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Edge of chaos in reaction-diffusion CNN models

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In this paper a polynomial Cellular Neural Network (CNN) model of Newell-Whitehead equation is introduced. Local activity domain and edge of chaos domain of the parameter space is found for the model. Numerical simulations of the CNN dynamics confirm the so called phenomena edge of chaos and help the better understanding of genesis and emergence of complexity in Newell-Whitehead equation.

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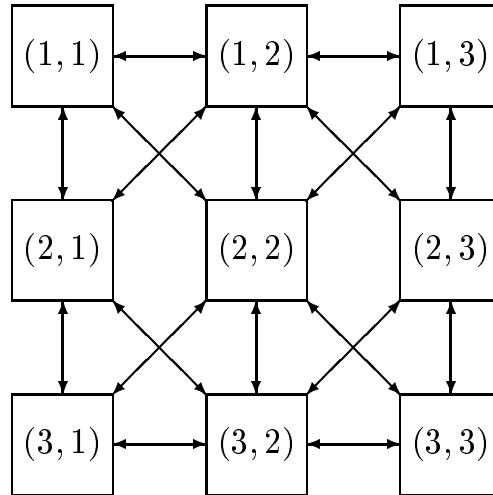
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1. INTRODUCTION

Reaction-diffusion type of equations are widely used to describe phenomena in different fields, as biology-Fisher model [1], FitzHugh-Nagumo nerve conduction model [1,10], Vector-disease model, chemistry - Brusselator model, physics - Sine-Gordon model [9], etc. In his pioneering work, Fisher [1] used a logistic-based reaction-diffusion model to investigate the spread of an advantageous gene in a spatially extended population. The generalized diffusion equation with a nonlinear source term which encompasses the Fisher, Newell-Whitehead and Fitzhugh-Nagumo equations as particular forms and appears in a wide variety of physical and engineering applications. Modulation equations play an essential role in the description of systems which exhibit patterns of nearly periodic nature. The so called Newell-Whitehead equation [7] is derived to describe the envelope of modulated roll-solutions with two large extended or unbounded space direction.

Cellular Neural Networks (CNNs) are complex nonlinear dynamical systems, and therefore one can expect interesting phenomena like bifurcations and chaos to occur in such nets. It was shown that as the cell self-feedback coefficients are changed to a critical value, a CNN with opposite-sign template may change from stable to unstable [3]. Namely speaking, this phenomenon arises as

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FIGURE 1. 3×3 neighborhood CNN.

the loss of stability and the birth of a limit cycles [3]. Moreover, the appearance of a strange attractor in a periodically driven two-cell CNN have been reported. In a three-cell autonomous CNN this attractor has properties similar to the double scroll attractor [3].

Let us consider a two-dimensional grid with 3×3 neighborhood system as it is shown on Fig.1.

The squares are the circuit units - cells, and the links between the cells indicate that there are interactions between linked cells. One of the key features of a CNN is that the individual cells are nonlinear dynamical systems, but that the coupling between them is linear. Roughly speaking, one could say that these arrays are nonlinear but have a linear spatial structure, which makes the use of techniques for their investigation common in engineering or physics attractive.

We will give the general definition of a CNN which follows the original one [2]:

Definition 1. *The CNN is a*

- a). *2-, 3-, or n- dimensional array of*
- b). *mainly identical dynamical systems, called cells, which satisfies two properties:*
- c). *most interactions are local within a finite radius r , and*
- d). *all state variables are continuous valued signals.*

Definition 2. An $M \times M$ cellular neural network is defined mathematically by four specifications:

- 1). CNN cell dynamics;
- 2). CNN synaptic law which represents the interactions (spatial coupling) within the neighbor cells;
- 3). Boundary conditions;
- 4). Initial conditions.

Now in terms of definition 4 we can present the dynamical systems describing CNNs. For a general CNN whose cells are made of time-invariant circuit elements, each cell $C(ij)$ is characterized by its CNN cell dynamics :

$$(1) \quad \dot{x}_{ij} = -g(x_{ij}, u_{ij}, I_{ij}^s),$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usually a scalar. In most cases, the interactions (spatial coupling) with the neighbor cell $C(i+k, j+l)$ are specified by a CNN synaptic law:

$$(2) \quad I_{ij}^s = A_{ij,kl}x_{i+k,j+l} + \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \tilde{B}_{ij,kl} * u_{i+k,j+l}(t).$$

The first term $A_{ij,kl}x_{i+k,j+l}$ of (2) is simply a linear feedback of the states of the neighborhood nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood.

