Modeling tsunami waves via Cellular Nonlinear Networks

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The study of propagation of tsunami from their small disturbance at the sea level to the size they reach approaching the coast has involved the interest of several scientists. It is clear that in order to predict accurately the appearance of a tsunami it is fundamental to built up a good model. From this point of view the most important tool in the context of water waves is soliton theory [6]. Frequently in the literature it is stated that a tsunami is produced by a large enough soliton. Solitons arise as special solutions of a widespread class weakly nonlinear dispersive PDEs modeling water waves, such as the KdV or Camassa-Holm equation [1,5], representing to various degrees of accuracy approximations to the governing equations for water waves in the shallow water regime. How the tsunami is initiated? The thrust of a mathematical approach is to examine how a wave, once initiated, moves, evolves and eventually becomes such a destructive force of nature.
In Constantin and Johnson [4] the model of the motion of the water before arrival of a tsunami wave is proposed. They require that a flat free surface for the background state excludes linear vorticity functions, unless the flow is trivial. So, nonlinear vorticity distributions are introduced in order to admit nontrivial flows with a flat free surface. Consider the following dynamical system

$$\varphi_{tt} + \varphi_{xx} = -f(\varphi), \tag{1}$$

$$f(\varphi) = \begin{cases} \varphi - \varphi|\varphi|^{-1/2} & \text{if} \quad \varphi \neq 0 \\ 0 & \text{if} \quad \varphi = 0. \end{cases} \tag{2}$$

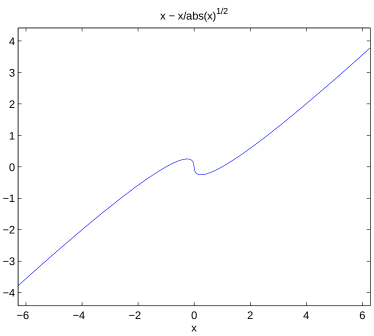

Fig.1. Bifurcation diagram of the nonlinear vorticity distribution.

By applying Cellular Neural Networks (CNN) approach [2,3,8] we shall study the wave propagation of the model (1), (2).
CNN model of our system (1), (2) will be the following:

$$\begin{aligned}\frac{dv_j}{dt} &= A_1 * u_j + f(u_j), \qquad (3)\\ \frac{du_j}{dt} &= v_j, 1 \le j \le N\end{aligned}$$

Our objective in this paper is to study the structure of the travelling wave solutions of the CNN model (3). There has been many studies on the travelling wave solutions of spatially discrete or both spatially and time discrete systems [7,9]. The study of travelling wave solutions can proceed as follows. Consider solutions of (3) of the form:

$$z_j = \Phi(j - ct),$$

$z_j = col(u_j, v_j)$ for some continuous functions $\Phi : \mathbf{R}^1 \to \mathbf{R}^1$ and for some unknown real number c. Denote $s = j - ct$. Let us substitute (4) in our CNN model (3). Then $\Phi(s)$ and c satisfies the system of the form:

$$\begin{aligned}-c\Phi^{'}(s) = G(\Phi(s + r_0), \Phi(s + r_1), \ldots, \Phi(s + r_n)+ \qquad (4)\\ +F(\Phi(s + r_0) = 0,\end{aligned}$$

here $r_0 = 0$, r_i are real numbers for $i = 1$ to n. Equation (5) is called bistable because it has three spatially homogeneous solutions $\Phi(s) \equiv z^-, z^0, z^+$ satisfying $z^- < z^0 < z^+$, and

$$G(z, z, \ldots, z) > 0 \text{ for } z \in (-\infty, z^-) \cup (z^0, z^+),$$

$$G(z, z, \ldots, z) < 0 \text{ for } z \in (z^-, z^0) \cup (z^+, \infty),$$

Recently, Mallet-Paret [7] showed that (5) has a unique monotone solutions satisfying the boundary conditions:

$$lim_{s\to-\infty}\Phi(s) = z^- \quad \text{and} \quad lim_{s\to\infty}\Phi(s) = z^+. \qquad (5)$$

More precisely, it is proved that under some assumptions, there is a unique c^* such that (5) has a monotone solutions satisfying (6) iff $c = c^*$, and such solution is also unique up to a phase shift if $c = c^* \neq 0$. Indeed, the solution $\Phi(s)$ of (5) and $lim_{s\to\infty}\Phi(s) = z^+$ can be represented as

$$\Phi(s) = z^+ - \gamma e^{\sigma s} - \tilde{\Phi}(s)e^{2\sigma s},$$

for $s \gg 1$, $\sigma^+ < 0$, $\gamma > 0$, $\tilde{\Phi}(s)$ is a bounded and C^1-function.
Suppose that our CNN model (3) is a finite circular array of $L = N.N$ cells. For this case we have finite set of frequences [2,3]:

$$\Omega = \frac{2\pi k}{L}, \ 0 \le k \le L - 1. \qquad (6)$$

The following proposition then hold:

Proposition 1. *Suppose that $z_j(t) = \Phi(j - ct)$ is a travelling wave solutions of the CNN model (3) with $\Phi \in C^1(\mathbf{R^1}, \mathbf{R^1})$ and $\Omega = \frac{2\pi k}{L}$, $0 \le k \le L-1$. Then there exist constants $c_* < c^* < 0$ such that*
(i) if $c \le c_*$ then $\Phi(s; c)$ is nondecreasing and satisfies

$$lim_{s\to-\infty}\Phi(s) = z^0 \quad \text{and} \quad lim_{s\to\infty}\Phi(s) = z^+; \tag{7}$$

