. Antiplane internal crack normal to the edge of a functionally graded piezoelectric/piezomagnetic half plane. *Meccanica*, 43:307–ï¿½325, 2008.

[2] C. C. Ma and J. M. Lee. Theoretical analysis of in-plane problem in functionally graded nonhomogeneous magnetoelectroelastic bimaterials. *Int. J. Solids Struct.*, 46:4208–4220, 2009.

[3] A. K. Soh and J. X. Liu. On the constitutive equations of magnitoelectroelastic solids. *J. Intell. Mater. Syst. Struct.*, 16:597–602, 2005.

[4] C. Y. Wang and C. Zhang. 2D and 3D dynamic Green's functions and time-domain BIE formulations for piezoelectric solids. *Engng. Anal. BE*, 29:454–465, 2005.

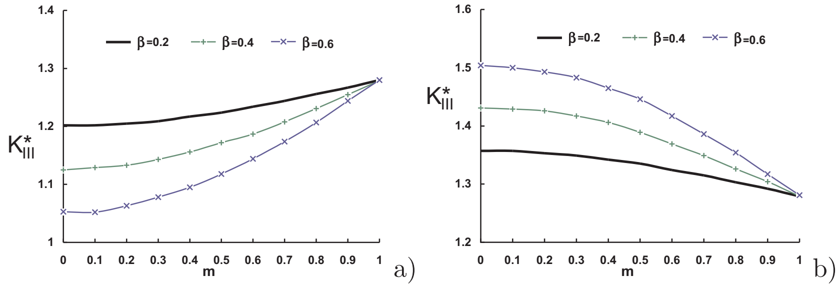

Figure 2: K^*_{III} versus the direction of material inhomogeneity $\alpha = m\pi/2$, $m = 0.0, 0.1 \cdots 1.0$ at a wave propagation direction $\theta = \pi/2$ for different values of the magnitude β and normalized frequency $\Omega = 1.0$: (a) right crack–tip; (b) left crack–tip.

[5] T. Rangelov, P. Dineva, and D. Gross. Effect of material inhomogeneity on the dynamic behavior of cracked piezoelectric solids: a BIEM approach. *ZAMM-Z. Angew. Math. Mech.*, 88:86–99, 2008.

[6] Y. Stoynov and T. Rangelov. Time-harmonic behaviour of anti-plane cracks in inhomogeneous magnetoelectroelastic solid. *Compt. Rend. Acad. Bulg. Sci.*, 62 (2):175–186, 2008.

[7] Y. Stoynov and T. Rangelov. Time-harmonic crack problems in magnetoelectroelastic plane by BIEM. *J. Theor. Appl. Mech.*, 39(4):73–92, 2009.

[8] Z. F. Song and G. C. Sih. Crack initiation behavior in magneto - electro - elastic composite under in - plane deformation. *Theor. Appl. Fract. Mech.*, 39:189–207, 2003.

[9] C. H. Daros. On modelling SH–waves in a class of inhomogeneous anisotropic media via the Boundary Element Method. *ZAMM-Z. Angew. Math. Mech.*, 90: 113–ï¿$\frac{1}{2}$121, 2010.