Complete stability, i.e. convergence of each trajectory towards some stationary state, is a fundamental dynamical property in order to design CNN's for accomplishing important tasks including image processing problems, the implementation of content addressable memories and the solution of combinatorial optimization problems [4]. The most basic result on complete stability is certainly the one requiring that the CNN interconnection matrix \tilde{A} be symmetric [2]. Also some special classes of nonsymmetric CNN's such as cooperative (excitatory) CNN's, were shown to be completely stable [4]. In the general case, however, competitive (inhibitory) CNN's may exhibit stable nonlinear oscillations [4].

It is known [3] that some autonomous CNNs represent an excellent approximation to nonlinear partial differential equations (PDEs). In this paper we will present the receptor-based model by a reaction-diffusion CNNs. The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. Consider the following well-known PDE, generally referred to us in the literature as a reaction-diffusion equation [1]:

$$\frac{\partial u}{\partial t} = f(u) + D\nabla^2 u,$$

where $u \in \mathbf{R}^N$, $f \in \mathbf{R}^N$, D is a matrix with the diffusion coefficients, and $\nabla^2 u$ is the Laplacian operator in \mathbf{R}^2 . There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate A -template [8,9].

In this paper we shall study the Newell-Whitehead equation [7] of the form:

$$(3) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^3,$$

where a and b are positive parameters. Partial differential equations of diffusion type have long served as models for regulatory feedbacks and pattern formation. Such systems cause some difficulty, since both existence and behavior of the solutions are more difficult to establish. Many aspects of qualitative behavior have to be investigated numerically. For this purpose we apply the Cellular Neural Networks (CNN) approach for studying such models.

In Section 2 we shall construct our Polynomial Cellular Neural Network (PCNN) model for Newell-Whitehead equation (3). In Section 3 we shall define the local activity parameter domain as well as edge of chaos phenomena. This will help us to understand better the genesis and emergence of complexity. Numerical simulations of the CNN dynamics will show the so called phenomena edge of chaos [5,6].

2. POLYNOMIAL CELLULAR NEURAL NETWORK MODEL

In a recently proposed VLSI development [11] a first CNN based hardware implementation with polynomial weight functions has been presented. Let us consider a m -layer polynomial CNN with cells $C^l(i)$ (arranged in a one-dimensional grid on each layer) assuming only coupling between adjacent layers with the following state equation

$$(4) \quad \dot{x}_i^l(t) = f^l(x_i^1, \dots, x_i^l, \dots, x_i^m) + \sum_{i_1=-1}^1 \dots \sum_{i_n=-1}^1 x_{i+i_n}^l a_{i_n},$$

where $l = 1, \dots, m$, i_j is the j th component of the index vector i (with $i_j \in \{-1, 0, 1\} \forall j \in \mathbf{N}$), i is the spatial coordinate vector and a_{i_n} are the elements of

the following A -template:

$$a_{i_n} = \begin{cases} -2n & i_1 = i_2 = \dots = i_n = 0 \\ 1 & \sum_{j=1}^n |i_j| = 1 \\ 0 & \sum_j^n |i_j| > 1. \end{cases}$$

For the purpose of modelling, a concrete representation of $f^l(\cdot)$ is needed that covers a wide class of possible functions. Consider the following simplified representation of $f^l(\cdot)$

$$(5) \quad f^l(x_i^1, \dots, x_i^m) = P_{l,l}(x_i^l) + P_{l,l-1}(x_i^{l-1}) + P_{l,l+1}(x_i^{l+1}),$$

with functions $P_{l,l'(\cdot)}$ defined by the polynomial

$$P_{l,l'}(x_i^{l'}) = \sum_{k=0}^K b_{ll'}^{(k)} (x_i^{l'})^k$$

For the reaction-diffusion Newell-Whitehead equation (3), the corresponding polynomial CNN model can be write as the following system:

$$(6) \quad \frac{du_j}{dt} = A_1 * u_j + f(u_j), \quad 1 \leq j \leq N,$$

where $A_1 : (1, -2, 1)$ is one-dimensional discretized Laplacian template, $*$ is convolution operator, the variable u from (3) is mapped to e reaction-diffusion CNN with polynomial order three and the cell states u_j represent the solution u leading to state equations given by (6). Here,

$$f(u_j) = b_1(u_j) + b_2(u_j)^2 + b_3(u_j)^3 + v_1$$

is Taylor series expansion of the functions $f(u) = au - bu^3$.