(ii) if $c = c^* > c_*$, then $\Phi(s; c)$ is nondecreasing and satisfy (6);
(iii) if $c^* < c < 0$, then $\Phi(s; c)$ is nondecreasing and unbounded.
Let us introduce the following energy function for our CNN model (3):

$$E(u_j, v_j) = \frac{1}{2}u_j^2 + \frac{1}{2}v_j^2 - \frac{2}{3}|v_j|^{3/2}. \tag{8}$$

We obtain the following simulation results in the plane (u, v)which present tow closed curves representing the solution set of the equation $u_j^2 = \frac{4}{3}|v_j|^{3/2} - v_j^2$:

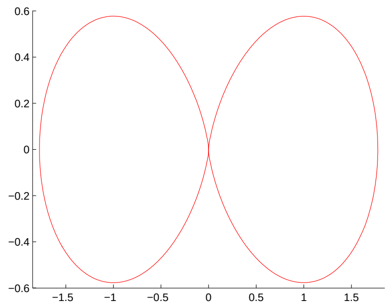

Fig.2.

The stationary points of (3) are $(\pm1, 0)$, or $(0, 0)$. If the energy function $E < 0$ the set of solutions consists of the interiors of the two closed curves given on Fig.2. By virtue of (12), once a solution of (1) intersects the boundary of these two closed curves at a point other than $(0, 0)$, it will remain in that set having the asymptotic limit $(-1, 0)$, or, correspondingly $(1, 0)$. Notice that on the boundary of the two closed curves (Fig.2) we have $E = 0$, while within these sets $E < 0$, with the minimum attained at $(\pm1, 0)$ where $E = -\frac{1}{6}$.
Let us consider the following initial conditions for our CNN model (3):

$$\left| \begin{array}{l} u_j(0) = a \\ v_j(0) = 0. \end{array} \right. \tag{9}$$

If $z^+ \to a$, for $s \to \infty$, the number of intersections approaches infinity. Denote by Ω_+, Ω_- the sets of points $(a, 0)$, where $a \in (-\infty, 0) \cup (0, \infty)$ and suppose that the corresponding solution of (3) has asymptotic limit $(1, 0)$, $(-1, 0)$ respectively. It is easy to prove that all intersections being transversal to the horizontal axis are

stable under small perturbations [5]. Therefore, for any integer $M \geq 1$ by continuous dependence on the initial data it might be proved that as $a \to \infty$ the number M of intersections approaches infinity.
In our case $\Phi : [0, \infty) \to [0, \infty)$ is defined implicitly by

$$r = \int_{\Phi}^{1/4} \frac{dt}{\sqrt{\frac{4}{3}|t|^{3/2} - t^2}}, r \in [0, I]$$

where

$$I = \int_{0}^{1/4} \frac{dt}{\sqrt{\frac{4}{3}|t|^{3/2} - t^2}}$$

and extend to $s \geq 1$ by setting Φ equal to zero here. Simulations of the implicit solution Φ are given below:

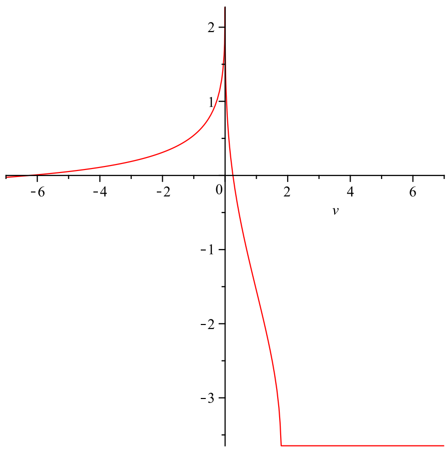

Fig.3. Real part of the solution Φ.

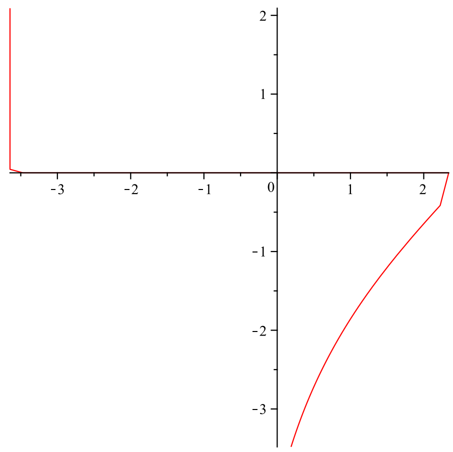

Fig.4. Implicitly solution Φ.

Remark 1. Notice that

$$\Phi_+^{''} + \Phi_+ - \Phi_+ |\Phi_+|^{-1/2} = 0,$$

with

$$\Phi_+(0) = \frac{1}{4}, \Phi_+^{'}(0) = -\frac{1}{4}\sqrt{\frac{5}{3}}, \Phi_+(I) = \Phi_+^{'}(I) = 0.$$

Since $lim_{s\to\infty}\Phi_0(s) = 0$, there exists some $s_0 > 0$ such that $|\Phi_0(s)| < \frac{1}{4}$ for $s \geq s_0$. So we claim

$$-\Phi_+(s - s_0) \leq \Phi_0(s) \leq \Phi_+(s - s_0), s \geq s_0.$$

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