In order to use a general CNN structure for a broad class of nonlinearities, a Taylor serie expansion (TSP) of the feedback function may be helpful, leading directly to above mentioned CNN with polynomial weight functions. Therefore the template coefficients in equation (6) are identified by the series expansion coefficients of the feedback function.

A numerical integration of (6) with the initial conditions $u_j(0) = 0$ leads to the solution which is shown on Fig.2:

Remark 1. In order to model a system represented by solutions of (3) using PCNN (6) the coefficients b_i have to be determined in an optimization process.

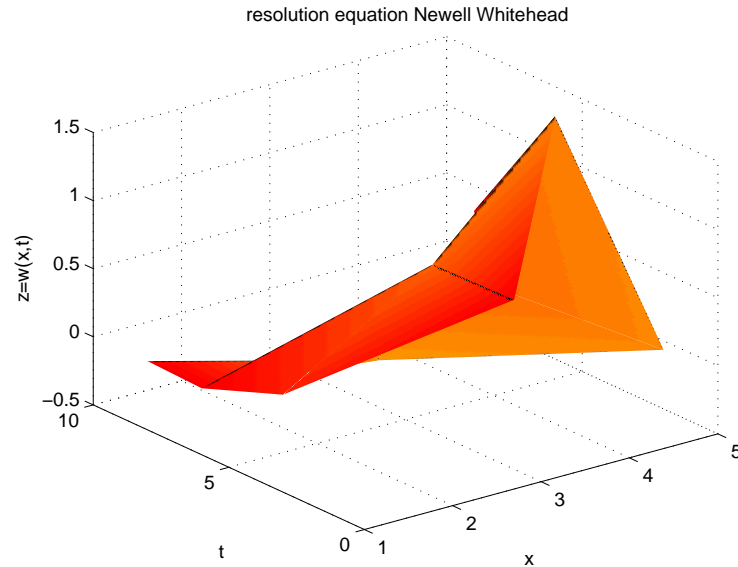


FIGURE 2. Simulation of PCNN model (6).

Thereby the solutions in Fig.3 was used as reference. During the optimization process the mean square error

$$e_{mse} = \sum_i \sum_j \sum_{l=1}^2 \frac{(u_j^l - \tilde{u}_j^l)^2}{n}$$

can be minimized using Powells method and Simulated Annealing [11]. In each step e_{mse} is calculated by taking the reference $u_j^l(t)$ and the output \tilde{u}_j^l of PCNN obtained by simulation system MATCNN applying 4th- order Runge-Kutta integration. In order to minimize the computational complexity and to maximize the significance of the mean square error only outputs of 10 cells are taken into account.

3. EDGE OF CHAOS IN THE NEWELL-WHITEHEAD CNN MODEL

The theory of local activity provides a definitive answer to the fundamental question: what are the values of the cell parameter for which the interconnected system may exhibit complexity? The answer is given in [4,5] - the necessary condition for a nonconservative system to exhibit complexity is to have its cell locally active. The theory which will be presented below and which follows [5] offers a constructive analytical method for uncovering local activity.

In particular, for diffusion CNN model, one can determine the domain of the cell parameters in order for the cells to be locally active, and thus potentially capable of exhibiting complexity. This precisely defined parameter domain is called the edge of chaos [5,6].

We apply the following constructive algorithm:

1. Map Newell-Whitehead equation (3) into the following associated discrete-space version which we shall call Newell-Whitehead PCNN model:

$$(7) \quad \frac{du_j}{dt} = A_1 * u_j + f(u_j), 1 \leq j \leq N$$

2. Find the equilibrium points of (7). According to the theory of dynamical systems the equilibrium points u^* of (7) are these for which :

$$(8) \quad A_1 * u^* + f(u^*) = F(u^*) = 0$$

Equation (8) may have one, two or three real roots u_1^* , u_2^* , u_3^* , respectively. In general, these roots are functions of the cell parameters a, b . In other words, we have $u_k^* = u_k^*(a, b)$, $k = 1, 2, 3$.

3. Calculate now the Jacobian matrix of (8) about each system equilibrium point $E_1^* = (u_1^*)$, $E_2^* = (u_2^*)$, $E_3^* = (u_3^*)$. In our particular case the associate linear system in a sufficient small neighborhood of the equilibrium points E_i^* can be given by

$$\frac{dz}{dt} = DF(E_i^*)z, i = 1, 2, 3,$$

$DF(E_i^*) = J$ is the Jacobian matrix of each of the equilibrium point and can be computed by:

$$(9) \quad J_{ps} = \left. \frac{\partial F_p}{\partial u_s} \right|_{u=E_i^*}, 1 \leq p, s \leq N.$$

In our particular case the Jacobian matrix is:

$$(10) \quad J = A_1 + (a - 3b(u^*)^2)I_d,$$

I_d is the identity matrix.

4. Calculate the trace $Tr(E_k^*) = \sum_{q=1}^N \lambda_q = \text{trace} [A_1 + (a - 3b(u^*)^2)I_d]$.
5. We shall identify the cell state variables u_j as follows: u_j is associated with the node-to-datum voltage at node (j) of a two-dimensional grid G of linear resistors. The importance of the circuit model is not only in the fact that we have a convenient physical implementation, but also in the fact that well-known results from classic circuit theory can be used

to justify the cells' local activity [4]. In this sense, if there is at least one equilibrium point for which the circuit model of the cell acts like a source of "small signal" power, in a precise sense defined in [4], i.e. if the cell is capable of injecting a net small-signal average power into the passive resistive grids then the cell is said to be locally active.

Definition 3. *Stable and Locally Active Region $SLAR(E_k)$ at the equilibrium point E_k for Newell-Whitehead PCNN model (7) is such that $Tr < 0$.*

In our particular case we have:

$$(11) \quad Tr(E_1^* = 0) = -2 + a$$

$$(12) \quad Tr(E_{1,2}^* = \pm \sqrt{\frac{a+1}{b}}) = -5 - 2a$$

In order the condition of Definition 3 to fulfill it is necessary the cell parameters to satisfy the following inequalities: $a > -5/2, b > 0$. Then

$$(13) \quad SLAR(E_k) : a > -5/2, b > 0$$

6. Edge of chaos.

In the literature [4,5,6] the so called edge of chaos (EC) means a region in the parameter space of a dynamical system where complex phenomena and information processing can emerge. We shall try to define more precisely this phenomena till now known only via empirical examples. Moreover, we shall present an algorithm for determining the edge of chaos for reaction-diffusion CNN models as the Newell-Whitehead PCNN model (7).

We determine the Stable Local Activity Region for each point in the cell parameter space by (13). We shall identify the edge of chaos domain EC in the cell parameter space by using the following definition [4,5]:

Definition 4. *A reaction-diffusion CNN is said to be operating on the edge of chaos EC iff there is at least one equilibrium point $E_k, k = 1, 2, 3$ which belongs to $SLAR(E_k)$.*

The following theorem then hold:

Theorem 1. *PCNN model of Newell-Whitehead equation (3) is operating in the edge of chaos regime iff $a > -5/2, b > 0$. For this parameter values there is at least one equilibrium point which belongs to $SLAR(E_k)$.*

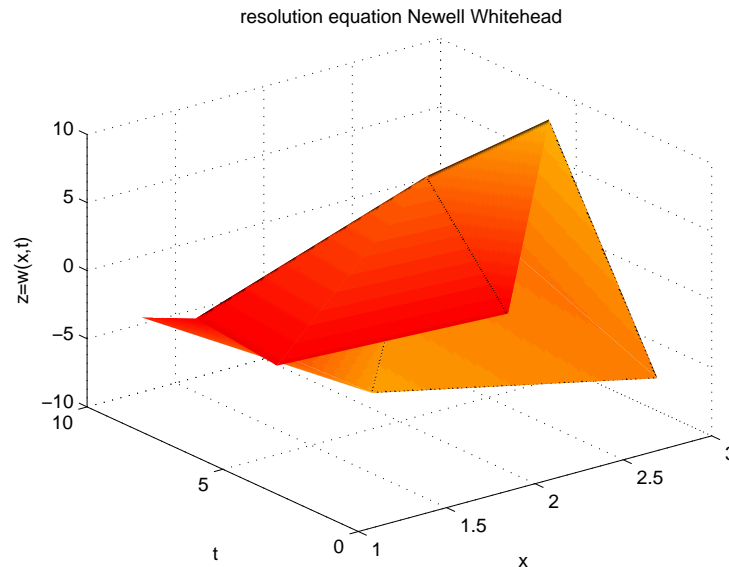


FIGURE 3. EC domain for Newell-Whitehead PCNN model.

The edge of chaos domain EC for the Newell-Whitehead PCNN model (7) is given on Fig.3:

Remark 2. By introducing the edge of chaos domain we determine the exact cell parameter values for which the equilibrium points of the Newell-Whitehead PCNN model (7) are both locally active and stable. In other words there exists a domain in which our PCNN model exhibits complexity.

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